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Classification of Topological Types of Isolated Quasi-homogeneous Two Dimensional Hypersurface Singularities

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§ 1 Introduction

In 1982, Mather and the second author [9] proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are biholomorphically equivalent if and only if their moduli algebras are isomorphic . It is a natural question to ask for a necessary and sufficient condition for two germs of complex analytic hypersurfaces with isolated singularities (V_1, p_1) and (V_2, p_2) of the same dimension to have the same topological type. We say that (V_1, p_1) and (V_2, p_2) in $\mathbf{C}^{n+1}$ have the same topological type if $(\mathbf{C}^{n+1}, V_1, p_1)$ is homeomorphic to $(\mathbf{C}^{n+1}, V_2, p_2)$. Recall that a hypersurface singularity $(V, 0) = \{(z_0, ..., z_n) : f(z_0, ..., z_n) = 0\} \subseteq \mathbf{C}^{n+1}$ is quasi-homogeneous if f is in the Jacobian ideal of f i.e. $f \in (\frac{\partial f}{\partial z_0}, ..., \frac{\partial f}{\partial z_n})$. By a theorem of Saito, after a biholomophic change of coordinates, f becomes a weighted homogenous polynomial.

A polynomial $h(z_0, ..., z_n)$ is weighted homogeneous of type $(w_0, ..., w_n)$, where $(w_0, ..., w_n)$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} ... z_n^{i_n}$ for which $\frac{i_0}{w_0} + ... + \frac{i_n}{w_n} = 1$. $(w_0, ...w_n)$ is called the weights of h.

Orlik and Wagreich [13] and Arnold [1] showed that if $h(z_0, z_1, z_2)$ is a weighted homogeneous polynomial in $\mathbf{C}^3$ and $V = \{h(z) = o\}$ has an isolated singularity at the origin, then V can be deformed into one of the following seven classes below while keeping the link $K_V = S_\epsilon^{2n+1} \cap V$ differentially constant.

ClassI $V(a_0, a_1, a_2; 1) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$

ClassII $V(a_0, a_1, a_2; 2) = \{z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}, \quad a_1 > 1$

ClassIII $V(a_0, a_1, a_2; 3) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} = 0\}, \quad a_1 > 1, a_2 > 1$

ClassIV $V(a_0, a_1, a_2; 4) = \{z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\} \quad a_0 > 1$

ClassV $V(a_0, a_1, a_2; 5) = \{z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2} = 0\}$

ClassVI $V(a_0, a_1, a_2; 6) = \{z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} = 0\} \quad (a_0 - 1)(a_1 b_2 + a_2 b_1) =$

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$a_0a_1a_2$

$Class VII \qquad V(a_0,a_1,a_2;7)=\{z_0^{a_0}z_1+z_0z_1^{a_1}+z_0z_2^{a_2}+z_1^{b_1}z_2^{b_2}=0\}$

$(a_0-1)(a_1b_2+a_2b_1)=a_2(a_0a_1-1)$

In §3 we prove that the above deformation is actually a topological trivial deformation. Therefore any weighted homogeneous singularity has the same topological type of one of the seven classes above. Let $\Delta_V(z)$ denote the characteristic polynomial of the Milnor fibration of $(V,0)$.

Theorem If $(V,0)$ and $(W,0)$ are among the seven classes above, then $(\mathbf{C}^3,V,0)$ is biholomorphic to $(\mathbf{C}^3,W,0)$ if and only if $(\mathbf{C}^3,V,0)$ is homeomorphic to $(\mathbf{C}^3,W,0)$ With the possible exceptions being listed in Table 3. $(\mathbf{C}^3,V,0)$ is homeomorphic to $(\mathbf{C}^3,W,0)$ if and only if $\pi_1(K_V)=\pi_1(K_W)$ and $\Delta_V(z)=\Delta_W(z)$

Corollary 1 Let $(V,0)$ and $(W,0)$ be two isolated quasi-homogeneous surface singularities in $\mathbf{C}^3$. Then $(\mathbf{C}^3,V,0)$ is homeomorphic to $(\mathbf{C}^3,W,0)$ if and only if $\pi_1(K_V)\cong\pi_1(K_W)$ and $\Delta_V(z)=\Delta_W(z)$.

Notice that in general for any isolated hypersurface singularitie $(V,0)$ and $(W,0)$ of the same dimension and same topological type, we have $\pi_1(K_V)\cong\pi_1(K_W)$ and $\Delta_V(z)=\Delta_W(z)$, (See Milnor[9], Le[2], Perron [14] and Yau [19]). In fact in [19], the second author made the following conjecture:

Conjecture: Corollary 1 remains true even if the quasi- homogeneous condition is removed.

Corollary 2 Let $(V,0)$ be an isolated singularity defined by a weighted homogeneous polynomial in $\mathbf{C}^3$ with weights (w_0,w_1,w_2). Then the topological type of $(V,0)$ determines and is determined by its weights (w_0,w_1,w_2).

Corollary 3 Let $(V,0)$ be an isolated singularity defined by a weighted homogeneous polynomial in $\mathbf{C}^3$ with weights (w_0,w_1,w_2). Then the fundamental group of the link $\pi_1(K_V)$ and the characteristic polynomial $\Delta_V(z)$ determines and are determined by the weights (w_0,w_1,w_2).

This follows immediately from Corollary 1 and Corollary 2. The original motivation of our paper is to prove the Zariski conjecture (cf. [21]) for isolated quasi-homogeneous surface singularities in $\mathbf{C}^3$: multiplicity is an invariant of topological type. As a corollary to the proof of our main

theorem, we have

Corollary 4 Let $(V,0)$ and $(W,0)$ be two isolated quasi-homogeneous surface singularities in $\mathbf{C}^3$. If $(\mathbf{C}^3,V,0)$ is homeomorphic to $(\mathbf{C}^3,W,0)$, then V and W have the same multiplicity at origin.

We would like to mention that in the above Corollary 4 if there is a Milnor number constant family connecting $(V,0)$ and $(W,0)$,the Zariski conjecture is proved independently by Greuel [4] and O'Shea [13]. (See also Laufer [6])

In §2 we recall the definitions of zeta function and characteristic polynomial of isolated hypersurface singularity. Varchenko's result is used to compute the zeta function of the above seven classes weighted homogeneous singularities explicitly. We also recall the result of Orlik and Wagreich [12] on description of resolution in terms of weights for weighted homogeneous singularities.

The above results were annouced in [20]. After this paper was finished , we received a preprint [15] from Professor Osamu Saeki, where Corollary 2 and Corollary 4 are stated and proved independently by him. However in the proof of his main result, Theorem 3, of [15], he quoted that by Orlik and Wagreich [12], (S^5,K_f) is the same knot type as the algebrac knot associated with one of the seven classes of polynomials. Actually, if one reads the paper of Orlik and Wagreich, one will see that they only proved that K_f as an abstract manifold is diffeomorphic to the link of the seven classes of polynomials. Even if one applies a result of M. Oka (Deformation of Milnor Fiberings, J. Fac. Sci. Univ. Tokyo 29, 1973, 397-400), one can only conclude that S^5-K_f is homeomorphic to the complement in S^5 of one of the link of the seven classes of polynomials. On the other hand it is well known that at least for non-algebraic knot (S^5,K) the complement S^5-K does not determine the knot type (S^5,K). Therefore he has to make use of our results in section 3 that (S^5,K_f) as a pair is diffeomorphic to the algebraic knot associated with one of the seven classes of polynomials, otherwise his proofs appear to be incomplete.

