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CURVATURE PROPERTIES OF 6-PARAMETRIC ROBOT MANIPULATORS

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The Lie group C_6 of all orientation preserving congruences of the Euclidean space E_3 has a natural invariant pseudo-Riemannian structure determined by the Klein quadratical form of its Lie algebra. In the paper we connect the acceleration properties of a 6-parametric robot manipulator with the properties of the Levi-Civita connection and its curvature tensor of the pseudo-Riemannian structure of C_6. The paper is a continuation of [3].

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1. Introduction

A pseudo-Riemannian manifold M is a manifold equipped with a nondegenerated scalar product $\langle\ ,\ \rangle$ in each tangent space $T_x, x\in M$ (all structures are C^∞ structures).

Let $\mathcal{F}(M)$ be the set of all functions on a manifold M, $\mathfrak{X}(M)$ be all vector fields on M. An affine connection on M is a mapping ∇ : $\mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) : (X,Y) \rightarrow \nabla_X Y$, which is $\mathcal{F}$-linear in the first variable, R-linear in the second variable and

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y \quad \text{for } f\in\mathcal{F}.$$

For any connection ∇ we have the torsion tensor

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and the curvature tensor

$$R_{XY} = \nabla_{[X,Y]} - \nabla_X \nabla_Y - \nabla_Y \nabla_X .$$

On any pseudo-Riemannian manifold M there exists a unique symmetric connection such that the parallel displacement preserves the scalar product (see [1], p.48).

This connection is called the Levi-Civita connection of the pseudo-Riemannian manifold, let us denote it ∇. Then we have

$[V,W] = \nabla_V W - \nabla_W V$ (T=0, the symmetry condition),

$X \langle V,W \rangle = \langle \nabla_X V, W \rangle + \langle V, \nabla_X W \rangle$ (the covariant derivative of the scalar product is zero), for X,V,W from $\mathcal{X}(M)$.

The Levi-Civita connection is determined by the Koszul formula

$$2\langle \nabla_V W, X \rangle = V \langle W,X \rangle + W \langle X,V \rangle - X \langle V,W \rangle - \langle V,[W,X] \rangle + \langle W,[X,V] \rangle + \langle X,[V,W] \rangle . \tag{1}$$

If x_i, i=1,...,n, is a local system of coordinates, we define the Christoffel symbols of the connection by

$$\nabla_{X_i} X_j = \Gamma^k_{ij} X_k, \text{where} \quad X_i = \frac{\partial}{\partial x_i} . \tag{2}$$

Let us define

$$h_{ij} = \langle X_i, X_j \rangle , \quad \Gamma_{ij,k} = \Gamma^m_{ij} h_{mk} . \tag{3}$$

The Koszul formula yields

$$2\, \Gamma_{ij,k} = \frac{\partial}{\partial x_i} h_{jk} + \frac{\partial}{\partial x_j} h_{ik} - \frac{\partial}{\partial x_k} h_{ij} . \tag{4}$$

For any two independent vectors v and w from T_x we define

$$Q(v,w) = \langle v,v \rangle \langle w,w \rangle - \langle v,w \rangle^2 .$$

Then we can define the sectional curvature for non-degenerate 2-planes by

$$K(v,w) = \frac{\langle R_{vw} v, w \rangle}{Q(v,w)} . \tag{5}$$

We define the components of the curvature tensor by

$$R_{X_i X_j} X_k = R^m_{ijk} X_m .$$

The Ricci tensor is defined by its components as $R_{ij} = R^m_{ijm}$ and it will be denoted by Ric(X,Y). In an orthonormal basis E_i in the tangent space T_x the Ricci tensor is given by

$$\mathrm{Ric}(X,Y) = \sum_{m=1}^{m} \varepsilon_m < R_{XE_m} Y, E_m > \text{ ,where } \varepsilon_m = < E_m, E_m > = \pm 1. \quad (6)$$

Finally, the scalar curvature S is defined by $S = h^{ij} R_{ij} = h^{ij} R^k_{ijk}$,

in an orthonormal basis we obtain $S = 2 \sum_{i<j} K(E_i, E_j)$. (7)

2. The Cartan connection of C_6

Let G be a Lie group. An affine connection on G is called left invariant iff $\nabla_{L_g X}(L_g Y) = L_g \nabla_X Y$ for all g from G. Left invariant connections on Lie groups were characterized by K.Nomizu (see [1],p.92) in the following theorem:

There is a 1-1 correspondence between the set of all left invariant affine connections ∇ on G and the set of all bilinear maps

$\alpha : \underline{g} \times \underline{g} \to \underline{g}$, given by $\alpha(X,Y) = [\nabla_{\tilde{X}} \tilde{Y}]_e$,

where $\underline{g}$ is the Lie algebra of G, $X, Y \in \underline{g}$ and $\tilde{X}$ denotes the left invariant vector field on G determined by $X \in \underline{g}$. The following are equivalent:

a) $\alpha(X,X) = 0$,

b) The geodesic line starting at e with the tangent vector X at e is a one-parametric subgroup of G.

A left invariant connection on G is called invariant iff it is invariant with respect to right translations of G (which means that the map α must be AdG invariant).

<u>Example</u>. The map $\alpha(X,Y) = k[X,Y]$ for $k \in R$ defines an invariant affine connection on any Lie group G. For this connection we have

$$T(X,Y) = (2k-1)[X,Y], \quad R_{XY} = -k(k-1)\mathrm{ad}[X,Y].$$

Special cases are:

a) k=0 and k=1. They are the so called (+) and (-) connections with R=0. They define the so called left and right parallelism, which was used by W.Blaschke.

b) k = 1/2. Then T = 0 and R = (1/4)ad. This connection is called the Cartan connection.

Let C_6 be the group of all orientation preserving congruences of the Euclidean space E_3 and L be its Lie algebra. Let further R^3 be the ordinary 3-dimensional Euclidean vector space with the standart scalar product (x,y) and vector product $x \times y$. Then L can be considered as the algebra of all pairs $X = (y;z)$ of ordinary vectors y and z with the Lie bracket

$$[X_1,X_2] = [(y_1;z_1),(y_2;z_2)] = (y_1 \times y_2;\ y_1 \times z_2 + z_1 \times y_2).$$

Vectors $X = (y;0)$ from L form a subalgebra of L which is isomorphic with the Lie algebra of the Lie group SO(3). The Lie bracket is then the ordinary vector product, $[X_1,X_2] = (x_1 \times x_2;\ 0)$.

