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The Dirichlet Energy of Mappings with values into the sphere

Mariano Giaquinta, Giuseppe Modica, Jiří Souček

We discuss the relaxed functional of the Dirichlet energy. We also prove partial regularity of minimizers and concentration of the gradient on singular lines.

In **[6]**[**7**] we have shown that, when dealing with variational problems for vector valued mappings, and especially for mappings with values into a manifold, the most natural setting is the one of *cartesian currents* there introduced. In the special case of the Dirichlet energy

$$\mathcal{D}(u) := \frac{1}{2}\int_\Omega |Du(x)|^2\,dx$$

for mappings u from a bounded domain Ω of $\mathbb{R}^3$ into the unit sphere S^2 of $\mathbb{R}^3$, we were led to consider the parametric extension $\mathcal{D}(T)$ over the class $\mathrm{cart}^{2,1}(\Omega, S^2)$. The class $\mathrm{cart}^{2,1}(\Omega, S^2)$ is defined in [**7**], and can be characterized (by theorem 5.1 of [**7**]) as the class of 3-dimensional currents T in $\Omega \times S^2$ without boundary in $\Omega \times S^2$ for which there exist a unique function $u_T \in H^{1,2}(\Omega, S^2)$ and a unique

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1-dimensional integer rectifiable current L_T in Ω such that

$$T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!]$$

where $[\![G_{u_T}]\!]$ denotes the rectifiable current integration over the graph of u_T, cfr. [7]. The parametric extension of $\mathcal{D}(u)$ is then given by

$$\mathcal{D}(T,\Omega) := \frac{1}{2}\int_\Omega |Du_T(x)|^2\,dx + 4\pi\,\mathbf{M}_\Omega(L_T)\ , \tag{1}$$

where $\mathbf{M}_\Omega(L_T)$ denotes the *mass* of the current L_T in Ω.

Let φ be a boundary datum and assume, for the sake of simplicity, that it is smooth, say $C^\infty(\partial\Omega, S^2)$. Suppose moreover that φ has degree zero on $\partial\Omega$, then we can think of φ as the restriction of a smooth function still denoted by φ and defined on some open set $\widetilde{\Omega} \supset\supset \Omega$. The Dirichlet problem amounts then to the problem of minimizing $\mathcal{D}(T,\widetilde{\Omega})$ in the class

$$\operatorname{cart}^{2,1}_\varphi(\widetilde{\Omega}, S^2) := \left\{T \in \operatorname{cart}^{2,1}(\widetilde{\Omega}, S^2) \mid T = [\![G_\varphi]\!] \text{ on } (\widetilde{\Omega}\setminus\overline{\Omega}) \times S^2\right\}$$

The existence of a minimizer follows easily from the semicontinuity of $\mathcal{D}(T,\widetilde{\Omega})$ with respect to the weak convergence in $\operatorname{cart}^{2,1}(\widetilde{\Omega}, S^2)$ and from the weak compactness of energy bounded sets in $\operatorname{cart}^{2,1}(\widetilde{\Omega}, S^2)$, cfr. [7].

In [7] we conjectured that $\mathcal{D}(T,\widetilde{\Omega})$ is the *relaxed functional* of $\mathcal{D}(u)$ in $\operatorname{cart}^{2,1}_\varphi(\widetilde{\Omega}, S^2)$, i.e. that for all $T \in \operatorname{cart}^{2,1}_\varphi(\widetilde{\Omega}, S^2)$ there exists a sequence of smooth functions $\{u_k\}$, $u_k = \varphi$ on $\widetilde{\Omega}\setminus\overline{\Omega}$, such that $[\![G_{u_k}]\!] \rightharpoonup T$ and

$$\mathcal{D}(T,\widetilde{\Omega}) = \lim_{k\to\infty} \mathcal{D}(u_k) \quad ;$$

consequently

$$\begin{aligned} \mathcal{D}(T,\widetilde{\Omega}) : = \inf\{&\liminf_{k\to\infty} \mathcal{D}(u_k) \mid u_k \text{ smooth}, u_k = \varphi \text{ on } \widetilde{\Omega}\setminus\overline{\Omega}, \\ &\sup_k \mathcal{D}(u_k) < +\infty, [\![G_{u_k}]\!] \rightharpoonup T \text{ in } \operatorname{cart}^{2,1}(\widetilde{\Omega}, S^2)\} \end{aligned} \tag{2}$$

Recently F. Bethuel, H. Brezis, J.M. Coron [**1**] have considered a different extension of $\mathcal{D}(u)$ on $H^{1,2}_{\varphi}(\Omega, S^2) := \{\, u \in H^{1,2}(\Omega, S^2) \mid u = \varphi$ on $\partial\Omega\}$ given by

$$(3) \qquad F(u,\Omega) := \frac{1}{2}\int_\Omega |Du(x)|^2\,dx + 4\pi L(u)$$

where $L(u)$ is defined as

$$(4) \qquad L(u) := \frac{1}{4\pi} \sup_{\substack{\xi:\Omega\to\mathbf{R} \\ \|D\xi\|_\infty \le 1}} \left\{ \int_\Omega \mathbf{D}(u)\cdot D\xi\,dx \;-\; \int_{\partial\Omega} \mathbf{D}(u)\cdot n\,\xi\,d\mathcal{H}^2 \right\} ,$$

$\mathbf{D}(u)$ is the vector field

$$\mathbf{D}(u) := (u\cdot u_{x^2}\wedge u_{x^3}, u\cdot u_{x^3}\wedge u_{x^1}, u\cdot u_{x^1}\wedge u_{x^2}) \ ,$$

and n denotes the outward normal to $\partial\Omega$. They have shown that $F(u,\Omega)$ is the relaxed functional of $\mathcal{D}(u)$ on $H^{1,2}_{\varphi}(\Omega, S^2)$.