In any event, our main theorem is a precise result which is not contained in Saeki's paper. Saeki's proof cannot be generalized beyond weighted homogeneous case because he uses Orlik's result on large Seifert 3-manifolds as well as Yoshinaga's result [20] on characterristic polynomial

for weighted homogeneous polynomials, whereas, the proof of our paper may eventualy work for any nondegenerate hypersurface singularities because we use the deep theorem of Neumann [11] which essentially allows us to assume that resolutions of two isolated 2-dimensional hypersurface singularities are topologically the same once they have isomorphic fundamental groups. We also use the deep theorem of Varchenko [17] which allows us to compute resolution and characteristic polynomial of monodromy from Newton's polygon for any nondegenerate hypersurface singularities.

Notation and convention: In the rest of this paper we shall assume that the weights of the weighted homogeneous polynomial are greater than or equal to two without loss of generality by [16](Notice that the weights we adopt here are reciprocal of those used by Saito). If a_1 and a_2 are integers and a_1 divides a_2, we shall write a_1/a_2.

§2 Zeta Function And Resolution For Weighted Homogeneous Singularity

We shall first recall a deep result due to Varchenko [17]. Let $f : (\mathbf{C}^{n+1}, 0) \mapsto (\mathbf{C}, 0)$ be an analytic function. Let ϵ, δ be positive numbers $0 < \delta << \epsilon << 1$. We define

$$T = \{t \in \mathbf{C} : \mid t \mid < \delta\}, \qquad B = \{z \in \mathbf{C}^{n+1} : \mid z_0 \mid^2 + \ldots + \mid z_{n+1} \mid^2 < \epsilon\},$$

$$X = B \bigcap f^{-1}(T), \qquad X(t) = B \bigcap f^{-1}(t)$$

Milnor proved that $f : X \setminus X(0) \mapsto T \setminus \{0\}$ is a locally trivial smooth fibre bundle. If f has an isolated singularity at the origin, the fibre $X(t)$ of this fibre bundle has the homotopy type of a bouquet of n-spheres. The generator of $\pi_1(T \setminus \{0\})$ (represented by a counter-clockwise oriented circle around the origin),induces the monodromy automorpism $h : H^*(X(t), \mathbf{C}) \mapsto H^*(X(t), \mathbf{C})$.

Definition: The zeta-function of the monodromy at the origin is the function

$$\zeta_f(z) = \Pi_{q \geq 0}\{det[Id - zh; H^q(X(t), \mathbf{C})]\}^{(-1)^q} \tag{2.1}$$

If f has an isolated singularity at the origin, then $H^q(X(t), \mathbf{C}) = 0$ for $q \neq 0, n$. In this case the characteristic polynomial $\Delta_f(z)$ of the monodromy $h : H^n(X(t), \mathbf{C}) \mapsto H^n(X(t), \mathbf{C})$ is expressed by means of $\zeta_f(z)$ according to the formula

$$\Delta_f(z) = z^{\mu}\left[\left(\frac{z}{z-1}\right)\zeta_f\left(\frac{1}{z}\right)\right]^{(-1)^n} \tag{2.2}$$

where $\mu = dim\ H^n(X(0), \mathbf{C})$ is the Milnor number.

In [17], Varchenko defined zeta-function ζ_Γ and characteristic polynomial $\Delta_\Gamma(z)$ of the Newton diagram Γ of f.

Theorem 2.1(Varchenko) Let f belong to the square of the maximal ideal of $\mathbf{C}\{x_0, ..., x_n\}$ and let the main part of the power series f be nondegenerate. Then the characteristic polynomial of the monodromy of f at the origin is equal to the characteristic polynomial $\Delta_\Gamma(z)$ of the Newton diagram Γ of f.

Let a, b, and c be integers. We shall denote (a, b, c) to be the greatest common divisor of a, b, and c.

Proposition 2.2 The following are the characteristic polynomials of the seven classes listed above. (I) Class I :

$$\Delta_I(z) = \frac{z}{z-1}(z^{a_0}-1)(z^{a_1}-1)(z^{a_2}-1)(z^{\frac{a_0a_1}{(a_0,\ a_1)}}-1)^{-(a_0,\ a_1)}$$
$$(z^{\frac{a_1a_2}{(a_1,\ a_2)}}-1)^{-(a_1,\ a_2)}(z^{\frac{a_0a_2}{(a_0,\ a_2)}}-1)^{-(a_0,\ a_2)}(z^{\frac{a_0a_1a_2}{(a_0a_1,\ a_1a_2,\ a_0a_2)}}-1)^{(a_0a_1,\ a_1a_2,\ a_0a_2)}$$

(II) Class II:

$$\Delta_{II}(z) = \frac{z}{z-1}(z^{a_0}-1)(z^{a_1}-1)(z^{\frac{a_0a_1}{(a_0,\ a_1)}}-1)^{-(a_0,\ a_1)}$$
$$(z^{\frac{a_1a_2}{(a_1-1,\ a_2)}}-1)^{-(a_1-1,\ a_2)}(z^{\frac{a_0a_1a_2}{(a_0a_2,\ a_1a_2,\ a_0(a_1-1))}}-1)^{(a_0a_2,\ a_1a_2,\ a_0(a_1-1))}$$

(III) Class III:

$$\Delta_{III}(z) = \frac{z}{z-1}(z^{a_0}-1)(z^{\frac{a_1a_2-1}{(a_1-1,\ a_2-1)}}-1)^{-(a_1-1,\ a_2-1)}$$
$$(z^{\frac{a_0(a_1a_2-1)}{(a_1a_2-1,\ a_0(a_2-1),\ a_0(a_1-1))}}-1)^{(a_1a_2-1,\ a_0(a_2-1),\ a_0(a_1-1))}$$

(IV) Class IV:

$$\Delta_{IV}(z) = \frac{z}{z-1}(z^{a_0}-1)(z^{\frac{a_0a_1}{(a_1,\ a_0-1)}}-1)^{-(a_1,\ a_0-1)}$$
$$(z^{\frac{a_0a_1a_2}{(a_1a_2,\ a_0a_1-a_0+1,\ a_2(a_0-1))}}-1)^{(a_1a_2,\ a_0a_1-a_0+1,\ a_2(a_0-1))}$$

(V) Class V:

$$\Delta_V(z) = \frac{z}{z-1}(z^{\frac{a_0a_1a_2+1}{(a_1a_2-a_2+1,\ a_0a_2-a_0+1,\ a_0a_1-a_1+1)}}-1)^{(a_1a_2-a_2+1,\ a_0a_2-a_0+1,\ a_0a_1-a_1+1)}$$

