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ON RINGS ADMITTING ORDERINGS AND 2-PRIMARY CHAINS OF ORDERINGS OF HIGHER LEVEL

Eberhard Becker and Danielle Gondard

Introduction

It was the notion of the real spectrum of a ring, introduced by Coste and Roy, cf. [9] , that made the meaning of orderings in real algebraic geometry even more apparent than it was already done by Artin's solution of the 17th problem of Hilbert. Technically, the real spectrum pulls back to a ring the orderings of all appropriate residue field of its prime ideals. Thus, the real spectrum may be seen as a notion globalizing the local studies of orderings.

The first author extended the ordinary Artin-Schreier theory of orderings to what is called the theory of orderings of higher levels, cf. [1], [3]. By definition, a subset $P \subset K$, K a field, is called an ordering if

$$P + P \subset P, P \cdot P \subset P, K^{2n} \subset P, -1 \notin P, K^{\times}/P^{\times} \text{ cyclic } (P^{\times} = P\setminus\{0\}).$$

Later on, in the study of real closures of higher level, it turned out that one also has to consider chains $(P_n)_{n\in\mathbb{N}_0}$ of orderings of higher level in fields, cf. [17]. The definition will be given in the first section.

It is quite tempting to search for the role of orderings and chains of orderings of higher level in real algebraic geometry. This present paper provides a ring-theoretic foundation. In analogy with the real spectrum we pull back orderings and chains of orderings of higher level from the various residue fields $k(\mathfrak{p}) = \mathrm{quot}(A/_{\mathfrak{p}})$ to the ring A itself. This paper contains several results about the existence of orderings and chains of orderings on rings as well as about the number of such structures on a given ring. As an application we derive a Nullstellensatz over certain fields. We have refrained from defining and studying a corresponding real spectrum of higher level. Such higher level spectra are presently the subject of investigations by several people including the authors, cf. [6],[7] e. g. These approaches have to be compared and studied in detail and will be published in due time. The plan of this paper is as follows:

1. Orderings and chains on fields
2. Orderings on rings
3. Spec^T and T-radicals
4. α-chains on rings
5. $\mathrm{Spec}^{T,\alpha}$ and (T,α)-radicals
6. A Nullstellensatz

1. Orderings and chains on fields

The definition of an ordering P in a field K was already given in the introduction. According to [2 , (2.3)] we have

<u>Proposition (1.1): The following statements are equivalent.</u>

i) K admits an ordering of level n,
ii) $-1 \notin \Sigma K^{2n}$,
iii) $-1 \notin \Sigma K^2$, i. e. K is formally real.

In this proposition we have used the following notation

$$\Sigma A^{2m} = \{x \mid x = \sum_1^N x_i^{2m} \text{ for some } N \in \mathbb{N}, x_1,\dots,x_N \in A\}$$

where A is any commutative ring.

If P is an ordering of K of level n then $[K^\times : P^\times] \mid 2n$. One obtains $[K^\times : P^\times] = 2s$ with $s \mid n$. The number s is called the <u>exact level</u> of P. The orderings P of level 1 are just the usual orderings of a field K, i. e. they are characterized by the axioms

$$P + P \subset P, P \cdot P \subset P, P \cap -P = \{0\}, P \cup -P = K.$$

Orderings with levels a power of 2 are called 2-primary.

As indicated in the introduction, the study of real closures relative to higher level orderings led Harman to the introduction of chains of orderings, cf. [17]. In this paper we confine ourselves to chains of 2-primary orderings. Additionally, for the purposes of this paper there is no need to turn to N. Schwartz's reformulation of the notion of a chain, cf. [20]. Following Harman, a sequence $(P_i)_{i\geq 0}$ of orderings of higher levels is called a <u>2-primary chain</u> if the following conditions are satisfied:

i) P_0, P_1 are distinct orderings of level 1,
ii) for every $i \geq 2$, P_i is an ordering of exact level 2^{i-1},
iii) for every $i \geq 1$:
$$P_i \cup -P_i = (P_{i-1} \cap P_0) \cup -(P_{i-1} \cap P_0).$$

In [12], [13] the following result was proved:

<u>Proposition (1.2): A field K admits a 2-primary chain iff there exists</u> $\alpha \in K$ <u>such that</u> (K,α) <u>satisfies</u>

i) the axioms of commutative fields,
ii) for every $m \geq 1$, the axiom
$$\forall x_1 \forall x_2 \ldots \forall x_m \neg (-1 = x_1^2 + \ldots + x_m^2)$$
(i. e. K is formally real)

iii) for every $m \geq 1$, the axiom

$$\forall x_1 \ldots \forall x_m \neg (\alpha^2 = x_1^4 + \ldots + x_m^4).$$

This result leads to the idea to fix an element $\alpha \in K$ and study chains in relation to the chosen element α. From now on, "chain" will always mean "2-primary chain".

Now, let $\alpha \in K$ be given. A chain $(P_i)_{i \geq 0}$ is called an α-chain if $\alpha^2 \notin P_2$ holds. By combining [2] with [17], we arrive at the following result.