LEMMA 1. *Let G be a connected Lie group. Then a left invariant affine connection on G is invariant iff*

$$[Z, \alpha(X,Y)] = \alpha([Z,X],Y) + \alpha(X,[Z,Y]) \tag{8}$$

for all $X,Y,Z \in \mathfrak{g}$.

Proof. The necessary condition is obvious. To prove the converse, we use the formula $\mathrm{Ad}\exp Z = e^{\mathrm{ad}\,Z}$, where $e^{\mathrm{ad}\,Z}$ is the usual exponential of a mapping. From (8) we derive

$$(\mathrm{ad}\,Z)^n \alpha(X,Y) = \sum_{i=0}^{n} \binom{n}{i} \alpha((\mathrm{ad}\,Z)^i X, (\mathrm{ad}\,Z)^{n-i} Y)$$

and the rest follows.

LEMMA 2. *The invariant connections on* SO(3) *are given by* $\alpha(X,Y) = k[X,Y]$, *where* $k \in R$.

Proof. We have $Z \times \alpha(X,Y) = \alpha(Z \times X,Y) + \alpha(X,Z \times Y)$.
For $X=Y=Z$ we obtain $X \times \alpha(X,X) = 0$, so $\alpha(X,X) = a(X).X$. We substitute Y for X and $X \times Y$ for Z in (8) to obtain

$(X \times Y) \times \alpha(X,X) = \alpha((X \times Y) \times X,X) + \alpha(X,(X \times Y) \times X)$.

This yields

$a(X)X^2Y + (X,Y)a(X)X = X^2(\alpha(X,Y) + \alpha(Y,X))$.

Now we interchange X and Y, multiply by Y^2 and by X^2 and subtract. We obtain

$(Y^2a(X) - (X,Y)a(Y))X^2Y - ((X,Y)a(X) - X^2a(Y))Y^2X = 0$.

Let X and Y be linearly independent. Then

$Y^2a(X) - (X,Y)a(Y) = 0$ and $(X,Y)a(X) - X^2a(Y) = 0$,

so $a(X) = 0$. This follows $\alpha(X,Y) = -\alpha(Y,X)$. Let us write

$\alpha(X,Y) = a(X,Y)X + b(X,Y)Y + c(X,Y)X\times Y$. Then for $Z = X\times Y$ we obtain $(X\times Y)\times\alpha(X,Y) = 0$, $\alpha(X,Y) = c(X,Y)X\times Y$ and we see easily that $c(X,Y) = \text{const}$.

Let us define the following map $D:L\to L:(y;z)\to(0;y)$. Then D is an Ad C_6 invariant linear map, $D^2 = 0$ and $D[X,Y] = [DX,Y]$.

THEOREM 1. *Every invariant connection on* C_6 *is given by the formula*

$$\alpha(X,Y) = (kE + rD)[X,Y],$$

where E *is the identity map*, $k,r\in R$.

Proof. Consider L as the 3-dimensional Lie algebra over dual numbers, $\pi = kE+rD, k,r\in R$, $X = x_1 + Dx_2$ for $X\in L, x_1,x_2\in R^3$. The Lie bracket becomes the D-extension of the ordinary vector product. Let us write $\alpha(X,Y) = \beta(X,Y) + D\gamma(X,Y)$ for the real and dual part of the mapping α. Then

$$[z_1,\beta(x_1,y_1)+D\gamma(x_1,y_1)] = \beta(z_1\times x_1,y_1)+\beta(x_1,z_1\times y_1)+D(\gamma(z_1\times x_1,y_1)+\gamma(x_1,z_1\times y_1)).$$

LEMMA 2 yields $\alpha(x_1,y_1) = (k+rD)x_1\times y_1$, E is left out.
By similar arguments we obtain

$$\alpha(x_1+Dx_2,y_1+Dy_2)=(k+rD)x_1\times y_1+(p+qD)x_2\times y_1+(m+nD)x_1\times y_2+(t+sD)x_2\times y_2,$$

where k,r,p,q,m,n,s,t are real numbers. For $Z = Dz$ we obtain from (8) $t=s=m=p=0, q=n=k$ and the statement follows.

COROLLARY of the THEOREM 1. *The only invariant symmetric connection on* C_6 *is the Cartan connection.*

Proof. We must have $\alpha(X,Y) - \alpha(Y,X) = [X,Y]$. Therefore $2k-1+2rD=0$ and $k=1/2, r=0$.

On C_6 we have two Ad C_6 invariant quadratical forms:

a) The Killing form $B(X,X) = (y,y)$,

b) the Klein form $\langle X,X\rangle = (y,z)$, where $X = (y;z)$.

The Klein form on L is a pseudo-Riemannian scalar product with signature (+++---). By it C_6 obtaines the structure of a pseudo-Riemannian manifold with Ad C_6 invariant scalar product. Such Lie groups are called pseudo-Riemannian Lie groups and they have the following properties (see [2],p.304):Let G be a connected Lie group with left and right invariant (pseudo-Riemannian) metric tensor $\langle\ ,\ \rangle$ .Then

a) the map $g \rightarrow g^{-1}$ is an isometry of G,

b) $\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle$,

c) the Levi-Civita connection is the Cartan connection,

d) all geodesic lines are translates of one-parametric subgroups.

This means that a pseudo-Riemannian Lie group is a pseudo-Riemannian symmetric space.

We have for any pseudo-Riemannian Lie group:

a) Let the 2-plane generated by X and Y be non-degenerated.Then

$$K(X,Y) = \frac{1}{4}\frac{\langle [X,Y],[X,Y]\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle\langle X,Y\rangle}$$

b) Ric = -(1/4)B,where B is the Killing form of $\underline{g}$.

LEMMA 3.*Let* H(X,Y) *be an* Ad G *invariant quadratical form on* $\underline{g}$, ∇ *be the Cartan connection.Then the parallel displacement determined by* ∇ *preserves the invariant tensor field* $\tilde{H}(X,Y)$ *on* G,*determined by* H(X,Y).