(VI) Class VI:

$$\Delta_{VI}(z) = \frac{z}{z-1}(z^{a_0}-1)(z^{\frac{a_0a_1}{(a_1,\ a_0-1)}}-1)^{-(a_1,\ a_0-1)}(z^{\frac{a_0a_2}{(a_2,\ a_0-1)}}-1)^{-(a_2,\ a_0-1)}$$

$$(z^{\frac{a_0a_1a_2}{(a_1a_2,\ a_2(a_0-1),\ a_1(a_0-1))}}-1)^{(a_1a_2,\ a_2(a_0-1),\ a_1(a_0-1))\frac{a_0}{(a_0-1)}}$$

(VII) Class VII:

$$\Delta_{VII}(z) = \frac{z}{z-1}(z^{\frac{a_0a_1-1}{(a_1-1,\ a_0-1)}}-1)^{-(a_1-1,\ a_0-1)}$$

$$(z^{\frac{a_2(a_0a_1-1)}{(a_2(a_1-1),\ a_2(a_0-1),\ a_1(a_0-1))}}-1)^{(a_2(a_1-1),\ a_2(a_0-1),\ a_1(a_0-1))\frac{a_0}{a_0-1}}$$

proof: Easy exercise.

Let f be a weighted homogeneous polynomial in $\mathbf{C}[z_0, z_1, z_2]$. Let (w_0, w_1, w_2) be weights of f. Let $w_i = u_i/v_i$ be the reduced fraction of w_i i.e. u_i and v_i are integers with $(u_i, v_i) = 1$. Define $d =< u_0, u_1, u_2 >$ the least common multiple of u_0, u_1 and u_2; $c = (u_0, u_1, u_2)$ the greatest common divisor of u_0, u_1, u_2; $q_i = d/w_i$; $c_0 = (u_1, u_2)/c$; $c_1 = (u_0, u_2)/c$; $c_2 = (u_0, u_1)/c$.

Finally, we define γ_0, γ_1 and γ_2 by $u_0 = cc_1c_2\gamma_0$, $u_1 = cc_0c_2\gamma_1$, $u_2 = cc_0c_1\gamma_2$.

The link $K_f = f^{-1}(0) \bigcap S^5$, where S^5 is a sphere with center at origin, is a Seifert fibered 3-manifold. Orlik and Wagreich [12] have calculated the Seifert invariants of K_f,

$$\{-b; g; n_0(\alpha_0, \beta_0), n_1(\alpha_1, \beta_1), n_2(\alpha_2, \beta_2), n_3(\alpha_3, \beta_3)\},$$

which are given as follows.

Table 1

	α_0	n_0	α_1	n_1	α_2	n_2	α_3	n_3
I	γ_0	cc_0	γ_1	cc_1	γ_2	cc_2		0
II	γ_0	$(cc_0-1)/v_2$	$v_2\gamma_0$	1	γ_2	c		0
III	γ_0	$(cc_0-v_1-v_2)/v_1v_2$	$v_2\gamma_0$	1	$v_1\gamma_0$	1		0
IV	γ_2	$(c-1)/v_1$	v_2	1	$v_1\gamma_2$	1		0
V	v_0	1	v_1	1	v_2	1		0
VI	γ_1	$(c-1)/v_1$	γ_2	$(c-1)/v_2$	$v_1\gamma_2$	1	$v_2\gamma_1$	1
VII	γ_2	$(c-v_0-v_1)/v_0v_1$	v_2	1	$v_1\gamma_2$	1	$v_0\gamma_2$	1

Table 2

	w_0	w_1	w_2
I	a_0	a_1	a_2
II	a_0	a_1	$\frac{a_1a_2}{a_1-1}$
III	a_0	$\frac{a_1a_2-1}{a_2-1}$	$\frac{a_1a_2-1}{a_1-1}$
IV	a_0	$\frac{a_0a_1}{a_0-1}$	$\frac{a_0a_1a_2}{a_0a_1-a_0+1}$
V	$\frac{a_0a_1a_2+1}{a_1a_2-a_2+1}$	$\frac{a_0a_1a_2+1}{a_0a_2-a_0+1}$	$\frac{a_0a_1a_2+1}{a_0a_1-a_1+1}$
VI	a_0	$\frac{a_0a_1}{a_0-1}$	$\frac{a_0a_2}{a_0-1}$
VII	$\frac{a_0a_1-1}{a_1-1}$	$\frac{a_0a_1-1}{a_0-1}$	$\frac{a_2(a_0a_1-1)}{a_1(a_0-1)}$

By the theorem of Saito [16], we may assume from now on that $w_i \geq 2$ for $i = 0, ..., n$. We have ,in particular, $u_i \geq 2$ for $i = 0, ..., n$.

§ 3 Reduction of Weighted Homogenous Polynomial With Isolated Critical Point to The One of The Seven Classes

The following proposition was due to Orlik and Wagreich [12] and corrected by Arnold [1].

Proposition 3.1 Suppose $h(z_0, z_1, z_2)$ is a polynomial and the locus V of h has an isolated singularity. Then $h(z_0, z_1, z_2) = f(z_0, z_1, z_2) + g(z_0, z_1, z_2)$ where f is one of the seven classes above and f and g have no monomial in common. If h is weighted homogenous of type (w_0, w_1, w_2), then so are f and g.

Definition 3.2 Suppose given a real manifold B of dimension m, and a family $\{(M_t, N_t) : t \in B, N_t$ is a closed submanifold of compact differentiable manifold $M_t\}$. We say that (M_t, N_t) depends C^∞ on t and that $\{(M_t, N_t) : t \in B\}$ is a C^∞ family of compact manifolds with submanifolds, if there is a C^∞ manifold $\mathcal{M}$, closed submanifold $\mathcal{N}$ and a C^∞ map ω from $\mathcal{M}$ onto B such that $\bar{\omega} = \omega/\mathcal{N}$ is also a C^∞ map from $\mathcal{N}$ onto B satisfying the following condition

(i) $M_t = \omega^{-1}(t) \supseteq N_t = \bar{\omega}^{-1}(t)$

(ii) The rank of the Jacobian of ω (respectively $\bar{\omega}$) is equal to m at every piont of $\mathcal{M}$ (respectively $\mathcal{N}$).