The aim of this paper is twofold. First we shall show that $F(u,\Omega)$ in (3) is the restriction of our functional $\mathcal{D}(T)$ to suitable currents. This provides an integral representation of $F(u,\Omega)$ and gives a precise geometric meaning to the term $L(u)$ in (4). Secondly we shall prove our conjecture, i.e. that (2) holds.

If T is a minimizer of the Dirichlet problem, then one easily sees that u_T is weakly harmonic. But T has to be stationary also with respect to variations of the domain parameters; this yields an extra equation expressing the energy conservation law, which gives at once a so-called monotonicity formula. We shall then show that, relying on the regularity theorem of R. Schoen and K. Uhlenbeck [**9**], the monotonicity formula allows us to prove easily a partial regularity theorem for the function u_T associated to the minimizer T.

We emphasize once more, cfr. [**7**] (see also [**8**]), that our minimizers T have in general line singularities, that in the approximation by smooth maps show up as lines where the gradient concentrates. This is precisely stated in theorem 6.

Let us state our results more precisely.

Given $u \in H^{1,2}_\varphi(\Omega, S^2)$, we think of u as being extended on $\widetilde{\Omega}\setminus\Omega$ by φ, i.e. as an element of $H^{1,2}_\varphi := \{u \in H^{1,2}(\widetilde{\Omega}, S^2) \mid u = \varphi \text{ on } \widetilde{\Omega}\setminus\overline{\Omega}\}$. Consider the class of currents

$$[u] := \{T \in \operatorname{cart}^{2,1}_\varphi(\widetilde{\Omega}, S^2) \mid u_T = u \text{ in } \widetilde{\Omega},\, T = [\![G_\varphi]\!] \text{ on } (\widetilde{\Omega}\setminus\overline{\Omega}) \times S^2\}$$

and set

$$\mathcal{D}([u],\widetilde{\Omega}) := \min_{T\in[u]} \mathcal{D}(T,\widetilde{\Omega})$$
$$F(u,\widetilde{\Omega}) := F(u,\Omega) + \frac{1}{2}\int_{\widetilde{\Omega}\setminus\Omega} |D\varphi(x)|^2\,dx\ .$$

Theorem 1. *Let* $u \in H^{1,2}_\varphi(\widetilde{\Omega}, S^2)$

(i) *We have*

$$F(u,\widetilde{\Omega}) = \mathcal{D}([u],\widetilde{\Omega})\ .$$

(ii) *More precisely, there exists a 1-dimensional rectifiable current L with* $\operatorname{spt} L \subset \overline{\Omega}$ *which minimizes* $\mathbf{M}(L)$ *under the condition*

$$-\partial L \times [\![S^2]\!] \;=\; \partial[\![G_u]\!] \llcorner (\widetilde{\Omega} \times S^2) \quad ;$$

moreover

$$L(u) \;=\; \mathbf{M}(L)$$

and

$$\mathcal{D}([u],\widetilde{\Omega}) \;=\; \frac{1}{2}\int_{\widetilde{\Omega}} |Du(x)|^2\,dx + 4\pi\,\mathbf{M}(L) \quad ;$$

in particular

$$F(u,\Omega) \;=\; \frac{1}{2}\int_{\Omega} |Du(x)|^2\,dx + 4\pi\,\mathbf{M}(L)$$

Theorem 2. *Let $T \in \mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$. Then there exists a sequence $\{u_k\}$ of smooth functions in $\widetilde{\Omega}$, with $u_k = \varphi$ on $\widetilde{\Omega} \setminus \overline{\Omega}$ such that*

$$[\![G_{u_k}]\!] \rightharpoonup T \qquad \text{in } \mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$$

and

$$\frac{1}{2}\int_{\widetilde{\Omega}} |Du_k(x)|^2\, dx \to \mathcal{D}(T, \widetilde{\Omega})$$

Let $T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!]$ be a minimizer of $\mathcal{D}(T, \widetilde{\Omega})$ in $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$. Consider the family of functions

$$u_t := \frac{u_T + t\psi}{|u_T + t\psi|}$$

For $\psi \in C^{\infty}_{o}(\widetilde{\Omega}, \mathbb{R}^3)$, $\operatorname{spt}\psi \subset \Omega$, and $|t|$ small we obviously have $[\![G_{u_t}]\!] + L_T \times [\![S^2]\!] \in \mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$. Thus we can conclude at once that u_T is weakly harmonic, i.e.

$$-\Delta\, u_T \;=\; u_T\, |Du_T|^2 \qquad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^3)\ .$$

We also have

Theorem 3. *Let $T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!]$, $L_T = \tau(\mathcal{L}, \theta, \zeta)$, be a minimizer of $\mathcal{D}(T, \widetilde{\Omega})$ in $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$. Then for all $\psi \in C^1_o(\Omega, \mathbb{R}^3)$ we have*

$$\int_{\Omega} \left(\frac{1}{2}|Du_T|^2 \operatorname{div}\psi - D_\alpha u^i_T\, D_\beta u^i_T\, D_\alpha \psi^\beta \right) dx +$$

(5)

$$+\, 4\pi \int \zeta_\alpha\, \zeta_\beta\, D_\alpha \psi^\beta\, \theta\, d\mathcal{H}^1 \llcorner \mathcal{L} = 0$$

Set now for $B_R(x_o) \subset\subset \Omega$

$$E_R := \frac{1}{2}\int_{B_R(x_o)} |Du_T(x)|^2\,dx + 4\pi\,\mathbf{M}(L_T \llcorner B_R(x_o))$$

$$\alpha_R := \frac{d}{dR}\left(\int_{B_R(x_o)} |\frac{\partial u_T}{\partial \nu}|^2 dx \;+ 4\pi \int_{B_R\cap\mathcal{L}} |\zeta - (\zeta,\nu)\nu|^2 \theta\, d\mathcal{H}^1\right)$$

Then an immediate consequence of (5) is the following

Theorem 4. *Under the assumptions of theorem 3 we have*

$$\frac{d}{dR}\left(\frac{1}{R}E_R\right) = \frac{1}{R}\alpha_R$$

in particular

$$\frac{1}{R}E_R$$

is increasing in R.