Proof.We have to show that $\nabla_Z(\tilde{H}(X,Y)) = 0$ for any $Z \in \mathfrak{X}(G)$,where ∇_Z denotes the covariant derivative with respect to ∇ .We have

$$(\nabla_Z\tilde{H})(X,Y) = -\tilde{H}(\nabla_Z X,Y) - \tilde{H}(X,\nabla_Z Y) + Z(\tilde{H}(X,Y)).$$

The last equation is $\mathcal{F}$-linear and so it is enough to verify it for left invariant vector fields.But then we have $\tilde{H}(X,Y) = $ const.,$Z(\tilde{H}(X,Y))=0$ and

$$\tilde{H}(\nabla_Z X,Y) + \tilde{H}(X,\nabla_Z Y) = \frac{1}{2}\{H([Z,X],Y) + H(X,[Z,Y])\} = 0$$

as H is Ad G invariant.

So far we have shown that C_6 is a pseudo-Riemannian symmetric space of index 3.We know that the parallel displacement preserves both the Killing and Klein forms and this means that the parallel displacement preserves the angular velocity and the pitch of a motion.Further,the geodesic lines are translates of one-parametric subgroups.One-parametric subgroups are screw-motions,rotations and translations.There exist three types of geodesic lines,(+,-,0),according to the sign of the Klein form.The (+) geodesic lines are right handed screw motions,(-)

geodesic lines are left handed screw motions, null geodesic lines are rotations for nonzero Killing form and translations if the Killing form is zero. We know that the Ricci tensor is given by the Killing form, which means that C_6 is not an Einstein space.

THEOREM 2. *A 2-plane generated by vectors $X,Y \in L$ with pitch v_1, v_2, distance d and angle φ is non-degenerated iff*

$$4v_1v_2 - (d \sin\varphi + (v_1+v_2)\cos\varphi)^2 \neq 0.$$

For such a plane it is

$$K(X,Y) = \frac{\sin\varphi\left[(v_1+v_2)\sin\varphi - d\cos\varphi\right]}{4v_1v_2 - [d\sin\varphi + (v_1+v_2)\cos\varphi]^2}.$$

The scalar curvature S of C_6 is zero.

Proof. Let the 2-plane be determined by two screw motions given by vectors $X=(x;z)$, $Y=(y;t)$. We may suppose $(x,x)=(y,y)=1$. Then $\langle X,X\rangle = (x,z) = v_1$ is the pitch of X, the axis of X is given by the vector $X_1 = (x; z-v_1x)$ and similarly for Y. $\langle X,Y\rangle = (1/2)((x,t)+(y,z))$. For the distance d and the angle φ of the two axes we obtain

$\cos\varphi = (x,y)$, $d\sin\varphi = 2\langle X_1,Y_1\rangle = (x,t)+(y,z)-\cos\varphi\,(v_1+v_2)$.

This yields

$\langle X,Y\rangle = (1/2)[d\sin\varphi + (v_1+v_2)\cos\varphi]$,

$\langle R_{XY}X,Y\rangle = (1/4)\langle [X,Y],[X,Y]\rangle = (1/4)\sin\varphi\left[(v_1+v_2)\sin\varphi - d\cos\varphi\right].$

For the scalar curvature we use (7). Let e_i, $i=1,2,3$, be the canonical basis of R^3. Then we obtain an orthonormal basis of L given by vectors $X^i_\varepsilon = (e_i; \varepsilon e_i)$, where $\varepsilon = \pm 1$, as $\langle X^i_{\varepsilon_1}, X^j_{\varepsilon_2}\rangle = (1/2)(\varepsilon_1+\varepsilon_2)\delta_{ij}$.

Further we obtain

$K(X^i_\varepsilon, X^i_{-\varepsilon}) = 0$, $K(X^i_{\varepsilon_1}, X^j_{\varepsilon_2}) = (1/4)\varepsilon_1\varepsilon_2(\varepsilon_1+\varepsilon_2)$ for $i \neq j$. This yields

$$K(X^i_\varepsilon, X^j_{-\varepsilon}) = 0,\ K(X^i_\varepsilon, X^j_\varepsilon) = \varepsilon/2 \text{ and } S = 0.$$

3. Application to 6-parametric robot manipulators

Let us have a 6-parametric robot manipulator, determined by vectors $X_1,\dots,X_6$ from L. By definition it is the mapping $g: R^6 \to C_6$ given by the rule $g(u_1,\dots,u_6) = g_1(u_1).\ \dots\ .g_6(u_6)$, where we denote $g_i(u_i) = \exp(u_i X_i)$, see [3]. Let us suppose that vectors $X_1,\dots,X_6$ are linearly independent in L. Then the mapping g is a local system of coordinates around 0 in R^6 (see for instance [1] or [3]). In this local system of coordinates $u_1,\dots,u_6$ we have the following

LEMMA 3. *Let a 6-parametric robot manipulator be given by independent vectors $X_1,\dots,X_6$ from L. Let us denote* $Y_i = L_{g^{-1}}\left(\frac{\partial}{\partial u_i}\right)$. *Then*

$$Y_i = (\mathrm{Ad}g_6^{-1}.\ \dots\ .\mathrm{Ad}g_{i+1}^{-1})X_i .$$

The Christoffel symbols of the Cartan connection are given by

$$\nabla_{\frac{\partial}{\partial u_i}}\left(\frac{\partial}{\partial u_j}\right) = (1/2)\,\varepsilon_{ij} L_g [Y_i, Y_j]$$

or equivalently by

$$\Gamma^k_{ij} Y_k = (1/2)\,\varepsilon_{ij} [Y_i, Y_j] ,$$

where there is no summation over i *and* j *and*

$$\varepsilon_{ij} = \begin{cases} 1 & \text{for } i < j \\ 0 & \text{for } i = j. \\ -1 & \text{for } i > j \end{cases}$$

Proof. Vectors $X_1,\dots,X_6$ form a basis of L. The tangent vector $\frac{\partial}{\partial u_i}$ at the point $g(u_1,\dots,u_6)$ is given by the matrix

$$g_1(u_1).\ \dots\ .g_{i-1}(u_{i-1})g_i'(u_i)g_{i+1}(u_{i+1}).\ \dots\ .g_6(u_6).$$

Let us compute the coordinates of the vector $\frac{\partial}{\partial u_i}$ with respect to the basis $L_g X_i$ at the point $g = g(u_1,\dots,u_6)$.

