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RICCI CURVATURE AND QUASICONFORMAL DEFORMATIONS OF A RIEMANNIAN MANIFOLD

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The theory of quasiconformal deformations of a Riemannian manifold (M, g) of dimension n leads in a natural way to the Ahlfors S operator being the symmetric and trace free part of the Levi-Civita connection ∇ on 1-forms, and to the Ahlfors Laplacian $L = S^*S = 2\delta d + \frac{4n-4}{n} d\delta - 4R$, where R is the Ricci action. It is well known that there are no conformal deformations on compact Riemannian manifolds M with negative Ricci curvature. The question arises, how close to being conformal a deformation on such M can be, i.e. the question on the minimal constant of quasiconformality. Using spectral properties of L, we derive several lower bounds for the constant of quasiconformality for the normalized deformations of compact manifolds with the positive definite, negative definite, or vanishing Ricci tensor R. As a result, we also obtain that there are no conformal deformations if the Ricci curvature is positive definite but small enough (Corollary 5.17).

0. Introduction.

The theory of conformal or, more general, quasiconformal deformations, leads in a natural way to the first order differential operator $SZ = \mathcal{L}_Z g - \frac{2}{n} divZ \cdot g$, where Z is a vector field (= deformation) on a Riemannian manifold (M, g) of dimension n and $\mathcal{L}_Z$ and $divZ$ denote the Lie derivative and the divergence of Z, respectively (cf. [5]). By the natural duality between vector fields Z and 1-forms $\alpha : \alpha(X) = g(Z, X)$, S may be regarded as an operator acting on 1-forms:

$$S\alpha = 2\nabla^S \alpha + \frac{2}{n} \delta\alpha g$$

where ∇^S is the symmetrized Levi-Civita connection ∇ of the Riemannian metric g:

$$\nabla^S \alpha(X, Y) = \frac{1}{2}[(\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X)],$$

and δ denotes the codifferential operator $\delta\alpha = -\nabla^i\alpha_i$. S will be called the Ahlfors operator. The operator S^* formally adjoint to S is of the form

$$S^*\varphi = 2\delta\varphi$$

for every symmetric trace-free two-form φ. There is also some interest in investigating the second-order differential operator $L = S^*S$, especially in the compact case (cf. [3]). It may be written in the form

$$L\alpha = S^*S\alpha = 2\delta d\alpha + \frac{4n-4}{n}d\delta\alpha - 4R\alpha \tag{0.1}$$

where R is the Ricci tensor of M and $R\alpha(X) = R(Z, X)$.

The aim of this paper is to derive some spectral properties of L by using the decomposition formula (0.1) and to interpret them in the language of quasiconformal deformations. In particular, we derive several lower bounds for the constant of quasiconformality for normalized deformations in the case of a compact Riemannian manifold with positive definite, negative definite, or vanishing Ricci tensor R. The results are in some sense complementary to that of [3] where upper bounds for the constant of quasiconformality have been given.

The following notation will be used throughout the paper.

M denotes a compact Riemannian manifold of dimension n with a Riemannian metric g. For simplicity we assume that all manifolds and mappings are smooth, i.e. of class C^∞. ∇ denotes the Levi-Civita connection of g extended in a natural way to the whole tensor algebra of M. $\mathcal{X}$, $\mathcal{D}$, and $\mathcal{M}$ denote the spaces of all vector fields, 1-forms, and symmetric trace free two-forms on M, respectively. Λ^k $(k = 0, \cdots, n)$ denotes the space of skew-symmetric k-forms. In particular $\Lambda^0 = C^\infty$ and $\Lambda^1 = \mathcal{D}$. At each point $x \in M$, T_x and T_x^* denote the tangent and cotangent spaces at p, respectively.