Theorem 4 in conjunction with the partial regularity theorem of [9] easily yields

Theorem 5. *Let $T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!]$ be a minimizer of $\mathcal{D}(T, \tilde{\Omega})$. Then u_T has locally Hölder-continuous first derivatives in some open set $\Omega_o \subset \Omega$, moreover the possible singular set $\Sigma := \Omega \setminus \Omega_o$ has Hausdorff dimension not greater than one.*

Theorem 6. *Let $\{u_k\}$ be a sequence of smooth functions such that $[\![G_{u_k}]\!] \rightharpoonup [\![G_u]\!] + L \times [\![S^2]\!]$ in $\mathrm{cart}_\varphi^{2,1}(\tilde{\Omega}, S^2)$ and $\mathcal{D}(u_k, \tilde{\Omega}) \to \mathcal{D}(T, \tilde{\Omega})$. Denote by $\mathrm{e}(T)$ the "energy density" of $\mathcal{D}(T)$, i.e.*

$$\mathrm{e}(T) = \frac{1}{2}|Du_T|^2\,\mathcal{H}^3 \;+\; 4\pi\|L_T\|$$

where $\|L_T\| = \theta(x)\,\mathcal{H}^1 \llcorner \mathcal{L}$. Then we have

(i) $\frac{1}{2}|Du_k|^2\,\mathcal{H}^3$ *converge as measures to* $\mathrm{e}(T)$.

(ii) *For all neighborhoods U of $\operatorname{spt} T$, u_k converges to u strongly in $H_\varphi^{1,2}(\tilde{\Omega} \setminus \overline{U}, S^2)$.*

Remark 1. As shown in [7], we can also consider the Dirichlet problem in $\widetilde{\Omega} \setminus \bigcup_{i=1}^{N}\{a_i\}$ where $\{a_i\}$ are points in Ω, prescribing T on $(\widetilde{\Omega} \setminus \overline{\Omega}) \times S^2$ and on each $\{a_i\} \times S^2$ as

$$\partial T \llcorner (\{a_i\} \times S^2) \;=\; d_i \,[\![\{a_i\} \times S^2]\!]$$

with d_i integers, $\sum_{i=1}^{N} d_i = 0$. A minimizer $T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!]$ exists, and one easily sees that u_T is a weakly harmonic mapping in Ω, as smooth variations do not vary the boundaries of T at $\{a_i\} \times S^2$. This way one may hope to find infinitely many weak harmonic mappings with the same values on $\partial\Omega$ by solving *different* Dirichlet's problems. Unfortunately it is not clear that the u_T corresponding to different singularities at the a_i's are different, as the boundary condition can be realized as boundary of $L_T \times [\![S^2]\!]$. In [1] it is shown, by means of a refined energy argument, that this procedure in fact produces infinitely many distinct weakly harmonic mappings with the same boundary value on $\partial\Omega$.

Remark 2. As shown in [7], we can also consider the following *weak* Dirichlet problem: minimize

$$\mathcal{D}(T,\Omega) \;:=\; \frac{1}{2}\int_{\Omega} |Du_T(x)|^2\, dx + 4\pi\, \mathbf{M}_{\Omega}(L_T)$$

among $T = [\![G_{u_T}]\!] + L_T \times [\![S^2]\!] \in \operatorname{cart}^{2,1}(\Omega, S^2)$ with $u_T = \varphi$ on $\partial\Omega$. In this case φ need not be of degree zero on $\partial\Omega$. Of course φ might have no smooth extension in Ω.

As previously, set for $u \in H^{1,2}_{\varphi}(\Omega, S^2)$

$$[u] := \{T \in \operatorname{cart}^{2,1}(\Omega, S^2) \mid u_T = u \text{ in } \Omega\}$$

and define

$$\mathcal{D}([u],\Omega) \;:=\; \min_{T\in[u]} \mathcal{D}(T,\Omega)$$

Then one can show that the above minimum is realized by a current $T_o = [\![G_u]\!] + L_o \times [\![S^2]\!]$ and that

$$\widetilde{F}(u,\Omega) = \frac{1}{2}\int_\Omega |Du(x)|^2\,dx + 4\pi\,\mathbf{M}_\Omega(L_o) = \mathcal{D}(T_o,\Omega)$$

where $\widetilde{F}$ is the functional considered in [1] and given by

$$\widetilde{F}(u,\Omega) = \frac{1}{2}\int_\Omega |Du(x)|^2\,dx + \widetilde{L}(u)$$

with

$$\widetilde{L}(u) \;:=\; \sup_{\substack{\xi:\Omega\to\mathbb{R}\\ \|D\xi\|_\infty\le 1\\ \xi=0 \text{ on } \partial\Omega}} \int_\Omega \mathbf{D}(u)\cdot D\xi\,dx$$

Remark 3. We observe that with the notations of [7], theorem 2 implies that

$$\mathrm{cart}^{2,1}_\varphi(\widetilde{\Omega},S^2) \;=\; \mathrm{Cart}^{1,2}_\varphi(\widetilde{\Omega},S^2)$$