Let us write

$\frac{\partial}{\partial u_i} = m_i^j L_g X_j$, where m_i^j are functions of $u_1,\dots,u_6$. The left translation to the unit element e of C_6 yields

$$L_{g^{-1}}\left(\frac{\partial}{\partial u_i}\right) = \mathrm{Ad}\, g_6^{-1}.\ \dots\ .\mathrm{Ad}\, g_{i+1}^{-1}\, X_i = Y_i = m_i^j\, X_j .$$

The derivative with respect to u_i yields

$$\frac{\partial Y_i}{\partial u_j} = 0 \text{ for } j \leq i, \quad \frac{\partial Y_i}{\partial u_j} = [Y_i, Y_j] \text{ for } j > i.$$

Further we obtain

$$\nabla_{\frac{\partial}{\partial u_i}} \left(\frac{\partial}{\partial u_j}\right) = \nabla_{m_i^\alpha L_g X_\alpha}(m_j^\beta L_g X_\beta) = m_i^\alpha \nabla_{L_g X_\alpha}(m_j^\beta L_g X_\beta) =$$

$$= m_i^\alpha m_j^\beta \nabla_{L_g X_\alpha}(L_g X_\beta) + m_i^\alpha (L_g X_\alpha)(m_j^\beta).L_g X_\beta =$$

$$(1/2) m_i^\alpha m_j^\beta [L_g X_\alpha, L_g X_\beta] + (L_g(m_i^\alpha X_\alpha))(m_j^\beta).L_g X_\beta = (1/2) L_g [Y_i, Y_j] + \frac{\partial m_j^\beta}{\partial u_i} L_g X_\beta =$$

$$L_g((1/2)[Y_i, Y_j] + \frac{\partial Y_j}{\partial u_i}) = (1/2)\, L_g\, \varepsilon_{ij} [Y_i, Y_j]$$ and this finishes the proof.

Let us denote $g_{ij} = B(Y_i, Y_j)$, $h_{ij} = \langle Y_i, Y_j \rangle$ .This means that h_{ij} are coefficients of the metric tensor of C_6 and g_{ij} are (up to a constant factor) coefficients of the Ricci tensor in coordinates $u_1, \ldots, u_6$,which are the joints coordinates of the robot manipulator.

The geometrical meaning of coefficients g_{ij} and h_{ij} is the following:Let an actual position of the robot manipulator be given by vectors $Y_1, \ldots, Y_6 \in L$,which correspond to rotational axes $p_1, \ldots, p_6$.Let φ_{ij} be the angle between p_i and p_j,d_{ij} be the distance of p_i and p_j.Then $g_{ij} = \cos \varphi_{ij}$, $h_{ij} = (1/2) d_{ij} \sin \varphi_{ij}$.

For Christoffel symbols $\Gamma_{ij,m}$ defined by (3) we have

$$\Gamma_{ij,m} = (1/2)\, \varepsilon_{ij} \langle [Y_i, Y_j], Y_m \rangle . \tag{13}$$

To see it,we multiply (12) by Y_m and we obtain

$$\Gamma^k_{ij} \langle Y_k, Y_m \rangle = \Gamma^k_{ij} h_{km} = \Gamma_{ij,m} = (1/2)\, \varepsilon_{ij} \langle [Y_i, Y_j], Y_m \rangle .$$

As the metric tensor h_{ij} is Ad C_6 invariant,we have

$$\langle [Y_i, Y_j], Y_m \rangle = \langle Y_i, [Y_j, Y_m] \rangle . \tag{14}$$

(14) yields

$$\Gamma_{ij,m} \varepsilon_{ij} = (1/2) \langle [Y_i, Y_j], Y_m \rangle = (1/2) \langle [Y_j, Y_m], Y_i \rangle = \varepsilon_{jm} \Gamma_{jm,i} .$$

The Christoffel symbols $\Gamma_{ij,m}$ therefore have the following properties:

a) $\Gamma_{ij,m} = \Gamma_{ji,m}$,

b) $\Gamma_{ij,m} = 0$ if two of the indices i,j,m are equal,

c) for $i<j<m$ we have $\Gamma_{ij,m} = -\Gamma_{im,j} = \Gamma_{mj,i}$ and the Koszul

formula (4) yields in this case

$$\Gamma_{ij,m} = (1/2)\left(\frac{\partial}{\partial u_j} h_{im}\right),$$

because h_{im} is a function of $u_{i+1},\dots,u_{m-1}$ only.

The equations of geodesic lines are

$$\frac{d^2u_i}{dt^2} + \Gamma^i_{jk}\frac{du_j}{dt}\cdot\frac{du_k}{dt} = 0 \tag{15}$$

The multiplication by h_{im} yields

$$\frac{d^2u_i}{dt^2} h_{im} + \Gamma_{jk,m}\frac{du_j}{dt}\cdot\frac{du_k}{dt} = 0. \tag{16}$$

LEMMA 4. Let $u_i = u_i(t)$ be a curve on a 6-parametric robot manipulator $g(u_1,\dots,u_6)$. Then this curve is a translate of a one-parametric subgroup iff

$$\sum_{i=1}^{6} Y_i\frac{d^2u_i}{dt^2} + \sum_{\substack{i,j=1\\ i<j}}^{6}[Y_i,Y_j]\frac{du_i}{dt}\cdot\frac{du_j}{dt} = 0 \tag{17}$$

Proof. Let $g(t) = g_1(u_1(t)).\ \dots\ .g_6(u_6(t))$ be a curve on the robot manipulator. Then $g(t)$ is a translate of a one-parametric subgroup iff $g^{-1}(t)g'(t) = Y = \text{const}$. Computation yields

$$Y = \sum_{i=1}^{6} Y_i u_i' = \text{const}. \tag{18}$$

The derivative of (18) yields (17).