If $e_1, \cdots, e_n$ is a basis for T_x and $\omega^1, \cdots, \omega^n$ its dual basis in T_x^*, we define the pointwise scalar product of covectors $v = v_i\omega^i$, $w = w_j\omega^j$ (summation convention) as follows

$$g(v, w) = g_x(v, w) = g^{ij}v_iw_j, \tag{0.2}$$

where (g^{ij} is the inverse matrix of $(g_{ij}) = (g(e_i, e_j))$). Similarly, if v, w are symmetric tensors $v = v_{ij}\omega^i \otimes \omega^j$, $w = w_{kl}\omega^k \otimes \omega^l$, we define $g(v, w)$ as follows

$$g(v, w) = g_x(v, w) = g^{ik}g^{jl}v_{ij}w_{kl}. \tag{0.3}$$

The fact that we use the same letter g for the extended metric should not be confusing.

We also define the global scalar product $(\cdot,\cdot)$:

$$(V,W) = \int_M g(V,W), \qquad V,W \in \mathcal{X} \text{ (or } \mathcal{D}, \text{or } \mathcal{M}). \tag{0.4}$$

The integral is taken with respect to the Lebesgue measure on M generated by g.

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1. Quasiconformal Deformations and Transformations.

Definition 1.1. *A vector field Z on M is said to be a k-quasiconformal (shortly k-q.c.) deformation if for its dual 1-form α the inequality*

$$\|S\alpha\|_\infty \leq k \tag{1.2}$$

holds, where $\| \ \|_\infty$ denotes the supremum norm:

$$\|S\alpha\|_\infty = \sup_{x\in M} \|S\alpha\|_x = \sup_{x\in M} (g_x(S\alpha, S\alpha))^{1/2}.$$

If this is the case we will also say that α defines a k-q.c. deformation.

Every deformation Z generates a 1-parameter group of transformations F_t ($t \in \mathbf{R}$) of M, defined by the relations $(\partial/\partial t)F_t = Z_{F_t}$ and $F_0(x) = x$. One can ask if k-q.c. deformations generate groups of transformations whose quasiconformality is controlled by k. The answer is positive (cf. [4], Theorem 3). However, since another pointwise norm for symmetric two-forms was used in [4], we are going to state an analogous result corresponding to the norms defined by (0.2) and (0.3). To this aim, let us introduce the notion of quasiconformal transformation.

Definition 1.3. *A transformation (= diffeomorphism) $F : M \to M$ is said to be a K-quasiconformal (shortly K-q.c.) transformation, $K \geq 1$, if*

$$\|B^F\|_\infty \leq n^{1/2}K \tag{1.4}$$

where B^F is a symmetric two-form defined as follows:

$$B^F(X,Y) = g(J_F^{-1/n} DFX, J_F^{-1/n} DFY), \qquad X,Y \in \mathcal{X}, \tag{1.5}$$

c.f. the definition given in [4] and [5].

The factor $n^{1/2}$ in (1.2) forces F to be a conformal transformation (in the classical sense) if and only if F is 1-quasiconformal. Moreover, since the norms used here and in [4] or [5] are equivalent, quasiconformal transformations in the sense of the above Definition 1.3 have the same properties as these in [4] or [5]. Some estimations may however hold with other constants. Let us estimate the constant of quasiconformality for a 1-parameter family of transformations generated by a k-q.c. deformation (cf. [1] and [6]).

Theorem 1.6. *Let Z be a k-q.c. (complete) deformation Z on M (not necessarily compact) and let F_t, $t \in \mathbf{R}$ be the 1-parameter family of transformations generated by Z. Then, for every $t \in \mathbf{R}$, F_t is an K_t-q.c. transformation with*

$$K_t = exp(k \cdot |t|). \tag{1.7}$$

Proof: For every F_t, consider the symmetric two-form B^{F_t} defined by (1.5). According to Definition 1.3, we have to estimate the norm $\|B^{F_t}\|_\infty$. By Corollary 1 in [4], for every point $x \in M$,

$$\frac{d}{dt} B_x^{F_t}(X,Y)\Big|_{t=s} = S\alpha_{F_s(x)}(J_{F_s}^{-1/n} DF_s X, J_{F_s}^{-1/n} DF_s Y) \qquad X,Y \in \mathcal{X}.$$