Proposition (1.3): Let K and $\alpha \in K$ be given. Then the following statements are equivalent.

i) (K,α) satisfies the axiom i)-iii) in (1.2),

ii) K admits an α-chain,

iii) K admits an ordering P of exact level 2 such that $\alpha^2 \notin P$.

Proof. i) $\Rightarrow$ iii) By [2 , (2.18)] we would get $\alpha^2 \in \Sigma K^4$ if iii) is not true. iii) $\Rightarrow$ ii) By [17,(1-4)],given an ordering P of exact level 2 there is a chain $(P_i)_{i \geqslant 0}$ with $P = P_2$. ii) $\Rightarrow$ i) In particular, $\alpha \notin P_2$, hence $\alpha^2 \notin \Sigma K^4$.

As an example we may consider $K = \mathbb{Q}(t)$. It is readily checked that $\mathbb{Q}(t)$ admits a t-chain.

2. Orderings of higher level on rings

Beginning with this section we generalize the concepts and results of the last section to the setting of rings. All rings considered here are commutative and have a unit. If $\mathfrak{p}$ is a prime ideal of the ring A then $k(\mathfrak{p}) := \mathrm{quot}(A/_{\mathfrak{p}})$ is called the residue field of $\mathfrak{p}$. A subset $P \subset A$ is called an ordering of level n if the following conditions are satisfied:

(2.1) i) $P + P \subset P$, $P \cdot P \subset P$, $A^{2n} \subset P$,

ii) $P \cap -P = \mathfrak{p}$ is a prime ideal of A,

iii) if $xy^{2n} \in P$ then $x \in P$ or $y \in P \cap -P$,

iv) $\overline{P} := \{\sum_{\text{finite}} a_i^{2n}\, \overline{p} \mid a_i \in k(\mathfrak{p}),\ p \in P\}$ is an ordering of level n of $k(\mathfrak{p})$ (where $\overline{p} = p + \mathfrak{p}$).

If $n = 1$ then i) - iv) can be replaced by the axioms

i)' $P + P \subset P$, $P \cdot P \subset P$, $P \cup -P = A$,

ii)' $P \cap -P$ is a prime ideal of A

as it is readily checked. Thus, our general definition coincide with the usual one in this case, cf. [9].

Note also that $\overline{P} = \{\frac{\overline{p}}{\overline{s}^{2n}} \mid p \in P,\ s \in A \setminus \mathfrak{p}\}$ holds.

As in the case of orderings of level 1 there is an alternative definition. To derive it let $\pi : A \to A/_{\mathfrak{p}} \to k(\mathfrak{p})$ denote the canonical homomorphism. If P is an ordering of level n in A then $P = \pi^{-1}(\overline{P})$. Conversely, if $\mathfrak{p}$ is a prime ideal and P' an ordering of level n in $k(\mathfrak{p})$ then, by setting $P := \pi^{-1}(P')$, we get an ordering of level n in A satisfying $\overline{P} = P'$. Therefore, an ordering of higher level can also be thought of as a pair $(\mathfrak{p}, \overline{P})$, $\mathfrak{p} \in \operatorname{Spec} A$, $\overline{P}$ an ordering of higher level in $k(\mathfrak{p})$. If it is convenient we will switch from one definition to the other.

A subset $T \subset A$ is called a <u>semiring of level n</u> if (2.2) is satisfied:

(2.2) $T + T \subset T$, $T \cdot T \subset T$, $A^{2n} \subset T$.

If, additionally,

(2.3) $-1 \notin T$

holds then T is called a <u>preordering of level n</u>.

Then, as usual, we have

Proposition (2.4): Let T be a semiring of level n in A.

Then the following statements are equivalent:

i) $-1 \notin T$,
ii) there exists an ordering P of level n in A with $T \subset P$.

Proof: ii) $\Rightarrow$ i) is clear. i) $\Rightarrow$ ii) Consider the multiplicative semigroup $S = 1 + T$. Choose by Zorn's lemma a prime ideal $\mathfrak{p}$ subject to $\mathfrak{p} \cap S = \emptyset$ and maximal with this property. Next pass over to $\overline{T} := \{\frac{\overline{t}}{\overline{s}^{2n}} | t \in \overline{T}, s \in A \setminus \mathfrak{p}\}$. If $-1 \notin \overline{\overline{T}}$ then there is an ordering $\overline{P} \supset \overline{T}$, $\overline{P}$ of level n by [2 , 1.5]. Then set $P = \pi^{-1}(\overline{P})$ to get an ordering as wanted. If $-1 \in \overline{T}$ we find $x^{2n} + t \in \mathfrak{p}$ for some $x \notin \mathfrak{p}$, $t \in T$. The maximality of $\mathfrak{p}$ shows the existence of $y \in A$ with $1 + t \equiv yx \bmod \mathfrak{p}$ for some $t \in T$. Then

$$(1 + t)^{2n} = 1 + t' \equiv y^{2n}x^{2n} \bmod \mathfrak{p}, \; t' \in T$$

leading to $1+(t'+y^{2n}t) \equiv 0 \bmod \mathfrak{p}$: a contradiction.

Corollary (2.5): The following statements are equivalent:

i) A admits an ordering of level n,
ii) $-1 \notin \Sigma A^{2n}$,
iii) $-1 \notin \Sigma A^2$.