Remark. If we multiply (17) by Y_k, we obtain immediately equations of geodesic lines (16). This is something we already know, as the geodesic lines are translates of one-parametric subgroups. (18) is a first integral of (17), the vector Y is given by initial conditions. The geodesic lines of the Levi-Civita connection are therefore in this case determined by 1. order differential equation -which is also to be expected.

(17) has two natural first integrals which are obtained by using the Killing and Klein forms, they give the angular velocity ω and the pitch v of the corresponding one-parametric subgroup by the formula

$$g_{ij}\frac{du_i}{dt}\cdot\frac{du_j}{dt} = \omega \quad , \quad h_{ij}\frac{du_i}{dt}\cdot\frac{du_j}{dt} = v\omega^2.$$

If t is the time, then the curve $g(t)$ determines an actual motion of the robot manipulator. Such a motion is a translation iff $B(\frac{d}{dt},\frac{d}{dt}) = 0$. This means that all translations are solutions of the equation $g_{ij}du_idu_j = 0$, or in an equivalent way, they are solutions of the system of

3 Pfaffian equations $DY_i du_i = 0$ (because for $X \in L$ we have $B(X,X) = 0$ iff $DX = 0$).

Similarly, the motion $g(t)$ of the robot manipulator is rotational (a rolling) iff it satisfies the equation $h_{ij} du_i du_j = 0$.

The singular set of the robot manipulator is given either by the equation $\det(h_{ij}) = 0$ or by the equation $\det(Y_1,\dots,Y_6) = 0$, because it is the set of all points in R^6, for which the axes of the robot manipulator are linearly dependent.

Further, for a 6-parametric robot manipulator we can define the volume integral, as it is defined on any pseudo-Riemannian manifold:

Let Ω be any (reasonably defined, for instance open or closed) set in R^6. Then we define

$$V(g,\Omega) = \int_\Omega |\det(h_{ij})|^{\frac{1}{2}}\, du_1 \dots du_6 .$$

$V(g,\Omega)$ is of course left and right invariant, which means that it does not depend on the choice of the matrix representation of the motion of the robot manipulator. $V(g,\Omega)$ is a measure for reaching ability of the robot manipulator if parameters change in Ω. If Ω is a maximal set in which g is a 1-1 mapping, we obtain the volume of the whole manipulator.

Remark. We have defined $g_{ij} = B(Y_i,Y_j)$, $h_{ij} = \langle Y_i,Y_j \rangle$. This expression is not very convenient for direct computation of g_{ij} and h_{ij}. As forms B and $\langle\ ,\ \rangle$ are Ad C_6 invariant, we have $B(Y_i,Y_j) = B(AdgY_i, AdgY_j)$ for any g from C_6 and similarly for $\langle\ ,\ \rangle$. Let us define

$\tilde{Y}_i = Ad(g_4 g_5 g_6)Y_i$. Then

$\tilde{Y}_1 = Adg_3^{-1} Adg_2^{-1} X_1, \tilde{Y}_2 = Adg_3^{-1} X_2, \tilde{Y}_3 = X_3, \tilde{Y}_4 = X_4, \tilde{Y}_5 = Adg_4 X_5,$

$\tilde{Y}_6 = Adg_4 Adg_5 X_6.$

Vectors $\tilde{Y}_i$ can be computed with reasonable effort and from them we obtain g_{ij} and h_{ij}.

4. Velocities and accelerations

Let $\hat{g}(t)$ be a space motion given by its matrix representation $g(t)$ by the formula $\hat{g}(\bar{R}) = R.g(t)$, where $\bar{R}$ is an orthonormal frame in the effector space $\bar{E}_3$, R is an orthonormal frame in the base space E_3 of the robot manipulator. Then for the trajectory $A(t)$ of a point $\bar{A} = \bar{R}\bar{X}$ from $\bar{E}_3$ we have $\hat{g}(\bar{A}) = g(\bar{R}\bar{X}) = Rg(t)\bar{X} = RX(t)$, where $\bar{X}$ are coordinates of $\bar{A}$

in $\bar{R}$ and $X(t)$ are coordinates of $A(t)$ in R,so $X(t) = g(t)\bar{X}$.For the velocity of $\bar{A}$ at the instant t at the place $X = X(t)$ we obtain

$$v_X = g'(t)\bar{X} = g'(t)g^{-1}(t)X.$$

So if Ω_1 is the velocity operator,we have $v_X = \Omega_1 X$ with $\Omega_1 = g'(t)g^{-1}(t) \in L$.

The velocities at the instant t are given as velocities of the instantaneous motion of the motion $g(t)$,which is in general a screw motion determined by the vector Ω_1 from L.The angular velocity v is given by the Killing form B,the pitch v_o is given by the Klein form:

$$v^2 = B(\Omega_1,\Omega_1),\ v_o v^2 = \langle \Omega_1,\Omega_1 \rangle .$$

The instantaneous axis is determined by Ω_1 itself.

The acceleration a_X of $\bar{A}$ at X is given by

$$a_X = v_X' = (\Omega_1 X)' = \Omega_1' X + \Omega_1 X' = (\Omega_1' + \Omega_1^2)X.$$

The operator Ω_2 of the acceleration is therefore given by the formula

$$\Omega_2 = \Omega_1' + \Omega_1^2.$$

Let now parameters u_i of the robot manipulator be given as functions of time t,$u_i = u_i(t)$.Then we obtain an actual motion of the robot manipulator.Let $du_i/dt = v^i$ be the angular or translational velocity of the i-th joint of the robot manipulator.Then

$$\Omega_1 = \sum_{i=1}^{6} Z_i v^i,\text{where } Z_i = Ad(g_1 \ldots g_{i-1})X_i.$$

We observe that $B(Z_i,Z_j) = B(Y_i,Y_j)$ and similarly for the Klein form, as they are Ad C_6 invariant.This means that for the instantaneous angular velocity of the motion of the effector of the robot manipulator we have $v^2 = g_{ij}v^i v^j$ and $v_o v^2 = h_{ij}v^i v^j$ for the instantaneous pitch.