Theorem 3.3 Let$((\mathcal{M}, \mathcal{N}), (\omega, \bar{\omega}), B)$ be a C^∞ family of compact manifolds with submanifolds, with B connected. Then $(M_t, N_t) = (\omega^{-1}(t), \bar{\omega}^{-1}(t))$ is diffeomorphic to $(M_{t_0}, N_{t_0}) = (\omega^{-1}(t_0), \bar{\omega}^{-1}(t_0))$ for any $t \in B$.

proof In view of the existence of a tubular neighborhood of $\mathcal{N}$ in $\mathcal{M}$, the proof is basically

the same as the proof of the classical case when $\mathcal{N}$ is empty. For the convenience of the readers , we include the detail of the proof. First we construct a C^∞ vector field θ on a neighborhood of N_{t_0} in $\mathcal{N}$ such that $\bar{\omega}$ induces $\bar{\omega}_*(\theta) = \partial/\partial s$ where s is a member of a coordinate system $(s, t^2, ..., t^m)$ in a neighborhood of the piont $t_0 \in B$ choosen as follows.We connect t_0 and t by an embedded arc $\phi : (-\varepsilon, \quad 1+\varepsilon) \mapsto \{\phi(s) : s \in (-\varepsilon, \quad 1+\varepsilon)\}$. A compactness argument shows that we can assume that t and t_0 lie in the same coordinate patch and since ϕ is an embedding we can find a chart with coordinate $(s, t_2, ..., t_m)$ around t_0 ($t_0 = (0, ..., 0), t = (1, ..., 0)$). Because of the rank condition, $\bar{\omega}^{-1}(\phi) = \bar{\omega}^{-1}\{(s, 0, ..., 0) : -\varepsilon < s < 1+\varepsilon\}$ is a submanifold of $\mathcal{N}$, and we can assume that $(s, x_j^2, ..., x_j^n)$ are coordinate of $\mathcal{N}$ for a given piont of $\bar{\omega}^{-1}(\phi)$ in some neighborhood $\mathcal{U}_j$ of the piont. Then the vector field $(\partial/\partial s)_j$ on $\mathcal{U}_j$ satisfies $\omega_*(\partial/\partial s)_j = \partial/\partial s$. Then if $\{\rho_j\}$ is a partition of unity subordinate to $\{\mathcal{U}_j\}$ ($\bigcup \mathcal{U}_j$ is a neighborhood of N_{t_0} in $\mathcal{N}$), the vector field $\theta = \sum_j \rho_j (\partial/\partial s)_j$ satisties our requirment.

We next want to construct a C^∞ vector field Θ_0 on a tubular neighborhood of $\mathcal{N}$ in $\mathcal{M}$ such that $\omega_*(\Theta_0) = \partial/\partial s$ and $\Theta_0/\mathcal{N}$ is a vector field on $\mathcal{N}$. Notice that on $\mathcal{U}_j$, θ is given by

$$\theta_j = \sum_{\alpha=2}^{n} v_j^\alpha (s, x_j) \frac{\partial}{\partial x_j^\alpha} + \left(\frac{\partial}{\partial s}\right)_j \tag{3.1}$$

Let $\mathcal{V}_j$ be an open set in $\mathcal{M}$ such that $\mathcal{V}_j \bigcap \mathcal{N} = \mathcal{U}_j$ and $(s, x_j^2, ..., x_j^n, y_j^1, ..., y_j^l)$ a tubular neighborhood coordinate on $\mathcal{V}_j$ ($y_j^1, ..., y_j^l$ correspond to the fiber coordinate of the normal bundle of $\mathcal{N}$ in $\mathcal{M}$). Then with respect to the coordinate transformation

$$y_j^\alpha = \sum_{\beta=1}^{l} f_{jk}^{\alpha\beta}(s, x_k) y_k^\beta$$

$$x_j^\alpha = g_{jk}^\alpha (s, x_k)$$

We have

$$\left(\frac{\partial}{\partial s}\right)_k = \sum_{\alpha,\beta=1}^{l} y_k^\beta \frac{\partial f_{jk}^{\alpha\beta}}{\partial s}(s, x_k) \frac{\partial}{\partial y_j^\alpha} + \sum_{\gamma=2}^{n} \frac{\partial g_{jk}^\gamma}{\partial s}(s, x_k) \frac{\partial}{\partial x_j^\gamma} + \left(\frac{\partial}{\partial s}\right)_j \tag{3.2}$$

$$\frac{\partial}{\partial x_k^\alpha} = \sum_{\beta,\gamma=1}^{l} y_k^\gamma \frac{\partial f_{jk}^{\beta\gamma}}{\partial x_k^\alpha}(s, x_k) \frac{\partial}{\partial y_j^\beta} + \sum_{\delta=2}^{n} \frac{\partial g_{jk}^\delta}{\partial x_k^\alpha}(s, x_k) \frac{\partial}{\partial x_j^\delta} \tag{3.3}$$

Let $\{\sigma_j(s, x_j, y_j)\}$ be a partitrion of unity surbordinate to $\{\mathcal{V}_j\}$. Then $\Theta_0 = \sum_j \sigma_j(s, x_j, y_j)\theta$ is a C^∞ vector field on a tubular neighborhood $\mathcal{W}_0$ of $\mathcal{N}$ in $\mathcal{M}$. On $\mathcal{V}_j$ we have

$$\Theta_0 = \sum_k \sigma_k \theta_k = \sum_k \sum_{\alpha=2}^{n} v_k^\alpha(s, x_k)\rho_k [\sum_{\beta,\gamma=1}^{l} y_k^\gamma \frac{\partial f_{jk}^{\beta\gamma}}{\partial x_k^\alpha}(s, x_k)\frac{\partial}{\partial y_j^\beta}$$

$$+\sum_{\delta=2}^{n} \frac{\partial g_{jk}^\delta}{\partial x_k^\alpha}(s, x_k)\frac{\partial}{\partial x_j^\delta}] + \sum_k \sum_{\alpha,\beta=1}^{l} \rho_k y_k^\beta \frac{\partial f_{jk}^{\alpha\beta}}{\partial s}(s, x_k)\frac{\partial}{\partial y_j^\alpha}$$

$$+\sum_k \sum_{\gamma=2}^{n} \rho_k \frac{\partial g_{jk}^\gamma}{\partial s}(s, x_k)\frac{\partial}{x_j^\gamma} + \sum_k \rho_k (\frac{\partial}{\partial s})_j$$

$$= \sum_k \sum_{\beta,\gamma=1}^{l} \rho_k y_k^\gamma [\frac{\partial f_{jk}^{\beta\gamma}}{\partial s}(s, x_k) + \sum_{\alpha=2}^{n} v_k^\alpha(s, x_k)\frac{\partial f_{jk}^{\beta\gamma}}{\partial x_k^\alpha}(s, x_k)]\frac{\partial}{\partial y_j^\beta}$$

$$+\sum_k \sum_{\delta=2}^{n} \rho_k [\frac{\partial g_{jk}^\delta}{\partial s}(s, x_k) + \sum_{\alpha=2}^{n} v_k^\alpha(s, x_k)\frac{\partial g_{jk}^\delta}{\partial x_k^\alpha}(s, x_k)] + (\frac{\partial}{\partial s})_j$$

Therefore $\omega_*(\Theta_0) = \frac{\partial}{\partial s}$ and the restriction of Θ_0 on $\mathcal{N}$ is a vector field on $\mathcal{N}$. Notice that Θ_0 is defined only in a tubular neighborhood $\mathcal{W}_0$ of $\mathcal{N}$ in $\mathcal{M}$.

Let $(s, z_j^2, ..., z_j^{n+l})$ be a coordinate system defined on an open set $\mathcal{W}_j$ where the closure of $\mathcal{W}_j$ is disjoint from $\mathcal{N}$ for $j \neq 0$. We simply take $\Theta_j = (\frac{\partial}{\partial s})_j$. Again let $\{\phi_j\}$ be a partition of unity subordinate to $\{\mathcal{W}_j\}$. Then

$$\Theta = \sum_j \phi_j \Theta_j \tag{3.4}$$

is a C^∞ vector field on $\mathcal{M}$. It is easy to see that $\omega_*(\theta) = \frac{\partial}{\partial s}$ and $\Theta/\mathcal{N}$ is a vector field on $\mathcal{N}$.