Remark 4. Suppose that $u_k \rightharpoonup u$ in $H^{1,2}_\varphi(\widetilde{\Omega},S^2)$ and that $L(u)>0$. Denote by Λ the set of 1-dimensional rectifiable currents L with $\mathrm{spt}\,L \subset \overline{\Omega}$ such that

$$-\partial L\times S^2 \;=\; \partial[\![G_u]\!] \llcorner (\widetilde{\Omega}\times S^2) \quad \text{and } \mathbf{M}(L) = L(u)\;.$$

For each $L = \tau(\mathcal{L},\theta,t)\in\Lambda$ we denote by $\mathrm{e}(T)$ the "energy density" of the current $T := [\![G_u]\!] + L\times[\![S^2]\!]$ i.e. $\mathrm{e}(T) := \frac{1}{2}|Du|^2\,\mathcal{H}^3 + \theta\,(\mathcal{H}^1 \llcorner \mathcal{L})$. We clearly have

(i) If $\#\Lambda = 1$, then $\Lambda = \{L\}$ and $[\![G_{u_k}]\!] \rightharpoonup T = [\![G_u]\!] + L\times[\![S^2]\!]$, $\frac{1}{2}|Du_k|^2 \rightharpoonup \mathrm{e}(T)$

(ii) If $\#\Lambda > 1$, then there exists a decomposition of the sequence $\{u_k\}$ into sequences $\{v_k^{(\alpha)}\}$

$$\{u_k\} \;=\; \bigcup_{\alpha\in A}\{v_k^{(\alpha)}\}$$

and a family of rectifiable 1-dimensional currents $\{L^{(\alpha)}\}_{\alpha \in A}$ such that for all $\alpha \in A$,

$$[\![G_{v_k^{(\alpha)}}]\!] \rightharpoonup T^{(\alpha)} = [\![G_u]\!] + L^{(\alpha)} \times [\![S^2]\!]$$

and $\frac{1}{2}|Dv_k^{(\alpha)}|^2 \mathcal{H}^3 \to \mathrm{e}(T^\alpha)$.

Thus while the weak convergence in H^1 and the functional $F(u)$ do not distinguish among different minimizers with the same u_T, our functional $\mathcal{D}(T)$ and the weak convergence in $\mathrm{cart}^{2,1}(\widetilde{\Omega}, S^2)$ does it: a typical situation is the one of two dipoles with degrees ± 1 prescribed on the opposite edges of a square in Ω.

Proof of theorem 1: First we observe that if $T = [\![G_u]\!] + L_T \times [\![S^2]\!]$ belongs to $[u]$ then $\operatorname{spt} L_T \subset \overline{\Omega}$.

(i) We claim that $L(u) \leq \mathbf{M}(L_T)$ for all $T \in [u]$. Let $\eta \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$ with $\|d\eta\|_{\infty,\overline{\Omega}} \leq 1$ and denote by $\widetilde{\eta}$ any extension of η with compact support in $\widetilde{\Omega}$. Since φ is regular then $\operatorname{div} \mathbf{D}(u) = 0$ in $\widetilde{\Omega} \setminus \overline{\Omega}$ so

$$\int_{\widetilde{\Omega}} \mathbf{D} \cdot D\eta \, dx - \int_{\partial\Omega} \mathbf{D} \cdot \nu \, \eta \, d\mathcal{H}^2 = \int_{\widetilde{\Omega}} \mathbf{D} \cdot D\widetilde{\eta} \, dx$$

On the other hand, compare [7]

$$\begin{aligned} \int_{\widetilde{\Omega}} \mathbf{D} \cdot D\widetilde{\eta} \, dx &= \pi_\# \left([\![G_u]\!] \llcorner \hat{\pi}^\# \omega_{S^2} \right) (d\widetilde{\eta}) = \\ &= \pi_\# \left(\partial [\![G_u]\!] \llcorner \hat{\pi}^\# \omega_{S^2} \right) (\widetilde{\eta}) = -\, 4\pi \, L_T(d\widetilde{\eta}) \end{aligned}$$

and (i) follows since $\operatorname{spt} L_T \in \overline{\Omega}$.

We shall now prove that

(ii) There exists $T \in [u]$ such that $\mathbf{M}_{\overline{\Omega}}(L_T) = L(u)$, i.e. $\mathcal{D}(T, \widetilde{\Omega}) = F(u, \widetilde{\Omega})$ and obviously this concludes the proof of theorem 1.

According to the result of [1], let $\{u_k\} \subset H^{1,2}_\varphi(\widetilde{\Omega}, S^2)$ be a sequence of smooth functions such that

$$u_k \rightharpoonup u \text{ in } H^{1,2}_\varphi(\widetilde{\Omega}, S^2)$$

and

$$\frac{1}{2}\int_{\widetilde{\Omega}} |Du_k(x)|^2\,dx \to F(u,\widetilde{\Omega}) .$$

Passing to a subsequence, we have that $[\![G_{u_k}]\!]$ converge weakly in $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega},S^2)$ to some $T \in [u]$ and from the semicontinuity of $\mathcal{D}$, compare [**7**]

$$\mathcal{D}(T,\widetilde{\Omega}) \le \liminf_{k\to\infty} \mathcal{D}([\![G_{u_k}]\!],\widetilde{\Omega}) = \\ \liminf_{k\to\infty} \frac{1}{2}\int_{\widetilde{\Omega}} |Du_k(x)|^2\,dx = F(u,\widetilde{\Omega})$$

q.e.d.

Proof of theorem 2: We shall now show that every T which belongs to $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega},S^2)$ can be approximated weakly and in energy by currents whose singular part is given by a finite number of disjoint segments, the conclusion then follows from the approximation theorem of [**1**], theorem 2. We shall split our proof into several steps.