For the angle ϑ_i of the instantaneous axis with the i-th axis of the robot manipulator we have $v \cos \vartheta_i = g_{ij}v^j$,for the distance D_i of the instantaneous axis from the i-th axis we have

$$v^2(D_i \sin \vartheta_i + v_o v \cos \vartheta_i) = 2h_{ij}v^j.$$

Moreover,the instantaneous motion is a translation iff $B(\Omega_1, \Omega_1) = 0$.

For the acceleration we have $\Omega_2 = \Omega_1' + \Omega_1^2$,where $\Omega_1' = Z_i' v^i + Z_i (v^i)'$.We splitt the acceleration operator into 3 parts:

$$\Theta_1 = Z_i' v^i, \quad \Theta_2 = Z_i a^i, \quad \Theta_3 = \Omega_1^2,$$

where $(v^i)' = a^i$ is the angular or translational acceleration of the i-th joint of the robot manipulator.

Θ_2 is an velocity operator for angular velocities equal to a^i, Θ_3 is the centrifugal acceleration,which is easy to understand,and so the only unknown part of the acceleration operator is $\Theta_1 = Z_i' v^i$.

Computation yields

$$\Theta_1 = (1/2)\, \varepsilon_{ij} [Z_i, Z_j] v^i v^j.$$

Let us write $\Theta_1 = Z_k m^k$.Then

$$[\Theta_1, Z_r] = \langle Z_k, Z_r \rangle m^k = \Gamma_{ij,k} v^i v^j.$$

This means that $m^k h_{kr} = \Gamma_{ij,k} v^i v^j$ and multiplication by the inverse matrix h^{kr} to h_{kr} leads to the formula $m^k = \Gamma^k_{ij} v^i v^j$.

This yields the expression

$$\Theta_1 = \Gamma^k_{ij} v^i v^j Z_k, \tag{19}$$

which connects the acceleration properties of the motion of the robot manipulator to the properties of the Cartan connection of the symmetric pseudo-Riemannian space C_6.

As $R_{ijk,m} = (1/4) \langle [Y_i, Y_j], [Y_k, Y_m] \rangle$,we have also

$$\langle \Theta_1, \Theta_1 \rangle = \varepsilon_{ij}\, \varepsilon_{km}\, R_{ijk,m}\, v^i v^j v^k v^m. \tag{20}$$

5. An example

To give an example for illustration,we shall choose a very simple type of a robot manipulator to compute components g_{ij},h_{ij},the Christoffel symbols and to show some other applications (without having any practical realization on mind).

Let us suppose that all the links of the robot manipulator are rotational (this follows $h_{ii} = 0$);the parameter u_i is the angle of rotation (which follows $g_{ii} = 1$).For the sake of convenience we shall denote by the same letter the axis of the robot manipulator and its corresponding vector from L.

As the starting position of the robot manipulator we take the position with all axes parallel to the (xz) plane. Let these axes be $X_1, \ldots, X_6$. By an axis of two straight lines X and Y ($X \neq Y$) we mean the straight line, which intersects both of them under right angles. If X and Y are not parallel, then the axis is determined by the vector $[X,Y] \in L$, in the case of parallel lines we take any such line.

Let us denote by α_i and d_i the oriented angle and distance of lines X_i and X_{i+1}, respectively. Δ_j be the oriented distance between axes of lines X_{j-1}, X_j and X_j, X_{j+1}. Then the constants $\alpha_i, d_i, i=1,\ldots,5$ and $\Delta_j, j=2,\ldots,5$, determine the robot manipulator uniquely (after some simple agreements).

As our example we choose the robot manipulator, for which $\alpha_i = \pi/2$, $d_i = 0$ for $i=1,\ldots,5$ and $\Delta_2 = \Delta_4 = 0$, $\Delta_3 \neq 0$, $\Delta_5 \neq 0$. The orthonormal frame in E_3 is chosen in such a way that X_1 is the x-axis.

For the motion of the robot manipulator we shall write

$g(u_1,\ldots,u_6) = g_1(u_1). \ \ldots \ .g_6(u_6)$, where $g_i(u_i) = \begin{pmatrix} 1 & , 0 \\ t_i(u_i), & \gamma_i(u_i) \end{pmatrix}$

is the matrix of the rotation around the axis X_i.

We obtain $\gamma_1 = \begin{pmatrix} 1, & 0, & 0 \\ 0, c_1 & , -s_1 \\ 0, s_1 & , c_1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} c_2, -s_2, 0 \\ s_2, \ c_2, 0 \\ 0 \ , \ 0 \ , 1 \end{pmatrix}$

and similarly for the others; we write simply c_i for $\cos u_i$ and s_i for $\sin u_i$. We have also

$$t_4 = \begin{pmatrix} (1-c_4)\Delta_3 \\ -s_4\Delta_3 \\ 0 \end{pmatrix}, \quad t_6 = \begin{pmatrix} (1-c_6)(\Delta_5-\Delta_3) \\ -s_6(\Delta_5-\Delta_3) \\ 0 \end{pmatrix},$$

remaining translations are equal to zero.

For $\widetilde{Y}_i$ we obtain

$$\widetilde{Y}_1 = \begin{pmatrix} c_2 & ; 0 \\ -s_2c_3 & ; 0 \\ -s_2s_3 & ; 0 \end{pmatrix}, \ \widetilde{Y}_2 = \begin{pmatrix} 0 & ; 0 \\ -s_3 & ; 0 \\ c_3 & ; 0 \end{pmatrix}, \ \widetilde{Y}_3 = \begin{pmatrix} -1 & ; 0 \\ 0 & ; 0 \\ 0 & ; 0 \end{pmatrix},$$

$$\widetilde{Y}_4 = \begin{pmatrix} 0 & ; 0 \\ 0 & ; -\Delta_3 \\ -1 & ; 0 \end{pmatrix}, \ \widetilde{Y}_5 = \begin{pmatrix} c_4 & ; 0 \\ -s_4 & ; 0 \\ 0 & ; \Delta_3 s_4 \end{pmatrix}, \ \widetilde{Y}_6 = \begin{pmatrix} -s_4s_5 & ; -\Delta_5s_4c_5 \\ -c_4s_5 & ; \Delta_3c_5-\Delta_5c_4c_5 \\ c_5 & ; \Delta_3c_4s_5-\Delta_5s_5 \end{pmatrix}$$

$$g_{ij}=\begin{pmatrix} 1 & 0 & -c_2, & s_2s_3, & c_2c_4+s_2s_4c_3, & g_{16} \\ 0 & 1 & 0, & -c_3, & s_3s_4 & c_3c_5+s_3c_4s_5 \\ -c_2 & 0 & 1, & 0, & -c_4 & s_4s_5 \\ s_2s_3 & -c_3 & 0, & 1, & 0 & -c_5 \\ c_2c_4+s_2c_3s_4, & s_3s_4 & -c_4, & 0, & 1 & 0 \\ g_{16} & c_3c_5+s_3c_4s_5, & s_4s_5, & -c_5, & 0 & 1 \end{pmatrix}$$

where $g_{16} = s_2c_3c_4s_5 - s_2s_3c_5 - c_2s_4s_5$.