Proof. (2.4) and (1.1) give the result.

If T is a preordering of level n in a field then T is the intersection of all orderings $P \supset T$ by [3]. In our more general situation we can prove the following result in the 2-primary case.

Proposition (2.6): Let T be a semiring of level $n = 2^m$ in A and $a \in A$. Then the following statements are equivalent:

i) $a \in P$ for all orderings P of level n with $T \subset P$,
ii) $at = a^{2nk} + t'$ for some $t,t' \in T$, $k \in \mathbb{N}$.

Proof. ii) $\Rightarrow$ i) Assume $a \notin P$, $P = (\mathfrak{p},\overline{P})$. Then pass over to $k(\mathfrak{p})$. We get $\overline{a}^{2nk} + \overline{t}' \in \overline{P}^x$, hence $\overline{at} \in \overline{P}^x$ and $\overline{a} \in \overline{P}$ implying $a \in P$. i) $\Rightarrow$ ii) If a is nilpotent then $a^{2nk+1} = 0$ for some k. Hence $0 = a \cdot a^{2nk} = a^{4nk}$ and ii) is verified. Next, let a be non-nilpotent. In A_a consider the semiring $T_a := \{\frac{t}{a^{2nk}} | t \in T, k \in \mathbb{N}\}$ of level n. If $-1 \in T_a$ then $a^l(a^{2nk} + t) = 0$ for some $l,k \in \mathbb{N}$, $t \in T$. Clearly, we may assume $l = 2nl'$. Hence, $-1 \in T_a$ implies $a^{2nk} + t = 0$ for some $k \in \mathbb{N}$, $t \in T$. Thus, the claim is proved in this case. Next assume $-1 \notin T_a$. By (2.4) there is an ordering $P' \supset T_a$. Its contraction $P = \{a \in A | \frac{a}{1} \in P'\}$ is readily seen to be an ordering over T where the prime ideal $\mathfrak{p} = P \cap -P$ is the contraction of $\mathfrak{p}' = P' \cap -P'$. $a \notin \mathfrak{p}$ as $\frac{a}{1}$ is a unit in A_a, moreover $\overline{P} = \overline{P}'$ in $k(\mathfrak{p}) = k(\mathfrak{p}')$. From our assumption we get that $\frac{a}{1} \in P'$ for all orderings $P' \supset T_a$. Thus we are facing a similar condition but, this time, the element in question is a unit. It is therefore enough to prove the following assertion:

(*) if a is a unit then, under the hypothesis of i), it follows that $at = 1 + t'$ holds for some $t,t' \in T$.

Namely, applying (*) to the above situation $A_a, T_a, \frac{a}{1}$, we find, $t,t' \in T_a$ such that $\frac{a}{1} \cdot t = 1 + t'$. Pulling this back to A we get $at = a^{2nk} + t'$ for some $t,t' \in T$, $k \in \mathbb{N}$.

To prove (*) we first note that there is $r \in \mathbb{N}$ with $a^{2^r} \in T$. Using this the proof proceeds by induction on r. If $r = 0$ then $a^{-1} \in T$ because of $a^{-1} = (a^{-1})^{2n}a^{2n-1}$. Hence $a \cdot a^{-1} = 1 + 0$, and the claim is proved. Now assume that (*) is proved for $r - 1 \geq 0$. As $a^{2^r} = (a^2)^{2^{r-1}} \in T$ then by induction hypothesis $a^2t = 1 + t'$ for some $t,t' \in T$. Raising this relation to a 2^{l-1}-th power we find for every $l \in \mathbb{N}$ $t,t' \in T$ (depending on l) such that $a^{2^l}t = 1 + t'$ holds.

Consider the semiring $T[-a] := \{\sum_{\text{finite}} t_i(-a)^i \mid \text{all } t_i \in T\}$. If $-1 \notin T[-a]$ then by (2.4) there is an ordering $P \supset T$ with $-a \in P$. By assumption also $a \in P$. As a is a unit this implies $-1 \in P$: a contradiction. Hence $-1 \in T[-a]$, i. e. $-1 = F(a^2) - aG(a^2)$ for some polynomials F,G with coefficients in T - for short: T-polynomials F and G.

So far we have shown the existence of an identity

(**) $1 + F(a^{2^l}) = aG(a^{2^l})$, F,G T-polynomials in the case $l = 1$. The general case is done by induction on l.

If $1 + F(a^{2^{l-1}}) = aG(a^{2^{l-1}})$ is assumed then pick an identity $a^{2^{l-1}}t = 1 + t'$ from above. Now we have

$$(a^{2^{l-1}})^{2k} \cdot a^{2^{l-1}}t = a^{2^l k}(1 + t')$$
$$(a^{2^{l-1}})^{2k+1} \cdot a^{2^{l-1}}t = a^{2^l (k+1)}t.$$

Hence, after multiplying the identity $1 + F(a^{2^{l-1}}) = aG(a^{2^{l-1}})$ by $a^{2^{l-1}} \cdot t$ we get an expression

$$1 + t' + F'(a^{2^l}) = aG'(a^{2^l})$$

with T-polynomials F',G'. Thus, the claim is proved for the exponent 2^l.