(A) First assume that ∂L_T is rectifiable, i.e. that ∂L_T is a finite combination with integer coefficients of points in $\widetilde{\Omega}$, and actually in Ω,

$$\partial L_T = \sum_{i=1}^{k} [\![\hat{p}_i]\!] - \sum_{i=1}^{k} [\![\hat{n}_i]\!]$$

and that

$$u_T \in C^\infty\left(\widetilde{\Omega} \setminus \bigcup_i \{p_i\} \cup \{n_i\}\right)$$

(i) *Approximation by polyhedral chains.* Using the approximation theorem of Federer [**5**], for all $\epsilon > 0$ we can find a polyhedral chain P_ϵ and a diffeomorphism $\phi_\epsilon : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{spt}\, P_\epsilon$ is contained in an ϵ-neighborhood U_ϵ of $\mathrm{spt}\, L_T$, $\mathrm{spt}\, P_\epsilon \subset U_\epsilon(\mathrm{spt}\, L_T)$, and

$$\mathrm{Lip}\,\phi_\epsilon\ ,\ \mathrm{Lip}\,\phi_\epsilon^{-1} \le 1+\epsilon\ ,\quad \phi_\epsilon = \mathrm{id\ on\ } \mathbb{R}^3 \setminus U_\epsilon(\mathrm{spt}\, L_T)$$

$$\mathbf{M}(P_\epsilon - \phi_{\epsilon\#}L_T) \;+\; \mathbf{M}(\partial P_\epsilon - \phi_{\epsilon\#}\partial L_T) < \epsilon$$

From the rectifiability of ∂P and ∂L, it follows that $\partial P_\epsilon = \phi_{\epsilon\#}\partial L_T$. Since $\operatorname{spt} L_T$ is a finite number of points, we can also find a diffeomorphism $\psi_\epsilon : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\psi_\epsilon(\phi_\epsilon(\Omega)) = \Omega$, $\psi_\epsilon = \phi_\epsilon = \mathrm{id}_{\partial\Omega}$, $\psi_\epsilon = \mathrm{id}$ on $\operatorname{spt}\partial P_\epsilon = \operatorname{spt}\phi_{\epsilon\#}\partial L_T$ and $\operatorname{Lip}\psi_\epsilon$, $\operatorname{Lip}\psi_\epsilon^{-1} \le 1 + c\epsilon$. If we now move the vertices of P_ϵ which are not in ∂P_ϵ by ψ_ϵ we finally find a new polyhedral chain $\widetilde{P}_\epsilon$ with $\operatorname{spt}\widetilde{P}_\epsilon \subset \Omega$, $\operatorname{spt}\widetilde{P}_\epsilon \subset U_{c\epsilon}(\operatorname{spt} P_\epsilon)$, $\partial\widetilde{P}_\epsilon = \partial P_\epsilon$ and clearly the currents

$$T_\epsilon := [\![G_{u_\epsilon}]\!] + \widetilde{P}_\epsilon \times [\![S^2]\!]\,, \quad u_\epsilon(x) := u_T(\psi_\epsilon(\phi_\epsilon(x)))$$

converge weakly for $\epsilon \to 0$ to T and $\mathcal{D}(T_\epsilon, \widetilde{\Omega}) \to \mathcal{D}(T, \widetilde{\Omega})$.

(ii) *Approximation by non autointersecting and density 1 polyhedral chains.* Let $T \in [\![G_{u_T}]\!] + P \times [\![S^2]\!] \in \operatorname{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$, P polyhedral, $\operatorname{spt} P \subset \Omega$ and $u_T \in C^\infty(\overline{\Omega} \setminus \operatorname{spt}\partial P)$. We have

$$P = \sum_i [\![(n_i, p_i)]\!]$$

where (n_i, p_i) is the oriented segment joining n_i to p_i, (the points n_i, respectively p_i, are not in general distinct). We claim that we can reorder the indeces i in such a way that if $p_i \notin \partial P$ then $p_i = n_{i+1}$. In fact, if $n_1 \in \partial P$ and there exist some $n_{\bar{i}} = p_1$, we rename n_i as $\tilde{n}_1$ and we consider $(\tilde{n}_1, \tilde{p}_1)$, $\tilde{p}_1 := p_{\bar{i}}$. If there exists $n_i \ne \tilde{n}_1$ with $n_i = \tilde{p}_1$, we rename n_i as $\tilde{n}_2$ and we continue this way until we are able to find points n_i different from the ones already choosed; this process, clearly finishes in a finite number of steps and the final $\tilde{p}_k$ we find obviously must belong to ∂P. Once the construction has been carried out for all $n_i \in \partial P$, we start with any n_i (if any is left) and we repeat the construction until we come back to some $p_k = n_i$ (observe that this must happen since we have already used all $n_i \in \partial P$, thus all $p_i \in \partial P$) and we continue this way. Observe that as a result of our construction on each point of ∂P chains either start or finish. Clearly we can now slightly move the $p_k = n_{k+1}$, which do not belong to ∂P, in such a way that the new $\overline{p}_k = \overline{n}_{k+1}$ belong to Ω and are distinct, the segments $(\overline{n}_k, \overline{p}_k)$ do not intersect in $\Omega \setminus \operatorname{spt} \partial P$ and finally $\widetilde{P} := \sum_k [\![(\overline{n}_i, \overline{p}_i)]\!] \subset U_\epsilon(\operatorname{spt} P)$. We therefore conclude that we can find a sequence of finite polyhedral lines $P^{(k)}$ (which are either closed or start and finish on ∂P) without autointersections such that $\partial P^{(k)} = \partial P$, $P^{(k)} \rightharpoonup P$ and $\mathbf{M}(P^{(k)}) \to \mathbf{M}(P)$. We emphasize that on each point of ∂P lines of $P^{(k)}$ either all start or all finish.