$$h_{ij}=(1/2)\begin{pmatrix} 0 & 0 & 0 & s_2c_3\Delta_3, & -s_2s_3s_4\Delta_3, & h_{16} \\ 0 & 0 & 0 & s_3\Delta_3, & c_3s_4\Delta_3, & h_{26} \\ 0 & 0 & 0 & 0 & 0 & s_4c_5\Delta_5 \\ s_2c_3\Delta_3 & s_3\Delta_3, & 0 & 0 & 0 & s_5\Delta_5 \\ -s_2s_3s_4\Delta_3, & c_3s_4\Delta_3, & 0 & 0 & 0 & 0 \\ h_{16} & h_{26} & s_4c_5\Delta_5, & s_5\Delta_5 & 0 & 0 \end{pmatrix}$$

where

$h_{16} = \Delta_5(s_2s_3s_5-c_2s_4c_5+s_2c_3c_4c_5) - \Delta_3(s_2c_3c_5+s_2s_3c_4s_5)$

$h_{26} = \Delta_5(s_3c_4c_5-c_3s_5) + \Delta_3(c_3c_4s_5-s_3c_5)$.

To compute the Christoffel symbols $\Gamma_{ij'k}$ it is enough to consider the case $i<j<k$, because the symbol is zero if two indices are equal and we have $\Gamma_{ij'k} = -\Gamma_{ik'j} = \Gamma_{jk'i}$. In this case we have $\Gamma_{ij'k} = (1/2)(\partial h_{ik}/\partial u_j)$. For instance we obtain

$$4\Gamma_{12'4} = c_2c_3\Delta_3,\quad 4\Gamma_{12'5} = -c_2s_3s_4\Delta_3.$$

Further, $\det(h_{ij}) = (1/64)(s_4^2s_2c_5\Delta_3^2\Delta_5)^2$.

For the volume function of this robot manipulator we have

$$V(\Omega) = (1/8)\Delta_3^2\Delta_5\int_{\Omega} s_4^2c_5s_2\, d\Omega,$$

where $\Omega\subset R^6$ is the domain of parameters u_i. Ω must be chosen in such a way that the map $g:\Omega \rightarrow C_6$ is one to one on an open and dense set. To compute the whole volume V of the robot manipulator we have to find a maximal domain Ω on which g is 1-1 mapping. To find such a domain we have to solve the "inverse" problem for the motion of the robot manipulator - for given position of the effector we have to find corresponding angles of rotation u_i, $i=1,\dots,6$ and look for domains, where the solution is unique.

We would like to solve the equation $g(u_1,\dots,u_6)=g(v_1,\dots,v_6)$ for given u_i and unknown v_i. Let us denote $g_i(v_i) = \bar{g}_i$ and similarly for γ_i and t_i. As $\bar{g}_6 = g_6(v_6)$ and $g_6(u_6)g_6^{-1}(v_6) = g_6(u_6-v_6)$, because g_i is a one-parametric subgroup, we may suppose $\bar{g}_6 = e$ and substitute $u_6-v_6 = w$ instead of u_6. Let us denote $k = -\Delta_5\Delta_3^{-1}$. Then we obtain the following equations:

$$\begin{pmatrix} c_6, s_6, 0 \\ -s_6, c_6, 0 \\ 0, 0, 1 \end{pmatrix} \begin{pmatrix} (c_4-1)+(1+k)(1-c_6) \\ s_4c_5 - (1+k)s_6 \\ -s_4s_5 \end{pmatrix} = \begin{pmatrix} \bar{c}_4 - 1 \\ \bar{s}_4\bar{c}_5 \\ -\bar{s}_4\bar{s}_5 \end{pmatrix}$$

The solutions are:

a) $\cos w = 1, \sin w = 0,\ \bar{c}_4 = c_4,\ \bar{s}_4 = s_4,\ \bar{c}_5 = c_5,\ \bar{s}_5 = s_5,$

b) $\cos w = \dfrac{(c_4+k)^2 - s_4^2c_5^2}{(c_4+k)^2+s_4^2c_5^2}, \quad \sin w = \dfrac{2(c_4+k)s_4c_5}{(c_4+k)^2+s_4^2c_5^2},$

$\bar{c}_4 = c_4,\ \bar{s}_4 = s_4,\ \bar{c}_5 = -c_5,\ \bar{s}_5 = s_5.$

This shows that we can choose

$$u_4 \in (0,\pi),\ u_5 \in (0,2\pi),\ u_6 \in (0,w).$$

If u_4, u_5, u_6 are fixed, then $u_1 \in (0,2\pi)$, $u_2 \in (0,2\pi)$, $u_3 \in (0,\pi)$, as they are Euler angles on $SO(3)$.

For the volume V of the robot manipulator we obtain

$$V = (16/3)\pi^4|\Delta_3|^3 \quad \text{for} \quad |k| \geq 1 \quad \text{and}$$
$$V = (8/3)\pi^4(3-k^2)|\Delta_3|^3 \quad \text{for} \quad |k| \leq 1.$$

The singular set of the robot manipulator is given by the equation $s_2s_4c_5 = 0$ with obvious solutions.