Finally, we seek a solution of the differential equation

$$\frac{d}{ds} z_j^\alpha(\tau) = \Theta_j^\alpha[z(\tau)] \qquad 1 \leq \alpha \leq n + l \tag{3.5}$$

Where Θ_j^α is the α-component of Θ in the coordinate patch $\mathcal{W}_j$, with initial conditions $z_j^\alpha(0) = a^\alpha$ where $(0, a^2, ..., a^{n+l})$ is some point close to $(0, ..., 0)$. If s is small enough and $| a |$ is small enough, (3.5) has a unique solution $z_j(\tau, a)$ on some small interval. By compactness, we can assume that $M_{t_0} \subset \bigcup \mathcal{W}_j$, a finite union of such patches, and that in each $\mathcal{W}_j$, (3.5) is satisfied for $| \tau | < \mu$ where μ is independent of j. If $z_j^\alpha(\tau, a)$ is such a solution, let $z_j = f_{jk}(z_k)$ and define $\xi_k^\beta[\tau, f_{kj}(0, a)]$ uniquely on $\mathcal{W}_j \bigcap \mathcal{W}_k$ by

$$z_j(\tau, a) = f_{jk}(\xi_k[\tau, f_{kj}(0, a)]) \tag{3.6}$$

Then

$$\frac{dz_j^\alpha(\tau,a)}{d\tau} = \sum_\beta \frac{\partial z_j^\alpha}{\partial z_k^\beta} \frac{\partial \xi_k^\beta[\tau, f_{kj}(0,a)]}{\partial \tau}$$

and by the uniqueness of the solution to (3.5),

$$z_j(\tau,a) = f_{jk}(z_k[\tau, f_{kj}(0,a)]) \tag{3.7}$$

Equation (3.7) implies that $z(\tau,a)$,$\mid \tau \mid < \mu, a \in M_0$ is a well-defined differentiable map defined on M_{t_0} for each τ,$\mid \tau \mid < \mu$ and $z(0,a) = a$.

Let $\psi_\tau(a) = z(\tau,a)$; then $\psi_0 = id$ (on M_0). It is also easy to check that $\omega[\psi_\tau(a)] = \varphi(\tau)$ since $\omega_*(\Theta) = \frac{d}{d\tau}$. Hence, ψ_τ maps M_{t_0} into $M_{\varphi(\tau)}$ and N_{t_0} into $N_{\varphi(\tau)}$ (for small τ) because Θ is tangential to $\mathcal{N}$. We can repeat this argument for $M_{\varphi(\tau)}$ and define $\zeta_\mu : M_{\varphi(\tau)} \mapsto M_{\varphi(\tau+\mu)}$ and by uniqueness get $\zeta_{-\tau} \circ \psi_\tau = id, \psi_\tau \circ \zeta_{-\tau} = id$. Since everything is differentiable, the theorem is proved. Q.E.D

Now we fix weights $(w_0, w_1 w_2)$ with $w_i \geq 2$. Suppose that there is a weighted homogeneous polynomial $f(z_0,z_1,z_2)$ with the weights (w_0,w_1,w_2) such that f has an isolated singularity at origin.

Let Δ be the intersection of the plane $X/w_0 + Y/w_1 + Z/w_2 = 1$ with the first quadrant of R^3. Let $\mathbf{C}[\Delta] = \{f \in \mathbf{C}[z_0,z_1,z_2] \mid supp\ f \subset \Delta\}$ where $supp\ f = \{(d_0,d_1,d_2) \in \mathbf{R}^3 \mid z_0^{d_0} z_1^{d_1} z_2^{d_2}$ occurs in $f\}$. Let N be the number of the integer pionts which are in Δ. There is a canonical correspondence between $\mathbf{C}[\Delta]$ and $\mathbf{C}^N$. Thus we may introduce a Zariski topology on $\mathbf{C}[\Delta]$.

Theorem 3.4 Notation as above. Let

$$U = \{f \in \mathbf{C}[\Delta] \mid f \text{ has an isolated singularity at origin}\}.$$

Then U is a nonempty Zariski open set of $\mathbf{C}[\Delta]$.

proof Let $\mathbf{C}^{N+3} = \mathbf{C}^3 \times \mathbf{C}[\Delta] \cong \mathbf{C}^3 \times \mathbf{C}^N$. Let $F = \sum_{(\alpha_0,\alpha_1,\alpha_2)\in\Delta} u_{(\alpha_0,\alpha_1,\alpha_2)} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2}$ and let Y be a subvariety in $\mathbf{C}^{N+3}$ defined by F, $\frac{\partial F}{\partial z_0}$, $\frac{\partial F}{\partial z_1}$, $\frac{\partial F}{\partial z_2}$. Let

$$\pi : \mathbf{C}^{N+3} = \mathbf{C}^3 \times \mathbf{C}^N \mapsto \mathbf{C}^N$$

be the projection onto the second factor. Let $Y_0 = \{(0,\ 0,\ 0)\} \times \mathbf{C}^N$. It is clear that $Y_0 \subseteq Y$. Let Y_0' be the irreducible component of Y containing Y_0. We shall prove that $Y_0 = Y_0'$. Since there is a $f = \sum_{(\alpha_0,\ \alpha_1,\ \alpha_2)\in\Delta} a_{(\alpha_0,\ \alpha_1,\ \alpha_2)} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2} \in \mathbf{C}[\Delta]$ which has origin as its only sigularity, we see that $Z_f = [\mathbf{C}^3 \times (..., a_{(\alpha_0,\ \alpha_1,\ \alpha_2)}, ...)] \cap Y$ is of dimension zero. As $Z_f \cap Y_0 \neq \emptyset$, $Z_f \cap Y_0'$ is a nonempty subvariety of dimension zero. In view of Z_f and Y_0' are irreducible and $dim Z_f = 3$, we have $dim Y_0' \leq N$ by Proposition 7.1 [5]. Note that $dim Y_0 = N \geq dim Y_0'$ and $Y_0 \subseteq Y_0'$ implies $Y_0 = Y_0'$.