(iii) *Adding small dipoles.* Let $T = [\![G_u]\!] + P \times [\![S^2]\!] \in \operatorname{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$ where P is a polyhedral chain as in the conclusion of (ii) and $u_T \in C^\infty(\widetilde{\Omega} \setminus \operatorname{spt} \partial P)$. Let $x_0 \in \Omega \setminus \operatorname{spt} \partial P$ and ϵ be a positive small number. We claim that for all x_1 in $B(x_0, \delta)$, δ small, there exists $v \in C^\infty(\widetilde{\Omega} \setminus (\operatorname{spt} \partial P \cup \{x_0\} \cup \{x_1\}))$, $v = u$ on $\Omega \setminus B(x_0, \delta)$ such that

$$\frac{1}{2} \int_{B(x_0,\delta)} |Dv(x)|^2\, dx \le \frac{1}{2} \int_{B(x_0,\delta)} |Du(x)|^2\, dx + \epsilon$$

$$\deg(v, x_0) = -\deg(v, x_1) = 1$$

In fact if δ is sufficiently small, the oscillation of u on $\partial B(x_0, \delta)$ is small, thus we can extend smoothly u to $B(x_0, \delta)$ as $\widetilde{u}$ with $\widetilde{u} = $ constant on $B(x_0, \delta/2)$ and with $\frac{1}{2} \int_{B(x_0,\delta)} |D\widetilde{u}(x)|^2\, dx$ small. Applying to $B(x_0, \delta/2)$ the dipole construction of [4], the claim follows.

Of course, if $x_0 = p_k = n_{k+1} \notin \partial P$, we can moreover require that $x_1 \in (n_{k+1}, p_{k+1})$, and obviously the current $\widetilde{T} := [\![G_v]\!] + [\![(x_1, x_0)]\!] \times [\![S^2]\!]$ belongs to $\operatorname{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$ and

$$\mathcal{D}(\widetilde{T}, B(x_0, \delta)) \le \mathcal{D}(T, B(x_0, \delta)) + 2\epsilon$$

We now apply the previous construction near every point $p_k = n_{k+1} \notin \partial P$ and we find points $\overline{n}_{k+1} \in (n_{k+1}, p_{k+1})$ and v smooth outside of $\operatorname{spt} \partial P$ and $\{p_k \mid p_k \notin \partial P\} \cup \{\overline{n}_k \mid \overline{n}_k \notin \partial P\}$ such that

$$\widetilde{T} := [\![G_v]\!] + \widetilde{P} \times [\![S^2]\!] \in \operatorname{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$$
$$\widetilde{P} := \sum_k [\![(\overline{n}_k, \overline{p}_k)]\!]$$

and

$$\mathcal{D}(\widetilde{T}, \widetilde{\Omega}) \;\leq\; \frac{1}{2}\int_{\widetilde{\Omega}} |Du_T(x)|^2\, dx + 4\pi\, \mathbf{M}_{\widetilde{\Omega}}(P) + c\epsilon$$

Decompose now $\operatorname{spt} \widetilde{P}$ into its convex components C_α and observe that for ϵ small the ϵ-neighborhoods $U_\epsilon(C_\alpha)$ of C_α are disjoint. Clearly either C_α is a segment or is a finite union of segments which all start (or finish) in some $x_0 \in \operatorname{spt} \partial P$. Thus C_α is a minimal connection in $U_\epsilon(C_\alpha)$. Applying theorem 2 of **[1]** on each $U_\epsilon(C_\alpha)$ the proof of our theorem is completed under the extra assumption (A).

(B) Let $T = [\![G_u]\!] + L \times [\![S^2]\!]$ be a generic element of $\operatorname{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$. From **[2]** we know that

$$R^\infty_\varphi := \Big\{\, u \in H^{1,2}_{\varphi}(\widetilde{\Omega}, S^2) \mid \exists\{a_i\} \subset \Omega \text{ finite},\; u \in C^\infty(\overline{\Omega} \setminus \{a_i\})\Big\}$$

is dense in $H^{1,2}_{\varphi}(\widetilde{\Omega}, S^2)$. We claim that for all $v \in R^\infty_\varphi$ there exists a 1-dimensional rectifiable current $L_{u,v}$ in $\mathbb{R}^3$ with $\operatorname{spt} L_{u,v} \subset \overline{\Omega}$ such that

$$\partial L_{u,v} \times [\![S^2]\!] = \partial[\![G_u]\!] - \partial[\![G_v]\!]$$

$$\mathbf{M}(L_{u,v}) = L(u,v) := \sup_{\substack{\xi:\mathbb{R}^3 \to \mathbb{R} \\ \|D\xi\|_\infty \leq 1}} \Big(\int_\Omega \mathbf{D}(u)\cdot D\xi\, dx - \int_\Omega \mathbf{D}(v)\cdot D\xi\, dx\Big) \tag{6}$$

We postpone the proof of the claim, completing first the proof of the theorem. For all $\epsilon > 0$ we can find $v \in R^\infty_\varphi$ such that $\|u - v\|_{H^1} < \epsilon$ and we extend v by ϕ on $\widetilde{\Omega}$; according to the claim, we also find $L_{u,v}$ satisfying (6). In particular, compare **[1]**, we have

$$\mathbf{M}(L_{u,v}) \leq c\ (\|Du\|_{L^2} + \|Dv\|_{L^2})\ \|Du - Dv\|_{L^2} \leq c\epsilon$$