Translations (motions of the effector with zero angular velocity) are given by equations

$$c_2du_1 + du_3 - c_4du_5 + s_4s_5du_6 = 0,$$
$$s_2s_3du_1 - c_3du_2 + du_4 - c_5du_6 = 0,$$
$$s_2c_3du_1 - c_3du_4 + s_3s_4du_5 + c_4s_5du_6 = 0,$$

which have the following 3 independent integrals:

$$(c_2s_4 - s_2c_3c_4)s_5 + s_2s_3c_5 = K_1,$$
$$(c_2c_4+s_2c_3s_4)c_6 + (c_2s_4-s_2c_3c_4)c_5s_6 - s_2s_3s_5s_6 = K_2,$$
$$(s_1s_2s_4+s_1c_2c_3c_4-c_1s_3c_4)s_5 - (s_1c_2s_3+c_1c_3)c_5 = K_3,$$

where K_i are integration constants.

Now we shall write down the system of differential equations for the screw-motion of the end effector around a chosen axis Z given by $Z = (p_1,p_2,p_3;p_4,p_5,p_6) \in L$. Let us denote by $a_i = du_i/dt$ the angular velocity of the i-th link. Then we obtain

$$c_2a_1-a_3+c_4a_5-s_4s_5b_6 = c_4y+s_4s_5z,$$
$$-s_2c_3a_1-s_3a_2-s_4a_5-c_4s_5b_6 = -s_4y+c_4c_5z,$$
$$-s_2s_3a_1+c_3a_2-a_4+c_5b_6 = s_5z,$$
$$-\Delta_5s_4c_5b_6 = c_4m+s_4c_5n+s_4s_5(q_6-\Delta_5z),$$
$$-\Delta_3a_4-c_5(\Delta_5c_4-\Delta_3)b_6 = -s_4m+c_4c_5n+c_4s_5(q_6-\Delta_5z)+\Delta_3s_5z,$$
$$\Delta_3s_4a_5-s_5(\Delta_5-\Delta_3c_4)b_6 = s_5n-c_5(q_6-\Delta_5z)-\Delta_3s_4y-c_4c_5\Delta_3z,$$

where

$$b_6 = a_6-p_3,\ y = p_1c_6-p_2s_6,\ z = p_1s_6+p_2c_6,\ m = p_4c_6-q_3s_6,\ n = p_4s_6+q_3c_6,$$
$$q_6 = p_2(\Delta_5-\Delta_3)-p_6,\ q_3 = p_3(\Delta_5-\Delta_3)+p_5.$$

As an application we shall consider a simpler case, the case for which $p_3 = q_3 = p_4 = q_6 = 0$. Then $a_6 = s_5z/c_5$, $a_4=0$, so $u_4=$ const., $a_5 = -\mu z/c_5 + y$, where $\mu=(k+c_4)/s_4$, $\cos u_5 = (\nu - \mu y)/z$, where $\nu =$ const., k=const.

Further we obtain

$c_3 = (\delta + \omega c_2)/s_2$, δ, ω are constant, $\omega = (1+c_4k)/s_4k$, which yields

$$(s_2^2-(\delta+\omega c_2)^2)^{-\frac{1}{2}}.s_2du_2 = (c_4-\mu s_4)(\nu - \mu y)^{-1}z^2du_6,$$
$$du_1 = (c_4-\mu s_4)(\delta + \omega c_2)s_2^{-2}.z^2(\nu - \mu y)du_6.$$

This shows that u_1 and u_2 can be computed by integration, finally

$$du_6 = z(\nu - \mu y)^{-1}(z^2-(\nu - \mu y)^2)\ dt.$$

As an another application we shall write the equations for plane motions of the effector in a plane perpendicular to the axis X_6. In this case we have $p_1=p_2=p_6= 0$, so $y=z=q_6= 0$. We obtain the following equations

$$a_5s_4c_5 + a_4c_4s_5 = 0,\ a_1s_2(c_3c_5+s_3c_4s_5) - a_2(c_3c_4s_5-s_3c_5) = 0,$$
$$a_4s_5 = a_1(c_2s_4c_5-s_2c_3c_4c_5-s_2s_3s_5) + a_2(c_3s_5-s_3c_4c_5) - a_3s_4c_5.$$

To express components Γ^i_{jk} we need the matrix h^{ij} inverse to h_{ij}. Let $\delta = (\det h_{ij})^{\frac{1}{2}} = s_4^2c_5s_2\Delta_3^2\Delta_5$. Then $h^{ij} = 2\Delta_3/\delta$.

$$\begin{pmatrix} 0 & ,0 & ,-c_3s_4s_5\Delta_5 & ,c_3s_4^2c_5\Delta_5 & ,-s_3s_4c_5\Delta_5 & ,0 \\ 0 & ,0 & ,-s_2s_3s_4s_5\Delta_5 & ,s_2s_3s_4^2c_5\Delta_5 & ,s_2s_4c_3c_5\Delta_5 & ,0 \\ -s_5c_3s_4\Delta_5 & ,-s_2s_3s_4s_5\Delta_5 & ,-2Ms_5 & ,Ms_4c_5 & ,N & ,s_2s_4\Delta_3 \\ c_3s_4^2c_5\Delta_5 & ,s_2s_3s_4^2c_5\Delta_5 & ,Ms_4c_5 & ,0 & ,0 & ,0 \\ -s_3s_4c_5\Delta_5 & ,s_2s_4c_3c_5\Delta_5 & ,N & ,0 & ,0 & ,0 \\ 0 & ,0 & ,s_2s_4\Delta_3 & ,0 & ,0 & ,0 \end{pmatrix}$$

where

$M = \Delta_5(c_2c_3s_4 - s_2c_4) + \Delta_3 s_2$, $N = \Delta_5(s_2s_5 - c_2s_3s_4c_5) - \Delta_3 s_2c_4s_5$.

By matrix multiplication we now obtain the components Γ^i_{jk} of the Levi-Civita connection and we are able to express the components of the relative acceleration Θ_2. For instance the contribution Θ_2^{12} of Θ_2 from the first two links of the robot manipulator $(u_3,\dots,u_6$ are constant) is:

$$\Theta_2^{12} = v_1v_2(c_2Y_1+Y_3)/2s_2,$$

where v_1 and v_2 are angular velocities around X_1 and X_2, respectively.

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