Consequently,

$$\left\| \frac{d}{dt} B^{F_t} \right\|_x \leq \|S\alpha\|_{F_t(x)} \cdot \|B^{F_t}\|_x.$$

Since Z is a k-q.c. deformation, then by (1.1) we get

$$\left\| \frac{d}{dt} B^{F_t} \right\|_x \leq k \cdot \|B^{F_t}\|_x,$$

and therefore

$$\left| \frac{d}{dt} \ell n \|B^{F_t}\|_x \right| \leq k.$$

Integrating the last inequality in the interval $(0,t)$ and applying the obvious inequality

$$\|B^{F_0}\|_x = \|g\|_x = \sqrt{n}$$

we get that

$$n^{-1/2}\,\|B^{F_t}\|_x \leq exp(k\cdot|t|).$$

Taking the supremum over all $x \in M$, we obtain the inequality

$$\|B^{F_t}\|_\infty \leq n^{1/2}\ exp(k\cdot|t|)$$

which, in the light of (1.4), is equivalent to our assertion (1.7).

Remark 1.8: Observe that 0-q.c. deformations are just conformal, i.e. conformal Killing vector fields. Since, as we observed before, 1-q.c. transformations are conformal then, in the particular case $k = 0$, Theorem 1.6 asserts that every conformal deformation (= conformal Killing vector field) generates an one-parameter group of conformal transformations.

2. Normalized Quasiconformal Deformations.

Deformations Z and $c\cdot Z$, where $c \neq 0$ is a constant, generate practically the same families of transformations: one is obtained from the other by reparametrization. Various kinds of normalization may then be accepted.

For our compact manifold M, denote by v the square root of its volume with respect to the Lebesgue measure generated by the Riemannian metric g:

$$v = (vol\ M)^{1/2} \tag{2.1}$$

For an arbitrary 1-form $\alpha \in \mathcal{D}$ denote by $\|\alpha\|_2$ its L_2-norm defined by

$$\|\alpha\|_2 = (\alpha,\alpha)^{1/2}, \tag{2.2}$$

cf. (0.2).

Definition 2.3. *Let Z be an arbitrary nonzero vector field on M and let α be the 1-form dual to Z. We say that Z is a k-quasiconformal normalized (shortly: k-q.c.n.) deformation if the "normalized norm" $(|S\alpha|)$ of $S\alpha$ defined by*

$$(|S\alpha|) = v\cdot\|S\alpha\|_\infty\ /\ \|\alpha\|_2 \tag{2.4}$$

is bounded by k, i.e. if the inequality

$$(|S\alpha|) \leq k \tag{2.5}$$

holds.

It follows directly from (2.4) that for an arbitrary constant $c \neq 0$,

$$(|S(c \cdot \alpha)|) = (|S\alpha|), \tag{2.6}$$

which may be stated as the following obvious

Proposition 2.7. *If Z is a k-q.c n. deformation then for an arbitrary constant $c \neq 0$, $c \cdot Z$ is a k-q.c.n. deformation. O-q.c.n.-deformations are simply conformal deformations, i.e. conformal Killing vector fields.*

Of course there are several possibilities of normalizing $\|S\alpha\|_\infty$. The normalization of the type (2.3) has, however, such an advantage that, as it will be shown later, the quasiconformality of a normalized deformation can be estimated relatively to the Ricci curvature of the manifold in question. In particular, some bounds may be obtained for manifolds with a definite or vanishing Ricci curvature tensor R.

Now we need some information on spectral properties of the Laplacian Δ and Ahlfors Laplacian L.