For any $f = \sum a_{(\alpha_0,\ \alpha_1,\ \alpha_2)} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2} \in \mathbf{C}[\Delta] \cong \mathbf{C}^N$, we have either $\pi^{-1}(f) \cap Y$ consists of a single point $(0, 0, 0, ..., a_{(\alpha_0,\ \alpha_1,\ \alpha_2)}, ...)$ or $\pi^{-1}(f) \cap Y$ is a subvariety of dimension at least one which contains $(0, 0, 0, ..., a_{(\alpha_0,\ \alpha_1,\ \alpha_2)}, ...)$. This follows immediately from the fact that the singular locus of the variety defined by f is invariant by the $\mathbf{C}^*$-action. In the former case f has an isolated singularity at origin and in the latter case f has nonisolated singularity at origin. Let

$$J = \{(..., a_{(\alpha_0,\ \alpha_1,\ \alpha_2)}, ...) \in \mathbf{C}^N : f = \sum_{(\alpha_0,\ \alpha_1,\ \alpha_2)\in\Delta} a_{(\alpha_0,\ \alpha_1,\ \alpha_2)} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2}$$
$$\text{has nonisolated singularity at origin.}\}$$

Then $Y = Y_0 \cup \pi^{-1}(J)$. Let $Y = \cup_{0\leq i\leq r} Y_i$ where Y_i are irreducible components of Y. It follows that $(\cup_{1\leq i\leq r} Y_i) \setminus Y_0 = \pi^{-1}(J) \setminus Y_0$. Observe that $Y_i \cap Y_0 = (\overline{Y_i \setminus Y_0}) \cap Y_0$ and $(\overline{\pi^{-1}(J) \setminus Y_0}) \cap Y_0 = \pi^{-1}(J) \cap Y_0$. Hence

$$\begin{aligned}
\pi^{-1}(J) \cap Y_0 &= (\overline{\pi^{-1}(J) \setminus Y_0}) \cap Y_0 \\
&= \overline{[(\cup_{1\leq i\leq r} Y_i) \setminus Y_0]} \cap Y_0 \\
&= [\cup_{1\leq i\leq r} (\overline{Y_i \setminus Y_0})] \cap Y_0 \\
&= \cup_{1\leq i\leq r} [(\overline{Y_i \setminus Y_0}) \cap Y_0] \\
&= \cup_{1\leq i\leq r} (Y_i \cap Y_0)
\end{aligned}$$

Hence $\pi^{-1}(J) \cap Y_0$ is a Zariski closed set. Since $\pi\ |_{Y_0}$: $Y_0 \mapsto \mathbf{C}^N$ is a biregular morphism, $J = \pi(\pi^{-1}(J) \cap Y_0)$ is a Zariski closed subset. Therefore $U = \mathbf{C}^N \setminus J$ is a Zariski open set. Q.E.D.

Theorem 3.5 Suppose that $f(z_0, z_1, z_2)$ and $g(z_0, z_1, z_2)$ are weighted homogeneous polynomials with the same weights (w_0, w_1, w_2). If the variety V of f and the variety W of g have isolated

singularities at origin, then $(\mathbf{C}^3, V, 0)$ is homeomorphically equivalent to $(\mathbf{C}^3, W, 0)$.

proof Let

$$U = \{f \in \mathbf{C}[\Delta] \mid f \text{ has an isolated singularity at origin.}\}.$$

By Theorem 3.4, U is a connected manifold of dimension $2N$. Let

$$\tilde{\omega} : \mathbf{C}^{N+3} \mapsto \mathbf{C}^N$$

be given by

$$(z_0, z_1, z_2, ..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...) \mapsto (..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...)$$

where $(\alpha_0, \alpha_1, \alpha_2) \in \mathbf{Z}^3 \bigcap \Delta$. Let

$$C = \{(z_0, z_1, z_2, ..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...) : \mid z_0 \mid^2 + \mid z_1 \mid^2 + \mid z_2 \mid^2 = 1\}$$

Let

$$F = \{(z_0, z_1, z_2, ..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...) : \sum a_{(\alpha_0,\alpha_1,\alpha_2)} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2} = 0\}$$

Let

$$\mathcal{M} = C \cap \tilde{\omega}^{-1}(U),$$

$$\mathcal{N} = F \cap \mathcal{M} = F \cap C \cap \tilde{\omega}^{-1}(U),$$

$$\omega = \tilde{\omega} \mid_{\mathcal{M}}, \quad \bar{\omega} = \tilde{\omega} \mid_{\mathcal{N}} \text{ and } \tilde{\omega}_1 = \tilde{\omega} \mid_F .$$

Using the same argument as in [12], we show that ($(\mathcal{M},\mathcal{N}), (\omega,\bar{\omega}), U$) is a C^∞ family as defined in Definition 3.2. Clearly we need only to show that ω and $\bar{\omega}$ has no critical point. It is suffices to prove that $\bar{\omega}$ has no critical point. Suppose

$$m = (z_0, z_1, z_2, ..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...) \in \mathcal{N}$$

Let $T_{\mathcal{N}}, T_F, T_C$ denote the tangent planes at m to $\mathcal{N}$,F and C respectively. T_F is the complex plane perpendicular to

$$v = (\partial k/\partial z_0, \partial k/\partial z_1, \partial k/\partial z_2, ..., \partial k/\partial a_{(\alpha_0,\alpha_1,\alpha_2)}, ...)_m$$

where

$$k(z_0, z_1, z_2, ..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...) = \sum a_{(\alpha_0,\alpha_1,\alpha_2)} z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2}$$

and T_C is the real plane perpendicular to

$$V' = (z_0, z_1, z_2, 0, ..., 0).$$

We must show that $(\textit{kernel}\ \tilde{\omega}) + T_{\mathcal{N}} = \mathbf{C}^{N+3}$. But $T_{\mathcal{N}} = T_F \bigcap T_C$ and $v' \in (\textit{kernel}\ \tilde{\omega})$. Hence it is sufficient to show that $(\textit{kernel}\ \tilde{\omega}) + T_F = \mathbf{C}^{N+3}$ or equivalently that $T_F \not\supset \textit{kernel}\ \tilde{\omega}$. Now suppose $\textit{kernel}\ \tilde{\omega} \subset T_F$. Then $(y_0, y_1, y_2, 0, ..., 0)$ is perpendicular to v for all $(y_0, y_1, y_2) \in \mathbf{C}^3$. But then

$$\frac{\partial k}{\partial z_0}(m) = \frac{\partial k}{\partial z_1}(m) = \frac{\partial k}{\partial z_2}(m) = 0$$

However $\tilde{\omega}_1^{-1}(..., a_{(\alpha_0,\alpha_1,\alpha_2)}, ...)$ has an isolated singularity at orgin. Hence we get a contradiction and our claim is proved. Denote the point corresponding to f (respectively g) by $m_f \in \mathbf{C}^N$ (respectively $m_g \in \mathbf{C}^N$). By Theorem 3.3 $(\omega^{-1}(m_f), \bar{\omega}^{-1}(m_f))$ is diffeomorphic to $(\omega^{-1}(m_g), \bar{\omega}^{-1}(m_g))$. It follows that $(\mathbf{C}^3, V, 0)$ is homeomorphically equivalent to $(\mathbf{C}^3, W, 0)$. This completes the proof. Q.E.D.

Corollary 3.6 Suppose that $h(z_0, z_1, z_2)$ is a weighted homogeneous polynimial with weights (w_0, w_1, w_2) and the variety $h^{-1}(0)$ has an isolated singularity at origin. Then $h = f + g$ where f and g have no monomials in common and f is one of the seven classes above. Moreover $h^{-1}(0)$ and $f^{-1}(0)$ have the same topological type.

proof This follows directly from Orlik–Wagreich [12] and Arnold [1], and Theorem 3.5. Q.E.D.