Since $a^{2^r} \in T$ we finally arrive at a relation $at = 1 + t'$ as desired, where $t,t' \in T$.

3. Spec^T and T-radical

By T we still denote a semiring of level n, n arbitrary. Let $\mathfrak{a}$ be any ideal of A. Following [5 , p. 81] we set

$$\mathrm{rad}_T\mathfrak{a} = \{f \in A \mid f^{2nk} + t \in \mathfrak{a} \text{ for some } k \in \mathbb{N},\ t \in T\}.$$

We call $\mathrm{rad}_T\mathfrak{a}$ the T-radical of $\mathfrak{a}$. By [5 , (2.4)] we know that $\mathrm{rad}_T\mathfrak{a} = \cap\, \mathfrak{p}$, $\mathfrak{p}$ ranging over all prime ideals subject to $\mathrm{rad}_T\mathfrak{p} = \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{p}$. In particular, $\mathrm{rad}_T\mathfrak{a}$ is an ideal that is T-radically closed, i. e. $\mathrm{rad}_T(\mathrm{rad}_T\mathfrak{a}) = \mathrm{rad}_T\mathfrak{a}$.

An ideal $\mathfrak{a}$ is called T-convex if the assumption $t + t' \in \mathfrak{a}$, $t,t' \in T$" always implies $t,t' \in \mathfrak{a}$.

Proposition (3.1): Let $\mathfrak{p} \in \mathrm{Spec}\, A$ be given. Then the following statements are equivalent:

i) $\mathfrak{p}$ is T-convex,
ii) $\mathrm{rad}_T\mathfrak{p} = \mathfrak{p}$,
iii) there is an ordering $P \supset T$ such that $\mathfrak{p} = P \cap -P$.

Proof. i) $\Rightarrow$ ii) clear, ii) $\Rightarrow$ iii) If not then $-1 \in \overline{T} = \{\frac{\overline{t}}{\overline{s}^{2n}} \mid t \in T,\ s \in A \setminus \mathfrak{p}\}$. This means $s^{2n} + t \in \mathfrak{p}$ yielding $s \in \mathrm{rad}_T\mathfrak{p}$, $s \notin \mathfrak{p}$. iii) $\Rightarrow$ i) Assume $t + t' \in \mathfrak{p}$, $t,t' \in T$. Passing over to $k(\mathfrak{p})$ and $\overline{P} \supset \overline{T}$ we find $\overline{t} + \overline{t}' = 0$. From $\overline{t},\overline{t}' \in \overline{P}$ we derive $\overline{t} = \overline{t}' = 0$, otherwise $-1 \in \overline{P}$. Hence, $t,t' \in \mathfrak{p}$.

According to the last result we set

$$\mathrm{Spec}^T A = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \mathfrak{p} = P \cap -P \text{ for some ordering } P \supset T\}.$$

Remark (3.2): From above we know that $\mathrm{rad}_T(0) = \{f \in A \mid f^{2nk} + t = 0 \text{ for some } k \in \mathbb{N},\ t \in T\}$. In the case $n = 2^m$, this also follows from (3.1) and the last section: if $f^{2nk} + t = 0$ then by (2.6) $f \in P \cap -P$ for all $P \supset T$, conversely, if $f \in P \cap -P$ for all such P's then either f is nilpotent or $-1 \in T_f$. In either case one finds a relation $f^{2nk} + t = 0$.

Remark (3.3): In the literature, in the case of $T = \Sigma A^2$ the radical $\mathrm{rad}_T\, \mathfrak{a} = \{f \in A \mid f^{2k} + \Sigma a_i^2 \in \mathfrak{a}$ for some $k \in \mathbb{N}$, $a_i \in A\}$ is called the real radical of $\mathfrak{a}$, sometimes denoted by r-rad $\mathfrak{a}$. Set $\Sigma^n := \Sigma A^{2n}$. Then we have the surprising result:

$$\text{r-rad}\ \mathfrak{a} = \mathrm{rad}_{\Sigma^n}(\mathfrak{a}) \text{ for all } n \in \mathbb{N}$$

To prove this it is enough to show that r-rad $\mathfrak{p} = \mathfrak{p}$ is equivalent to $\mathrm{rad}_{\Sigma^n}(\mathfrak{p}) = \mathfrak{p}$, for every prime ideal $\mathfrak{p}$ and $n \in \mathbb{N}$. By (3.1) we have to consider the following statements:

i) $k(\mathfrak{p})$ admits an ordering of level 1,

ii) $k(\mathfrak{p})$ admits an ordering of level n.

Now, [2 , (2.3)] shows them to be equivalent.