3. Spectral Properties of the Ahlfors Laplacian.

By formula (0.1), the spectral properties of L are related to these of the Laplace-Beltrami operator $\Delta = \delta d + d\delta$. Let us then observe some spectral properties of Δ acting on k-forms, and, in particular, on 1-forms $\alpha \in \mathcal{D}$. As a basic tool we will use the Hodge theorem asserting that every k-form α may uniquely be written as a sum:

$$\alpha = \alpha_0 + \alpha' + \alpha'', \tag{3.1}$$

where α_0 is a harmonic form ($\Delta\alpha = 0$), α' is a coclosed form ($\alpha' \in \operatorname{Im}\delta$) and α'' is a closed form ($\alpha'' \in \operatorname{Im} d$). Moreover, the summands are orthogonal (each to the other) with respect to the global scalar product (0.4) (cf. [7]).

Let λ be an arbitrary non-zero eigenvalue for Δ on k-forms. Then necessarily $\lambda > 0$ (Δ is a strongly elliptic, positively defined, self-adjoint operator). Take an arbitrary eigenform $\alpha \in \Lambda^k$ corresponding to λ, i.e. such a form α that

$$\Delta\alpha = \lambda\alpha. \tag{3.2}$$

Let us split α relatively to (3.1):

$$\alpha = \alpha' + \alpha''. \tag{3.3}$$

Of course $\alpha_0 = 0$, since $\lambda > 0$. Applying Δ to both sides of (3.3), we obtain by (3.2) that

$$\Delta\alpha' + \Delta\alpha'' = \lambda\alpha' + \lambda\alpha''. \tag{3.4}$$

By comparison of the parts which are cohomologous and homologous to zero in both sides of (3.3) we conclude, using again the Hodge theorem, that

$$\Delta\alpha' = \lambda\alpha' \text{ and } \Delta\alpha'' = \lambda\alpha''.$$

This means that λ is an eigenvalue on k-forms for $d\delta$ or δd (or both).

Moreover, if we assume now that $\lambda > 0$ is an eigenvalue for δd on coclosed k-forms, i.e. that there exists a k-form $\alpha' \in \operatorname{Im}\delta$ ($\delta\alpha' = 0$), $\alpha' \neq 0$ such that

$$\delta d\alpha' = \lambda\alpha', \tag{3.5}$$

then it follows that $d\alpha' \neq 0$ and, by applying d to both sides of (3.5),

$$d\delta(d\alpha') = \lambda(d\alpha').$$

This means that λ is an eigenvalue for $d\delta$ on closed $(k+1)$-forms.

Conversely, if we assume that $\lambda > 0$ is an eigenvalue for $d\delta$ on closed $(k+1)$-forms, i.e. that there exists a $(k+1)$-form $\beta'' \in \operatorname{Im} d$ ($d\beta'' = 0$), $\beta'' \neq 0$, such that

$$d\delta\beta'' = \lambda\beta'', \tag{3.6}$$

then it follows that $\delta\beta'' \neq 0$ and, by applying δ to both sides of (3.6), that

$$\delta d(\delta\beta'') = \lambda\delta\beta''.$$

This means that λ is an eigenvalue for δd on coclosed k-forms.

What is more, since the Hodge star operator $*$ (cf. [7] p. 79 for the definition) is an isomorphism between the spaces Λ^n and Λ^{n-k} and since it transmits coclosed forms into closed ones and conversely, we get that the non-zero spectra for δd on coclosed k-forms and for $d\delta$ on closed (n-k)-forms coincide.

Notice that the multiplicities of eigenvalues in the coinciding spectra are also the same.

The described situation is displayed on the Diagram 1 below. The subspaces corresponding to the Hodge decomposition are denoted by dots. The Diagram 1 is to be read as follows: If two dots are connected by a straight line then the Laplacian Δ restricted to each of the two spaces corresponding to the dots has the same spectrum including multiplicities.