Consider now the current

$$\widetilde{T} := [\![G_v]\!] + (L + L_{u,v}) \times [\![S^2]\!].$$

We have

$$\begin{aligned}
&\partial\widetilde{T} \llcorner (\widetilde{\Omega} \times S^2) = \\
&\qquad \partial[\![G_v]\!] \llcorner (\widetilde{\Omega} \times S^2) + (\partial L + \partial L_{u,v}) \times [\![S^2]\!] = \\
&\qquad\qquad \partial[\![G_v]\!] \llcorner (\widetilde{\Omega} \times S^2) + \partial L \times [\![S^2]\!] + \\
&\qquad\qquad\qquad \partial[\![G_u]\!] \llcorner (\widetilde{\Omega} \times S^2) - \partial[\![G_v]\!] \llcorner (\widetilde{\Omega} \times S^2) = 0
\end{aligned}$$

thus $\widetilde{T} \in \mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$ and

$$\mathcal{D}(\widetilde{T}, \widetilde{\Omega}) \leq \mathcal{D}(T, \widetilde{\Omega}) + 4\pi(1 + c)\epsilon$$

But $\widetilde{T}$ satisfies the extra assumption in (A), as $\partial(L + L_{u,v}) \times [\![S^2]\!] = \partial[\![G_v]\!] \llcorner (\Omega \times S^2)$, thus the proof is easily completed.

Finally let us prove our claim. Let $\{u_k\}$ be a sequence in R^{∞}_{φ} such that $\{u_k\}$ converge strongly in H^1 to u (and we extend u_k by ϕ on $\widetilde{\Omega}$). Denote by L_k the 1-dimensional rectifiable currents with $\operatorname{spt} L_k \subset \overline{\Omega}$ such that

$$\begin{aligned}
&\partial L_k \times [\![S^2]\!] = -\partial[\![G_{u_k}]\!] \llcorner (\Omega \times S^2) \\
&\mathbf{M}(L_{u_k}) = L(u_k)
\end{aligned}$$

which exist by theorem 1; by lemma 4 of [1] there exist 1-dimensional rectifiable currents L'_k with $\operatorname{spt} L'_k \subset \overline{\Omega}$ such that

$$\begin{aligned}
&\partial L'_k \times [\![S^2]\!] = \partial[\![G_{u_k}]\!] - \partial[\![G_v]\!] \\
&\mathbf{M}(L'_k) = L(u_k, v)
\end{aligned}$$

Obviously $\sup_k \mathbf{M}_{\overline{\Omega}}(L'_k) < +\infty$, thus, passing to a subsequence, $L'_k \to L'$, where L' is a 1-dimensional current with $\operatorname{spt} L' \subset \overline{\Omega}$. From the semicontinuity of $\mathbf{M}$ and theorem 1 of [1] we also get

$$\mathbf{M}(L') \leq \liminf_{k \to \infty} L(u_k, v) = L(u, v);$$

moreover, since $\partial[\![G_{u_k}]\!] \rightharpoonup \partial[\![G_u]\!]$ (as $u_k \to u$ strongly in H^1) and hence $\partial L'_k \rightharpoonup \partial L$, we have

$$\partial L' \times [\![S^2]\!] = \partial[\![G_u]\!] - \partial[\![G_v]\!].$$

As in theorem 1, one can also prove that $\mathbf{M}(L') \geq L(u,v)$, hence we deduce that $\mathbf{M}(L') = L(u,v)$. It remains to prove that L' is rectifiable. Consider the rectifiable currents $\widetilde{L}_k := L_k + L'_k$. we have $\partial\widetilde{L}_k \times [\![S^2]\!] = -\partial[\![G_v]\!] \llcorner (\Omega \times S^2)$ thus, by the closure theorem, passing to a subsequence, $\widetilde{L}_k$ converge weakly to a rectifiable current $\widetilde{L}$. From

$$T_k = [\![G_{u_k}]\!] + L_k \times [\![S^2]\!] \quad \rightharpoonup \quad T = [\![G_u]\!] + \overline{L} \times [\![S^2]\!]$$

and $[\![G_{u_k}]\!] \rightharpoonup [\![G_u]\!]$, we deduce that $L_k \rightharpoonup \overline{L}$, i.e.

$$L'_k = \widetilde{L}_k - L_k \quad \rightharpoonup \quad L' = \widetilde{L} - \overline{L}$$

As L' and $\overline{L}$ are rectifiable, the claim is proved.

q.e.d.

Proof of theorem 3: Set $\Psi_t(x) := x + t\psi(x)$. For $|t|$ small Ψ_t is a diffeomorphism of Ω into Ω. Set now

$$\eta_t(x,y) := (\Psi_t(x), y) \quad U_t(x) := u(\Psi_t^{-1}(x)) \quad x \in \Omega,\ y \in S^2$$

Obviously $U_t \in H^{1,2}(\Omega, S^2)$, $U_t = \varphi$ on $\partial\Omega$ and $\eta_{t\#}([\![G_u]\!]) = [\![G_{U_t}]\!]$. Thus from the minimality of T we deduce

$$0 = \frac{d}{dt}\mathcal{D}(\eta_{t\#}T)_{|t=0} = \frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |DU_t(x)|^2\,dx + 4\pi\,\mathbf{M}(\Psi_{t\#}L)\right\}_{|t=0}$$

Now from

$$\frac{1}{2}\int_\Omega |DU_t(x)|^2\,dx = \frac{1}{2}\int_\Omega |Du(x)\,D\Psi_t^{-1}(\Psi_t(x))|^2 \det D\Psi_t^{-1}\,dx$$

$$Du \cdot D\Psi_t^{-1}(\Psi_t) = Du\,(1 - tD\psi + o(t^2))\ , t \to 0$$

$$\det D\Psi_t \;=\; 1 + t\,\mathrm{div}\,\psi + o(t^2)$$

we deduce

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |DU_t(x)|^2\,dx\Big|_{t=0} =$$
$$= \int_\Omega \left(\frac{1}{2}|Du|^2 \operatorname{div}\psi - D_\alpha u^i\, D_\beta u^i\, D_\alpha \psi^\beta\right) dx$$

From **[10]** p.78 we get

$$\frac{d}{dt}\,\mathbf{M}(\Psi_{t\#}L)\Big|_{t=0} = \int \zeta_\alpha(x)\,\zeta_\beta(x)\,D_\alpha\psi^\beta(x)\,\theta(x)\,d\mathcal{H}^1 \llcorner \mathcal{L}(x)$$

q.e.d.