§4 Proof of The Theorem And Corollaries

Theorem 4.1 Suppose that f and g are among the seven classes in §1. If $\Delta_f(z) = \Delta_g(z)$ and $\pi_1(K_f) = \pi_1(K_g)$, then f, g belong to one of the rows listed in Table 3 below.

Table 3

Type of f	Type of g	Conditions on f and g
I	I	$\left\{\begin{array}{l} f = z_0^{a_0'} + z_1^{a_1'} + z_2^{a_2'} \\ g = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} \end{array}\right\}$ $a_0' = a_{i_0}, a_1' = a_{i_1}, a_2' = a_{i_2}$ where $\{i_0, i_1, i_2\} = \{0, 1, 2\}$
I	II	$\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{a_1} + z_2^{\frac{a_1 a_2}{a_1 - 1}} \\ g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ $(a_1 - 1)/a_2$
I	III	$\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{a_1+1} + z_2^{a_1+1} \\ g = z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_1} \end{array}\right\}$
I	IV	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_2^{tm(k+1)} \\ g = z_0^{k+1} + z_0 z_1^{mk} + z_1 z_2^{t(mk+m-1)} \end{array}\right\}$
I	V	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{k+1} + z_2^{k+1} \\ g = z_0^{k} z_1 + z_1^{k} z_2 + z_0 z_2^{k} \end{array}\right\}$
I	VI	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_2^{t(k+1)} \\ g = z_0^{k+1} + z_0 z_1^{mk} + z_0 z_2^{tk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$
I	VII	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{k+1} + z_2^{m(k+1)} \\ g = z_0^{k} z_1 + z_0 z_1^{k} + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$
II	II	$\left\{\begin{array}{l} f = z_0^{a_0'} + z_1^{a_1'} + z_1 z_2^{a_2'} \\ g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ $a_i' = a_i, i = 0, 1, 2$
		$\left\{\begin{array}{l} f = z_0^{a_1} + z_1^{a_0} + z_1 z_2^{\frac{a_1 a_2 (a_0 - 1)}{a_0 (a_1 - 1)}} \\ g = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ $a_0(a_1 - 1)/a_1 a_2 (a_0 - 1)$
		$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_1 z_2^{t(mk+m-1)} \\ g = z_0^{tm(k+1)} + z_1^{k+1} + z_1 z_2^{mk} \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{nk} \\ g = z_0^{n(k+1)} + z_1^{k+1} + z_1 z_2^{mk} \end{array}\right\}$

Type of f	Type of g	Conditions On f and g
II	III	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{k+1} + z_1 z_2^{mk} \\ g = z_0^{mk+1} + z_1^k z_2 + z_1 z_2^k \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{mk+1} + z_1^{mk+1} + z_1 z_2^k \\ g = z_0^{mk+1} + z_1^{m(k-1)+1} z_2 + z_1 z_2^k \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{mk+m+1} + z_1 z_2^{k+1} \\ g = z_0^{a_0} + z_1^{mk+1} z_2 + z_1 z_2^{k+1} \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_1^{mn+1} + z_1 z_2^{\frac{(mk+m+1)n}{mn+1}} \\ g = z_0^{mn+1} + z_1^{mk+1} z_2 + z_1 z_2^{k+1} \end{array}\right\}$ with $(mn+1)/(mk+m+1)$
II	IV	$\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{\frac{a_0 a_1}{a_0 - 1}} + z_1 z_2^{a_2} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ with $(a_0 - 1)/a_1$
		$\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{\frac{a_0 a_1 a_2}{a_0 a_1 - a_0 + 1}} + z_1 z_2^{\frac{a_0 a_1 a_2 - a_0 a_1 + a_0 - 1}{a_2(a_0 - 1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ with $(a_0 a_1 - a_0 + 1)/a_1 a_2$ and $a_2(a_0 - 1)/(a_0 a_1 a_2 - a_0 a_1 + a_0 - 1)$
		$\left\{\begin{array}{l} f = z_0^{\frac{a_0 a_1 a_2}{a_0 a_1 - a_0 + 1}} + z_1^{a_0} + z_1 z_2^{a_1} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ with $(a_0 a_1 - a_0 + 1)/a_1 a_2$
		$\left\{\begin{array}{l} f = z_0^{\frac{a_0 a_1}{a_0 - 1}} + z_1^{a_0} + z_1 z_2^{\frac{a_1 a_2 (a_0 - 1)}{a_0 a_1 - a_0 + 1}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ with $(a_0 - 1)/a_1$ and $(a_0 a_1 - a_0 + 1)/a_1 a_2 (a_0 - 1)$
II	V	$\left\{\begin{array}{l} f = z_0^{kl+1} + z_1^{kl+1} + z_1 z_2^l \\ g = z_0^{kl} z_1 + z_1^{k(l-1)+1} z_2 + z_0 z_2^l \end{array}\right\}$
II	VI	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_1 z_2^{t(mk+m-1)} \\ g = z_0^{k+1} + z_0 z_1^{mk} + z_0 z_2^{tmk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$

Type of f	Type of g	Conditions On f and g
II	VI	$\left\{\begin{array}{l} f = z_0^{k+1} + z_1^{m(k+1)} + z_1 z_2^{t(mk+m-1)} \\ g = z_0^{k+1} + z_0 z_1^{tmk} + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{a} \\ g = z_0^{k+1} + z_0 z_1^{mk} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{m(k+1)} + z_1^{k+1} + z_1 z_2^{a} \\ g = z_0^{k+1} + z_0 z_1^{a} + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$
II	VII	$\left\{\begin{array}{l} f = z_0^{a_0+1} + z_1^{a_0+1} + z_1 z_2^{a_2} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_0} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{a_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{n(mk+m+1)} + z_1^{mk+m+1} + z_1 z_2^{k+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{n(mk+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{n(mk+m+1)} + z_1^{mk+m+1} + z_1 z_2^{k+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{nm(k+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$
III	III	$\left\{\begin{array}{l} f = z_0^{a_0'} + z_1^{a_1'} z_2 + z_1 z_2^{a_2'} \\ g = z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2} \end{array}\right\}$ with $a_0 = a_0'$ and either $a_1 = a_1', a_2 = a_2'$ or $a_1 = a_2', a_2 = a_1'$
III	IV	$\left\{\begin{array}{l} f = z_0^{n(mk+1)} + z_1^{mk-m+1} z_2 + z_1 z_2^{k} \\ g = z_0^{mk+1} + z_0 z_1^{k} + z_1 z_2^{n(mk-m+1)} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{n(mk+1)} + z_1^{k} z_2 + z_1 z_2^{mk-m+1} \\ g = z_0^{mk+1} + z_0 z_1^{k} + z_1 z_2^{n(mk-m+1)} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{\frac{a_0 a_1 a_2 - a_0 a_1 + a_0 - 1}{a_2(a_0-1)}} z_2 + z_1 z_2^{a_2} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{a_0} + z_1^{a_2} z_2 + z_1 z_2^{\frac{a_0 a_1 a_2 - a_0 a_1 + a_0 - 1}{a_2(a_0-1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$