4. α-chains on rings

In accordance with the axiomatic approach described in the first section we fix $\alpha \in A$ and look for pullbacks $(P_i)_{i\in\mathbb{N}}$ of 2-primary chains $(\overline{P}_i)_{i\in\mathbb{N}_0}$ in some residue fields $k(\mathfrak{p})$ subject to $\overline{\alpha}^2 \notin \overline{P}_2$. We therefore define a 2-primary α-chain in A to be sequence of 2-primary orderings $(P_i)_{i\in\mathbb{N}_0}$ such that

i) $\mathfrak{p} := P_0 \cap -P_0 = P_i \cap -P_i$ for all $i \in \mathbb{N}_0$,

ii) $\overline{\alpha}^2 \notin \overline{P}_2$, $(\overline{P}_i)_{i\in\mathbb{N}_0}$ is a 2-primary chain in $k(\mathfrak{p})$.

Since, as stated in section 1, every ordering $\overline{P}_2$ of exact level 2 belongs to a 2-primary chain we have the following result:

Proposition (4.1): A admits a 2-primary α-chain iff A admits

<u>an ordering P of level 2 with</u> $\alpha^2 \notin P$.

Hence, one has to study the existence of orderings P of level 2 subject to $\alpha^2 \notin P$. To this end one must consider the quotient ring A_α and the contraction and extension of orderings. If P' is an ordering of level n in A_α then, as already remarked in the proof of (2.6), its <u>contraction</u>

$$P := P' \cap A := \{a \in A \mid \tfrac{a}{1} \in P'\}$$

is an ordering of level n in A with $\mathfrak{p}' = P' \cap -P'$ contracting to $\mathfrak{p} = P \cap -P$ and $\overline{P} = \overline{P}'$ in $k(\mathfrak{p}) = k(\mathfrak{p}')$. Moreover, $\alpha \notin \mathfrak{p}$. Conversely, if an ordering P of level n in A is given with $\alpha \notin P \cap -P = \mathfrak{p}$ then its <u>extension</u>

$$P' := P_\alpha := \{\tfrac{p}{\alpha^{2nk}} \mid p \in P,\ k \in \mathbb{N}\}$$

is an ordering of level n in A_α whose contraction is just P. One readily verifies <u>that contraction and extension constitute bijections, inverse to each other, between the set of all orderings of level n in A_α and the set of those in A satisfying $\alpha \notin P \cap -P$</u>.

Let in this section T denote a <u>semiring of level 2</u>. We will assume that <u>α is not nilpotent</u>, otherwise there cannot exist P of level 2 with $\alpha^2 \notin P$. Set

$$T_\alpha = \{\tfrac{t}{\alpha^{4k}} \mid t \in T,\ k \in \mathbb{N}\}.$$

T_α is again a semiring of level 2: the extension of T to A_α. With these notations and assumptions we have:

<u>Proposition (4.2):</u> <u>Contraction and extension yield bijections between the following two sets:</u>

i) $\{P \mid P \text{ ordering of level } 2,\ T \subset P,\ \alpha^2 \notin P\}$,
ii) $\{P' \mid P' \text{ ordering of level } 2,\ T_\alpha - \alpha^2 T_\alpha \subset P'\}$.

Proof. If P belongs to the set in i) then $T_\alpha \subset P'$, $-\alpha^2 \in P'$ where $P' = P_\alpha$ is the extension of P. Thus $T_\alpha - \alpha^2 T_\alpha \subset P'$. Conversely, if $P = P' \cap A$, $T_\alpha - \alpha^2 T_\alpha \subset P'$ then $\alpha^2 \notin P$ since otherwise the unit $\frac{\alpha^2}{1} = \alpha^2$ of A_α would satisfy $-\alpha^2 \in P'$ leading to $-1 \in P'$.

Consequently, with the help of (2.4), we can now characterize the existence of α-chains. To this end we call the semiring T an α-preordering if it satifies

(4.3) i) $-1 \notin T$ (i. e. T is preordering),
ii) $\forall k \in \mathbb{N}$ $\alpha^{4k+2} \notin T - \alpha^2 T$.

Proposition (4.4): The following statements are equivalent

i) T is an α-preordering,
ii) A admits an 2-primary α-chain with $T \subset P_2$.

Proof. By (4.2) and (2.4), the negation of the statement in ii) is equivalent to the statement "$-1 \in T_\alpha - \alpha^2 T_\alpha$" (note that $T_\alpha - \alpha^2 T_\alpha$ is a semiring since T and T_α are of level 2). The latter statement is directly seen to be equivalent to the existence of $k \in \mathbb{N}$ such that $\alpha^{4k+2} \in T - \alpha^2 T$.

Remark (4.5): As R. Berr pointed out to us, if α is a unit in A then i), ii) can be equivalently expressed by $-1 \notin T - \alpha^2 T$.

In the absolute case $T = \Sigma A^4$, the last proposition specializes as follows:

Corollary (4.6): i) A admits a 2-primary α-chain iff both conditions are satisfied:

1) $-1 \notin \Sigma A^2$,
2) for all $k \in \mathbb{N}$ a relation $\alpha^2(\alpha^{4k} + \Sigma a_i^4) = \Sigma b_i^4$ is impossible.

ii) If α is a unit then A admits a 2-primary α-chain iff -1 is not a sum of the following type $\Sigma a_i^4 - \alpha^2(\Sigma b_i^4)$.

Proof. All follows from (4.4) in view of (2.5) and the last remark (4.5).