Proof of theorem 4: Choosing in (5) $\psi(x) = (x - x_0)\eta(r)$, where $r = |x - x_0|$ and η is a smooth function in $[0, R]$ with $\eta(0) = 1$, $\eta(R) = 0$, $0 \le \eta(r) \le 1$, we find

$$\frac{1}{2}\int_{B_R(x_0)} |Du|^2(\eta + r\eta')\,dx + 4\pi\int_{B_R(x_0)} (\eta + r\eta')\theta(x)\,d(\mathcal{H}^1 \llcorner \mathcal{L}) =$$

$$= \int_{B_R(x_0)} r\eta'|u_{,r}|^2\,dx + 4\pi\int_{B_R(x_0)} |\zeta - (\zeta,\nu)\nu|^2\theta\,d(\mathcal{H}^1 \llcorner \mathcal{L})$$

by choosing a sequence of η's which approach the characteristic function of $(0, R)$ we then conclude

$$\frac{1}{2}\int_{B_R(x_0)} |Du(x)|^2\,dx - R\frac{d}{dR}\left(\frac{1}{2}\int_{B_R(x_0)} |Du(x)|^2\,dx\right) +$$
$$+ 4\pi\,\mathbf{M}(L \llcorner B_R(x_0)) - 4\pi R\frac{d}{dR}\,\mathbf{M}(L \llcorner B_R(x_0)) =$$
$$= R\frac{d}{dR}\int_{B_R(x_0)} |\frac{du}{dr}|^2\,dx +$$
$$+ 4\pi R\frac{d}{dR}\int_{B_R(x_0)} |\zeta - (\zeta,\nu)\nu|^2\theta\,d(\mathcal{H}^1 \llcorner \mathcal{L})$$

from which the result follows at once.

q.e.d.

Proof of theorem 5: Assume that for some point $x_0 \in \Omega$ and some R, $R < \operatorname{dist}(x_0, \partial\Omega)$ we have

$$\text{(7)} \qquad \frac{1}{R}\int_{B_R(x_0)} |Du_T|^2\, dx + \frac{4\pi}{R}\, \mathbf{M}(L_T \llcorner B_R(x_0)) < \epsilon_0$$

where ϵ_0 is positive and small. we claim that $L \llcorner B_{R/2}(x_0) = 0$. Otherwise there would exist $\overline{x} \in B_{R/2}(x_0)$ such that

$$\lim_{\rho\to 0} \frac{1}{2\rho}\, \mathbf{M}(L \llcorner B_\rho(\overline{x})) = 1$$

Consequently, for ρ small enough we would have, using the monotonicity formula,

$$4\pi \le \frac{1}{2\rho}\int_{B_\rho(\overline{x})} |Du_T|^2\, dx + \frac{4\pi}{\rho}\, \mathbf{M}(L_T \llcorner B_\rho(\overline{x})) \le$$

$$\le \frac{1}{R}\int_{B_{R/2}(\overline{x})} |Du_T|^2\, dx + \frac{8\pi}{R}\, \mathbf{M}(L_T \llcorner B_{R/2}(\overline{x})) \le$$

$$\frac{1}{R}\int_{B_R(x_0)} |Du_T|^2\, dx + \frac{8\pi}{R}\, \mathbf{M}(L_T \llcorner B_R(x_0)) \le 2\epsilon_0$$

a contradiction. We now deduce that if (7) holds, then u_T is an energy minimizing harmonic map in $B_{R/2}(x_0)$, thus if ϵ_0 is sufficiently small, u_T is regular thanks to the regularity result of [**9**]. The theorem is then proved since (7) holds on a set of Hausdorff dimension not greater than one. This follows as

$$\frac{1}{R}\int_{B_R(x_0)} |Du_T|^2\, dx \;\to\; 0 \qquad \mathcal{H}^1 \text{ a.e.}$$

while the density of L is zero except at most on a 1-dimensional set.

q.e.d.

Proof of theorem 6: Passing to a subsequence we have

$$\mathrm{e}([\![G_{u_k}]\!]) \rightharpoonup \mu_0$$

For all $\psi \in C_0^0(\widetilde{\Omega})$, $\psi \geq 0$ consider now the functional defined in $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$

$$\mathcal{E}(T) := \int \psi(x)\mathrm{e}(T)$$

We know from [7] that $\mathcal{E}$ is lower semicontinuous with respect to the convergence in $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{\Omega}, S^2)$, thus we conclude that

$$\mathrm{e}(T)(\psi) \leq \liminf_{k\to\infty} \mathcal{E}(u_k) \leq \mu_0(\psi)$$

i.e. $\mathrm{e}(T) \leq \mu_0$. As $\mathcal{D}(u_k, \widetilde{\Omega}) \to \mathcal{D}(T, \widetilde{\Omega})$ we then get $(\mu_0 - \mathrm{e}(T))(\widetilde{\Omega}) = 0$, i.e. $\mu_0 = \mathrm{e}(T)$. This concludes the proof of (i). The claim in (ii) then follows easily.

q.e.d.

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Mariano Giaquinta & Giuseppe Modica
Istituto di Matematica Applicata
Università di Firenze
Via S. Marta, 3
50139 Firenze
Italy

Jiří Souček
Československá Akademie
Věd Mathematický Ústáv
Praha
ČSSR

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