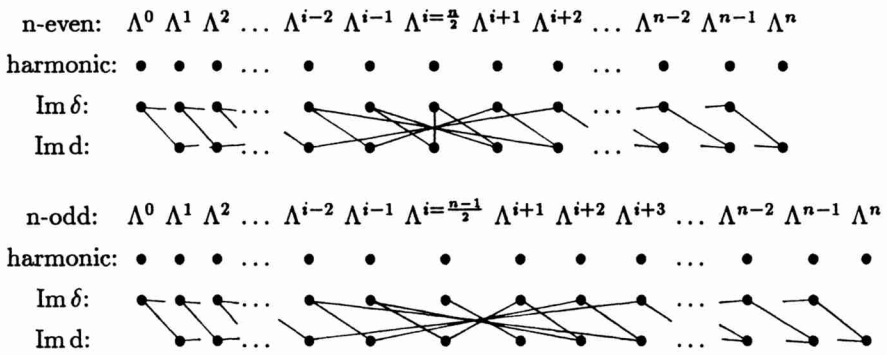

DIAGRAM 1

The observed above spectral properties of the Laplacian and the shape (0.1) of the Ahlfors Laplacian $L = S^*S$ make it possible to count the spectrum of L from the spectrum of Δ in many particular cases. The situation is surprisingly easy when the Ricci tensor R is relatively simple. Einstein manifolds are an example. Recall that an Einstein manifold is a Riemannian manifold (M, g) such that for some constant s

$$R = s \cdot g. \tag{3.7}$$

By (0.1) and the spectral properties of Δ we obtain directly the following

Proposition 3.8. *Assume that M is an Einstein manifold of dimension n with the Ricci tensor R satisfying (3.7). If $0 < \lambda_1' \leq \lambda_2' \leq \dots$ are eigenvalues of Δ on coclosed 1-forms and if $0 < \lambda_1'' \leq \lambda_2'' \leq \dots$ are eigenvalues of Δ on closed 1-forms (or on functions), then the spectrum of the Ahlfors Laplacian L is of the form*

$$\{0 < 2\lambda_1' - 4s \leq 2\lambda_2' - 4s \leq \cdots\} \cup \{0 \leq \frac{4n-4}{n}\lambda_1'' - 4s \leq \frac{4n-4}{n}\lambda_2'' - 4s \leq \cdots\}$$

and the multiplicities are the same as these for Δ.

In particular, Proposition 3.8 gives the spectrum of L for n-dimensional spheres $S^n \subset \mathbf{R}^{n+1}$, where $s = n - 1$, cf. Example 2.6 in [3] or for Ricci flat manifolds where $s = 0$.

Now, using the relation between the spectra of Δ on L, analogous to that of Proposition 3.8, we shall estimate the quasiconformality of normalized deformations in the following cases:

(1) Ricci flat manifolds: $R = 0$

(2) Manifolds with positive definite Ricci tensor

(3) Manifolds with negative definite Ricci tensor.

In what follows, λ_1' and λ_1'' will denote the first (smallest) eigenvalues of Δ restricted to

$$\mathcal{D}' = \mathcal{D} \cap Im\,\delta \quad \text{and} \quad \mathcal{D}'' = \mathcal{D} \cap Im\,d,$$

respectively.

4. Quasiconformal Deformations on Ricci Flat Manifolds.

If the Ricci curvature tensor vanishes: $R = 0$, the Ahlfors Laplacian is (by (0.1)) of the form

$$L = 2\delta d + \frac{4n-4}{n} d\delta. \tag{4.1}$$

Take an arbitrary 1-form $\alpha \in \mathcal{D}$. α defines a conformal deformation if and only if $S\alpha = 0$. Since M is compact this is equivalent to $L\alpha = S^*S\alpha = 0$. By the Hodge theorem and (4.1) it implies that $\delta d\alpha = d\delta\alpha = 0$, which is equivalent to

$$\delta\alpha = 0 \qquad \text{and} \qquad d\alpha = 0.$$

This means that α is harmonic and, since $\delta\alpha = 0$, that is in fact an isometric deformation (= infinitesimal isometry).

Conversely, each harmonic form α defines a conformal transformation which is an infinitesimal isometry.