5. $\mathrm{Spec}^{T,\alpha}$ and (T,α)-radical

We are keeping the notations and assumptions of the last section. As in section 3 we want to investigate the ideal theory related with α-chains. Given any ideal $\mathfrak{a} \lhd A$ we denote by $\mathfrak{a}' = \mathfrak{a}A_\alpha$ its extended ideal and if $\mathfrak{b} \lhd A_\alpha$ is given then $\mathfrak{b} \cap A$ denotes its contraction to A. We now set for an ideal $\mathfrak{a} \lhd A$:

(5.1) $\mathrm{rad}_{T,\alpha}\,\mathfrak{a} := (\mathrm{rad}_{T_\alpha - \alpha^2 T_\alpha}(\mathfrak{a}')) \cap A.$

Using the description of general T-radicals in section 3 we first obtain:

(5.2) $\mathrm{rad}_{T,\alpha}\mathfrak{a} = \{a \in A \mid (\alpha a)^{4k} + t - \alpha^2 t' \in \mathfrak{a}$ for some $k \in \mathbb{N},\ t,t' \in T\}$.

We call $\mathrm{rad}_{T,\alpha}\,\mathfrak{a}$ the (T,α)-radical of $\mathfrak{a}$. From the general theory, quoted in section 3, it follows that $\mathrm{rad}_{T,\alpha}\mathfrak{a}$ is a (T,α)-radically closed ideal. Moreover, from $\mathrm{rad}_{T'}\mathfrak{a}' = \cap\, \mathfrak{p}'$, $\mathfrak{p}' \in \mathrm{Spec}\, A_\alpha$, $\mathrm{rad}_{T'}\mathfrak{p}' = \mathfrak{p}'$, where $T' = T_\alpha - \alpha^2 T_\alpha$, we derive the following result:

Proposition (5.3): $\mathrm{rad}_{T,\alpha}\mathfrak{a} = \cap\, \mathfrak{p}$, $\mathfrak{p}$ ranging over all (T,α)-radically closed prime ideals $\mathfrak{p} \supset \mathfrak{a}$.

The proposition (3.1) can be generalized as follows:

Proposition (5.4): Let $\mathfrak{p} \in \mathrm{Spec}\, A$ be given. Then the following statements are equivalent:

i) $\mathrm{rad}_{T,\alpha}\mathfrak{p} = \mathfrak{p}$,

ii) there is a 2-primary α-chain $(P_i)_{i\in\mathbb{N}_0}$ in A such that $T \subset P_2$, $\alpha^2 \notin P_2$ and $\mathfrak{p} = P_i \cap -P_i$ for all $i \in \mathbb{N}_0$.

<u>Proof.</u> This follows from (4.2), (3.1) and the identity $(\mathfrak{b} \cap A)' = \mathfrak{b}$ for every ideal $\mathfrak{b}$ of A.

Motivated by this result and in analogy to section 3 we set

$$\mathrm{Spec}^{T,\alpha}A = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \mathrm{rad}_{T,\alpha}\mathfrak{p} = \mathfrak{p}\}.$$

In the case of $T = \Sigma A^4$ we simplify notation and set $\mathrm{rad}_\alpha \mathfrak{a} := \mathrm{rad}_{T,\alpha}\mathfrak{a}$, $\mathrm{Spec}^\alpha A := \mathrm{Spec}^{T,\alpha}A$. Thus, for every $\mathfrak{p} \in \mathrm{Spec}\, A$

$$\mathrm{rad}_\alpha\mathfrak{p} = \mathfrak{p} \text{ iff } k(\mathfrak{p}) \text{ admits a chain with } \overline{\alpha}^2 \notin \overline{P}_2,$$

and we have $\mathrm{rad}_\alpha \mathfrak{a} = \bigcap_{\mathfrak{p}\in \mathrm{Spec}^\alpha A} \mathfrak{p}$.

In the next section, we will apply this last result to obtain a certain Nullstellensatz.

6. A Nullstellensatz

The principles behind the proof of the following Nullstellensatz are well known. One first derives an abstract version of it like $\mathrm{rad}_\alpha \mathfrak{a} = \cap \mathfrak{p}$, $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. In the next step one exploits model theoretic properties of the various residue fields $k(\mathfrak{p})$. The general method of this approach is displayed in [8], [21], but see also [5], [7], [10] for more concrete versions.

In this section, K denotes a chain-closed field [17], [20] [12], [13]. Recall that we are only dealing with 2-primary chains. Thus K admits a unique chain $(P_i)_{i\geq 0}$, up to the order of P_0 and P_1, but allows no faithful extension of this chain to algebraic extensions $K \subsetneq L$. According to [3], these chain-closed fields are characterized as follows: they admit a henselian valuation with real closed residue field (of level 1) with odd-divisible value group Γ such that $[\Gamma : 2\Gamma] = 2$.

If K is chain-closed and $\alpha \in K$ with $\alpha^2 \notin P_2$ is specified we call K α-chain-closed.

The model theory of chain-closed fields is developed in [18] and [11], the latter paper giving special attention to α-chain-closed fields. One has

Proposition (6.1): Let K and L be α-chain-closed fields where $K \subset L$. If K has a unique henselian valuation with real closed residue field then $K \prec L$ (in ordinary field language added by a constant "α").

We are now ready to state and prove the Nullstellensatz. Let K be as above, $\mathfrak{a}$ an ideal in $K[\overline{X}] := K[X_1,\dots,X_n]$ and set as usual

$$V_K(\mathfrak{a}) = \{\overline{x} \in K^n \mid \forall f \in \mathfrak{a} : f(\overline{x}) = 0\}$$
$$I_K(W) = \{f \in K[\overline{X}] \mid \forall \overline{x} \in W : f(\overline{x}) = 0\}$$

where $W \subset K^n$. Recall

$$\mathrm{rad}_\alpha \mathfrak{a} = \{f \in K[\overline{X}] \mid (\alpha f)^{4k} + \Sigma g_i^4 - \alpha^2 \Sigma h_j^4 \in \mathfrak{a} \text{ for some } k \in \mathbb{N},\ g_i, h_j \in K[\overline{X}]\}.$$

<u>Theorem (6.2): Let K be an α-chain-closed field with a unique henselian valuation ring having a real closed residue field. Then, for every ideal</u> $\mathfrak{a} \lhd K[\overline{X}]$,

$$I_K V_K(\mathfrak{a}) = \mathrm{rad}_\alpha \mathfrak{a}.$$

<u>Remark:</u> Using a much more complicated description of the radical this Nullstellensatz can also be deduced from [5]. In the above given form it is also proved in [7], however the approach presented here is more ringtheory-oriented.

<u>Proof of 6.2:</u> We use [5 , (2.1)], but [10] may be applied as well. Thus, $I_K V_K(\mathfrak{a}) = \cap \mathfrak{p}$, $\mathfrak{p}$ ranging over all prime ideals $\mathfrak{p}$ such that K is existentially closed in $k(\mathfrak{p})$. Hence, we have to show that this latter property is equivalent to $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. Assume first $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. By (5.4) we get that $k(\mathfrak{p})$ admits an α-chain $(\overline{P}_i)_{i \geq 0}$. Let L denote a real closure of this chain, cf. [17]. From (6.1) we derive $K \prec L$ which implies that K is existentially closed in $k(\mathfrak{p})$. Conversely, we now assume K to be existentially closed in $k(\mathfrak{p})$. From (1.3) one derives that $k(\mathfrak{p})$ admits an α-chain which, in view of (5.4), implies $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. Thus, we have shown $I_K V_K(\mathfrak{a}) = \cap \mathfrak{p}$, $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. Hence, by (5.3), the chain is proved.

In the remaining part of this section we are going to study the conditions under which the above Nullstellensatz holds. The counterexample in proposition (6.4) is based on ideals of A. Prestel he communicated to the first author several years ago.

We first turn to the case of one indeterminate.

Proposition (6.3): Let K be an arbitrary α-chain-closed field and $\mathfrak{a}$ any ideal of $K[X_1]$. Then

$$I_K V_K(\mathfrak{a}) = \mathrm{rad}_\alpha \mathfrak{a}.$$

Proof: Following the proof of (6.2) one has only to show that $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$ implies $K = k(\mathfrak{p})$. Hence, assume $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. In the present case, $k(\mathfrak{p})$ is a finite extension of K admitting an α-chain $(\overline{P}_i)_{i \geq 0}$. This chain extends the unique chain of K. As K is chain-closed we get $K = k(\mathfrak{p})$.

Proposition (6.4): Let K be a chain-closed field admitting more than one henselian valuation ring with real closed residue field. Then for all $n \geq 2$ there is an ideal $\mathfrak{a}$ in $K[X_1,\dots,X_n]$ such that

$$I_K V_K(\mathfrak{a}) \supsetneqq \mathrm{rad}_\alpha \mathfrak{a}.$$

Remark: Quite generally we have $I_K V_K(\mathfrak{a}) \supset \mathrm{rad}_\alpha \mathfrak{a}$ since every maximal ideal $\mathfrak{m}$ belonging to a point $\overline{x} \in V_K(\mathfrak{a})$ satisfies $\mathrm{rad}_\alpha \mathfrak{m} = \mathfrak{m}$.

Proof of 6.4: It is sufficient to treat the case $n = 2$, the general case follows by extending the suitable ideal from $K[X_1,X_2]$ to $K[X_1,\dots,X_n]$. According to [5 , section 4, remark] the hypothesis amounts to $\Delta \neq \{0\}$ where Δ is the largest divisible convex subgroup of the value group $\Gamma = v(K^\times)$ belonging to $A(P_2)$ which in turn is the smallest henselian valuation ring with a real closed residue field. Hence, we may choose $t \in K$ such that $v(t) > 0$, $v(t) \in \Delta$ and we consider the polynomial $f(X) = (1 + X^2)(t^2 + X^2)$. We claim that $f(x) \in K^{\times 4}$ for all $x \in K$. To prove this we make use of the fact that the valuation v is henselian with a real closed residue field. If $v(x) < 0$ then $f(x) = x^4 \cdot \varepsilon$ where ε is a unit and a square in K - note K is pythagorean. The above cited property of v implies $\varepsilon = \eta^4$, $\eta \in K$, hence $f(x) \in K^{\times 4}$. If $0 \leq v(x) \leq v(t)$ then $f(x) = x^2 \cdot \varepsilon$, ε a unit. Since Δ is convex and $\Delta \subset 2\Gamma$ we see $x = y^2 \cdot \eta$, η a unit.