Now, take $\alpha \in \mathcal{D}'$, i.e. an 1-form α which is cohomologous to 0. By the definition of λ_1', we have

$$2\lambda_1' \|\alpha\|_2^2 = 2(\lambda_1'\alpha, \alpha) \leq (2\delta d\alpha, \alpha).$$

Using (4.1) we get

$$2\lambda_1' \|\alpha\|_2^2 \leq (L\alpha, \alpha) = (S^*S\alpha, \alpha) = (S\alpha, S\alpha). \tag{4.2}$$

Since

$$(S\alpha, S\alpha) \leq v^2 \|S\alpha\|_\infty^2, \tag{4.3}$$

then (4.2) and (4.3) imply that

$$v\|S\alpha\|_\infty / \|\alpha\|_2 \geq (2\lambda_1')^{1/2},$$

or, by (2.5), that

$$(|S\alpha|) \geq 2(\frac{1}{2}\lambda_1')^{1/2}.$$

This means, by Definition 2.3, that α defines a q.c.n. deformation of the constant at least equal to $2(\frac{1}{2}\lambda_1')^{1/2}$.

Finally, take $\alpha \in \mathcal{D}''$, i.e. an 1-form α which is homologous to 0. Then, repeating the above consideration, we get that

$$(|S\alpha|) \geq 2(\frac{n-1}{n}\lambda_1'')^{1/2}.$$

This means, by Definition 2.3, that α defines a q.c.n. deformation of the constant at least equal to $2(\frac{n-1}{n}\lambda_1'')^{1/2}$.

The results of this section may be recapitulated in the following

Theorem 4.4. *If M is a compact Riemannian manifold with vanishing Ricci curvature tensor, then*

(i) Each harmonic 1-form defines on M a conformal deformation which is in fact isometric.

(ii) Each cohomologous to zero 1-form defines on M a k-q.c.n. deformation with $k \geq 2(\frac{1}{2}\lambda_1')^{1/2}$.

(iii) Each homologous to zero 1-form defines on M a k-q.c.n. deformation with $k \geq 2(\frac{n-1}{n}\lambda_2'')^{1/2}$.

5. Manifolds with Positive Definite Ricci Tensor.

Throughout this section we will assume that the quadratic form $\alpha \to g(R\alpha, \alpha)$ is positive definite. Then, at each point $x \in M$, R defines a positive, self-adjoint linear operator in T_x^*. Denote by $\underline{\lambda}(x)$ and $\bar{\lambda}(x)$ its minimal and maximal eigenvalues and define two numbers

$$\underline{\lambda} = \min_{x \in M} \underline{\lambda}(x) \quad \text{and} \quad \bar{\lambda} = \max_{x \in M} \bar{\lambda}(x). \tag{5.1}$$

Both of them are positive, since M is compact. Take an arbitrary form $\alpha \in \mathcal{D}$. Since

$$(L\alpha, \alpha) = (S\alpha, S\alpha) \tag{5.2}$$

we have that

$$\int_M g(L\alpha, \alpha) - g(S\alpha, S\alpha) = 0. \tag{5.3}$$

If α' is now a coclosed eigenform of Δ corresponding to the eigenvalue λ_1' then by (0.1) the integrand of (5.3) is equal to

$$g(2\lambda_1' - 4R\alpha', \alpha') - g(S\alpha', S\alpha'). \tag{5.4}$$

If we had $2\lambda_1' < 4\underline{\lambda}$, the integral of (5.4) over M would be negative. This would contradict to (5.3). Therefore (cf. [2], §5.1.84), we have

$$\underline{\lambda} \leq \frac{1}{2}\lambda_1'. \tag{5.5'}$$

Moreover, if $\underline{\lambda} = \frac{1}{2}\lambda_1'$, then the first term in (5.4) is equal to zero, so by (5.3) the other is equal to zero too. It means that α' defines a conformal deformation which on account of $\delta\alpha' = 0$, is isometric. In addition λ_1' has to be an eigenvalue of R and α' has to be an eigenform of R corresponding to λ_1'.