Hence, $f(x) = y^4 \cdot \omega$, ω a unit. As above, this implies $f(x) \in K^{\times 4}$. In the remaining case $v(t) < v(x)$ we get $f(x) = t^2 \cdot \varepsilon$, ε a unit, and, using $v(t) \in \Delta \subset 2\Gamma$, we have $f(x) = s^4 \cdot \eta$, η a unit. Then again $f(x) \in K^{\times 4}$.

As a consequence we derive that the polynomial $g(X,Y) = (1 + X^2)(t^2 + X^2) - \alpha^2 Y^4$ has no zero in K since $\alpha^2 \notin K^4$. Thus, setting $\mathfrak{p} = (g)$, we have $\mathcal{I}_K V_K(\mathfrak{p}) = K[X,Y]$. In contrast to this we will show $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$. Hence, the Nullstellensatz (6.2) does not hold.

First note that $\mathfrak{p}$ is a prime ideal since g is irreducible. Therefore, by (5.4), we have to show that $k(\mathfrak{p})$ admits an ordering P' of level 2 with $\alpha^2 \notin P'$. One has $K(X) \subset k(\mathfrak{p})$. The valuation v on K can be extended to K(X) by setting

$$w(\Sigma a_j X^j) = \min\{(v(a_j),j) \mid a_j \neq 0\} \in \Gamma \times \mathbf{Z}$$

where $\Gamma \times \mathbf{Z}$ denotes the lexicographic product with $\mathbf{Z}$ as the smaller factor. Since $v(\alpha) + 2\Gamma$ generates $\Gamma/2\Gamma$ we derive

$$\Gamma \times \mathbf{Z}/_{4(\Gamma\times\mathbf{Z})} \cong \langle\overline{w}(X\alpha^{-1})\rangle \times \langle\overline{w}(\alpha)\rangle$$

where $\overline{w}(y) := w(y) + 4(\Gamma \times \mathbf{Z})$. Therefore we find a character $\eta : \Gamma \times \mathbf{Z} \to \mu(4) = \{\zeta \in \mathbf{C} \mid \zeta^4 = 1\}$ satisfying $\eta w(X\alpha^{-1}) = 1$, $\eta w(\alpha) = i$. Following [4] this leads to the construction of an ordering P_0 on K(X) of level 2 subject to $X\alpha^{-1} \in P_0$, $-\alpha^2 \in P_0$. Next, let $(L,\tilde{P})$ be a real closure of $(K(X),P_0)$. From $0 < w(X) < w(t)$ we get $f(X) = X^2 \cdot \varepsilon$ with $\varepsilon \in P_0 \cap \Sigma K(X)^2$. As $P_0 \subset \tilde{P}$, $\Sigma K(X)^2 \subset \Sigma L^2 = L^2$ and $\tilde{P} \cap L^2 = L^4$ as well as $(X\alpha^{-1})^2 \in \tilde{P} \cap L^2 = L^4$ we obtain $f(X) = X^2 \cdot \varepsilon = \alpha^2 \cdot a^4$. Hence, $g(X,a) = 0$, and $k(\mathfrak{p})$ can be embedded into L. Set $P' = \tilde{P} \cap k(\mathfrak{p})$, then $\alpha^2 \notin P'$, and $\mathrm{rad}_\alpha \mathfrak{p} = \mathfrak{p}$ follows.

<u>Remarks:</u> i) In the last proof K is not existentially closed in $k(\mathfrak{p})$ which is the function field of the hypersurface $g = 0$. By [7] or [18] this is equivalent to the fact that the Jacob ring of $k(\mathfrak{p})$ (relative to P') does not extend the

Jacob ring of K. It is exactly this fact observed by A. Prestel several years ago. ii) Even if the chain-closed field K does not meet the hypothesis of (6.2) there are Nullstellensätze of the type

$$I_K V_K(\mathfrak{a}) = \{f \in K[X_1,\ldots,X_n] \mid f^{2lk} + s \in \mathfrak{a} \text{ for some } k \in \mathbb{N},\ s \in S\}$$

where S is a certain semiring, cf. [5 , (4.9)]; here K has to be a generalized real closed field of level l. In the 2-primary case (this amounts to l = 2), R. Berr has found nice generators of S, cf. [7]. To derive these more general Nullstellensätze one has to study the behaviour of the Jacob ring under field extensions. As a consequence, a ring theoretic approach as presented in this paper will hardly work in case one has to consider the Jacob ring. At least, one needs the relative setup of R-algebras, R a certain ring, and a ring theoretic version of the notion of the Jacob ring.

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Eberhard Becker
University of Dortmund
Dept. of Mathematics
P.O. Box 50 05 00
4600 Dortmund 50
Fed. Rep. of Germany

Danielle Gondard
University of Paris VI
Institute of Mathematics
UER 47
4, Place Jussieu
75320 Paris Cedex 05
France

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