An analogous consideration may be repeated for a closed eigenform α''. Then, instead of (5.5'), we get

$$\underline{\lambda} \leq \frac{n-1}{n}\lambda_1''. \tag{5.5"}$$

Now assume that

$$\bar{\lambda} < \frac{1}{2}\lambda_1'. \tag{5.6}$$

Take an arbitrary $\alpha' \in \mathcal{D}'$. Since

$$(2\lambda_1' - 4\bar{\lambda})\|\alpha'\|_2^2 = (2\lambda_1\alpha' - 4\bar{\lambda}\alpha', \alpha')$$

and since

$$(\lambda_1'\alpha', \alpha') \leq (\delta d\alpha', \alpha') \quad \text{and} \quad (\bar{\lambda}\alpha', \alpha') \geq (R\alpha', \alpha')$$

we get that

$$(2\lambda_1' - 4\bar{\lambda})\|\alpha'\|_2^2 \leq (S\alpha', S\alpha').$$

Now, by (0.1) and (5.2) we get

$$(2\lambda_1' - 4\bar{\lambda})\|\alpha'\|_2^2 \leq (S\alpha', S\alpha').$$

Applying (4.3) and (2.5) we obtain

$$(|S\alpha'|) \geq 2(\frac{1}{2}\lambda_1' - \bar{\lambda})^{1/2}. \tag{5.7}$$

This means that α' defines a q.c.n. deformation of the constant at least equal to $2(\frac{1}{2}\lambda_1' - \bar{\lambda})^{1/2}$.

Assume finally that

$$\bar{\lambda} < \frac{n-1}{n}\lambda_1''. \tag{5.8}$$

Taking an arbitrary $\alpha \in \mathcal{D}''$, we get analogously that

$$(|S\alpha''|) \geq 2(\frac{n-1}{n}\lambda_1'' - \bar{\lambda})^{1/2}. \tag{5.9}$$

This means that α'' defines a q.c.n. deformation of the constant at least equal to $2(\frac{n-1}{n}\lambda_1'' - \bar{\lambda})^{1/2}$.

We have then proved the following

Theorem 5.10. *If M is a compact manifold with positive definite Ricci tensor, and if $\underline{\lambda}$ and $\bar{\lambda}$ are defined by (5.1), then:*

(i) If $\underline{\lambda} = \frac{1}{2}\lambda_1'$, then every coclosed eigenform of Δ corresponding to λ_1' defines a conformal (and in fact isometric) deformation which is, at the same time, an eigenform of R.

(ii) If $\underline{\lambda} = \frac{n-1}{n}\lambda_1''$ then every closed eigenform corresponding to λ_1'' defines a (pure) conformal deformation which is, at the same time, an eigenform of R.

(iii) If $\bar{\lambda} < \frac{1}{2}\lambda_1'$ then every coclosed 1-form on M defines a k-q.c.n. deformation with $k \geq 2(\frac{1}{2}\lambda_1' - \bar{\lambda})^{1/2}$.

(iv) If $\bar{\lambda} < \frac{n-1}{n}\lambda_1''$ then every closed 1-form on M defines a k-q.c.n. deformation with $k \geq 2(\frac{n-1}{n}\lambda_1'' - \bar{\lambda})^{1/2}$.

Making use of the fact that there are no harmonic forms on M, we can prove the following

Corollary 5.11. *Under the assumption of Theorem 5.10, if $\bar{\lambda} < \lambda$, where*

$$\lambda = min(\frac{1}{2}\lambda_1', \frac{n-1}{n}\lambda_1''), \tag{5.13}$$

every 1-form on M defines a k-q.c.n. deformation with

$$k \geq 2(\lambda - \bar{\lambda})^{1/2}.$$

Proof: Take an arbitrary $\alpha \in \mathcal{D}$. By the Bochner theorem (cf. [8], p. 42) there are no harmonic 1-forms other than zero on M. consequently α may be uniquely written as follows

$$\alpha = \alpha' + \alpha'', \tag{5.14}$$

where $\alpha \in \mathcal{D}'$ and $\alpha'' \in \mathcal{D}''$. The splitting is L_2-orthogonal, i.e.

$$(\alpha', \alpha'') = 0.$$

Consequently,

$$4(\lambda - \bar{\lambda})\|\alpha\|_2^2 = (4\lambda\alpha', \alpha') + (4\lambda\alpha'', \alpha'') - (4\bar{\lambda}\alpha, \alpha). \tag{5.15}$$

By (5.1), (5.13) and the definition of λ_1' and λ_1'' we get that the right hand side in (5.15) is less or equal to

$$(2\delta d\alpha', \alpha') + (\frac{4n-4}{n}\alpha'', \alpha'') - (4R\alpha, \alpha), \tag{5.16}$$

Using again the L_2-orthogonality of the decomposition (5.14) we see that (5.16) is equal to

$$(2\delta d\alpha, \alpha) + (\frac{4n-4}{n}\alpha, \alpha) - (4R\alpha, \alpha)$$

i.e., by (0.1), to $(L\alpha, \alpha) = (S\alpha, S\alpha)$. Summarizing, we obtain the inequality

$$4(\lambda - \bar{\lambda})\|\alpha\|_2^2 \leq (S\alpha, S\alpha)$$

which, by (4.3) implies that

$$v \cdot \|S\alpha\|_\infty / \|\alpha\|_2 \geq 2(\lambda - \bar{\lambda})^{1/2}$$

or, by (2.5), that

$$(|S\alpha|) \geq 2(\lambda - \bar{\lambda})^{1/2}.$$

This is equivalent to the assertion.

In particular, Corollary 5.11 says that if the Ricci curvature is small enough, then there are no conformal deformations on M. Let us state it more precisely:

Corollary 5.17. *Under assumptions of Corollary 5.11, there are no conformal deformations other than zero on M.*

6. Manifolds with Negative Definite Ricci Tensor.

Throughout this section we will assume that the quadratic form $\alpha \to g(R\alpha, \alpha)$ is negative definite at each point $x \in M$. Denote by $\underline{\lambda}(x)$ the minimal eigenvalue of $-R$ at x and define

$$\underline{\lambda} = \min_{x \in M} \underline{\lambda}(x). \tag{6.1}$$

$\underline{\lambda} > 0$, since M is compact.

Analogously, as in Section 5, we can prove the following

Theorem 6.2. *If M is a compact manifold with negative definite Ricci tensor and if $\underline{\lambda}$ is defined by (6.1), then*

(i) Every harmonic 1-form on M defines a k-q.c.n. deformation with $k \geq 2\underline{\lambda}^{1/2}$.

(ii) Every 1-form cohomologous to zero on M defines a k-q.c.n. deformation with $k \geq 2(\frac{1}{2}\lambda_1' + \underline{\lambda})^{1/2}$.

(iii) Every 1-form homologous to zero on M defines a k-q.c.n. deformation with $k \geq 2(\frac{n-1}{n}\lambda_1'' + \underline{\lambda})^{1/2}$.

Although we have no counterpart of the Bochner theorem at our disposal now, we can, by a reasoning similar to that of the proof of Corollary 5.1, estimate the constant of quasiconformality for arbitrary $\alpha \in \mathcal{D}$. Instead of the Bochner theorem we may use part (i) of Theorem 6.2. Since the proof is similar, we will only state the result.

Corollary 6.3. *Under the assumption of Theorem 6.2 every 1-form on M defines a k-q.c.n. deformation with $k \geq 2\underline{\lambda}^{1/2}$.*

Final Remark: In the light of Theorem 1.6, all estimations obtained in Sections 4–6 may be used to estimate constants of quasiconformality for one parameter families of transformations generated by deformations on M.

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