Type of f	Type of g	Conditions On f and g
III	V	$\left\{ \begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_1a_2-a_2+1}} + z_2z_1^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} + z_1z_2^{a_1} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array} \right\}$
		$\left\{ \begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_1a_2-a_2+1}} + z_1^{a_1}z_2 + z_1z_2^{\frac{a_0(a_1a_2-a_2+1)}{a_0a_1-a_1+1}} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array} \right\}$
		$\left\{ \begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_2-a_0+1}} + z_2z_1^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_2+1}} + z_1z_2^{a_2} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array} \right\}$
		$\left\{ \begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_2-a_0+1}} + z_1^{a_1}z_2 + z_1z_2^{\frac{a_1(a_0a_2-a_0+1)}{a_1a_2-a_2+1}} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array} \right\}$
		$\left\{ \begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_1-a_1+1}} + z_1^{a_0}z_2 + z_1z_2^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array} \right\}$
		$\left\{ \begin{array}{l} f = z_0^{\frac{a_0a_1a_2+1}{a_0a_1-a_1+1}} + z_2z_1^{\frac{a_2(a_0a_1-a_1+1)}{a_0a_2-a_0+1}} + z_1z_2^{a_0} \\ g = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2} \end{array} \right\}$
III	VI	$\left\{ \begin{array}{l} f = z_0^{k+1} + z_1^{mk+m-1}z_2 + z_1z_2^{mk+m-1} \\ g = z_0^{k+1} + z_0z_1^{mk} + z_0z_2^{mk} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$
		$\left\{ \begin{array}{l} f = z_0^{m(k+1)} + z_1^kz_2 + z_1z_2^k \\ g = z_0^{m(k+1)} + z_0z_1^k + z_0z_2^{mk} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$
III	VII	$\left\{ \begin{array}{l} f = z_0^{t(mnk+m+n)} + z_1^{nk+1}z_2 + z_1z_2^{mk+1} \\ g = z_0^{nk+1}z_1 + z_0z_1^{mk+1} + z_0z_2^{n(mk+1)} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$ with $(m,n) = 1$
		$\left\{ \begin{array}{l} f = z_0^{t(mnk+m+n)} + z_1^{nk+1}z_2 + z_1z_2^{nk+1} \\ g = z_0^{nk+1}z_1 + z_0z_1^{mk+1} + z_0z_2^{tn(mk+1)} + z_1^{b_1}z_2^{b_2} \end{array} \right\}$ with $(m,n) = 1$

Type of f	Type of g	Conditions On f and g
III	VII	$\left\{\begin{array}{l} f = z_0^{k+2} + z_2 z_1^{k+2-\frac{k+1}{a}} + z_1 z_2^a \\ g = z_0^{k+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^a + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ with $a/(k+1)$
		$\left\{\begin{array}{l} f = z_0^{(k+2)} + z_1^a z_2 + z_1 z_2^{k+2-\frac{k+1}{a}} \\ g = z_0^{k+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^a + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ with $a/(k+1)$
		$\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_1^{s(tm+1)+1} z_2 + z_1 z_2^{sm+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{sm+1} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ with $k = s(tm+1)+t$
		$\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_1^{sm+1} z_2 + z_1 z_2^{s(tm+1)+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{sm+1} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ with $k = s(tm+1)+t$
IV	IV	$\left\{\begin{array}{l} f = z_0^{a'_0} + z_0 z_1^{a'_1} + z_1 z_2^{a'_2} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2} \end{array}\right\}$ with $a_0 = a'_0$, $a_1 = a'_1, a_2 = a'_2$ or $a_1 = a'_2, a_2 = a'_1$
IV	V	$\left\{\begin{array}{l} f = z_0^{ms+m+1} + z_0 z_1^{s+1} + z_1 z_2^{sm+1} \\ g = z_0^{sm+1} z_1 + z_1^{s+1} z_2 + z_0 z_2^{m(s+1)} \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{st+s+t+2} + z_0 z_1^{st+s+t+1} + z_1 z_2^{s+1} \\ g = z_0^{st+s+t+1} z_1 + z_1^{st+s+1} z_2 + z_0 z_2^{s+1} \end{array}\right\}$ $t > 0$
		$\left\{\begin{array}{l} f = z_0^{mk+1} + z_0 z_1^k + z_1 z_2^a \\ g = z_0^{a_0} z_1 + z_1^a z_2 + z_0 z_2^k \end{array}\right\}$ $a_0 = (mk+1) + \frac{mk-m+1}{a}, a/(mk-m+1)$
		$\left\{\begin{array}{l} f = z_0^{sm+m+1} + z_0 z_1^{s+1} + z_1 z_2^{sm+1} \\ g = z_0^{s+1} z_1 + z_1^{m(s+1)} z_2 + z_0 z_2^{sm+1} \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{st+s+t+2} + z_0 z_1^{st+s+t+1} + z_1 z_2^{s+1} \\ g = z_0^{st+s+1} z_1 + z_1^{s+1} z_2 + z_0 z_2^{st+s+t+1} \end{array}\right\}$
		$\left\{\begin{array}{l} f = z_0^{mk+1} + z_0 z_1^k + z_1 z_2^a \\ g = z_0^a z_1 + z_1^k z_2 + z_0 z_2^{a_2} \end{array}\right\}$ $a_2 = (mk+1) + \frac{mk-m+1}{a}, a/(mk-m+1)$

Type of f	Type of g	Conditions On f and g
IV	V	$\left\{\begin{array}{l} f = z_0^{sm+m+1} + z_0 z_1^{s+1} + z_1 z_2^{sm+1} \\ g = z_0^{m(s+1)} z_1 + z_1^{sm+1} z_2 + z_0 z_2^{s+1} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{st+s+t+2} + z_0 z_1^{st+s+t+1} + z_1 z_2^{s+1} \\ g = z_0^{s+1} z_1 + z_1^{st+s+t+1} z_2 + z_0 z_2^{st+s+1} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk+1} + z_0 z_1^{k} + z_1 z_2^{a} \\ g = z_0^{k} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a} \end{array}\right\}$ $a_1 = (mk+1) + \frac{mk-m+1}{a}, a/(mk-m+1)$
IV	VI	$\left\{\begin{array}{l} f = z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{\frac{a_2(a_0 a_1 - a_0 + 1)}{a_1(a_0-1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\frac{a_1(a_0-1)}{(a_0-1, a_1)} / a_2$ $\left\{\begin{array}{l} f = z_0^{a_0} + z_0 z_1^{a_2} + z_1 z_2^{\frac{a_1(a_0 a_2 - a_0 + 1)}{a_2(a_0-1)}} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\frac{a_2(a_0-1)}{(a_0-1, a_2)} / a_1$
IV	VII	$\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{k+1} + z_1 z_2^{\frac{a(mk+1)}{m(k+1)}} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $m(k+1)/(mk+1)a$ $\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{k+1} + z_1 z_2^{a} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mst+1} + z_0 z_1^{tm} + z_1 z_2^{k+1-\frac{s-1}{m}} \\ g = z_0^{m(st-1)+1} z_1 + z_0 z_1^{st} + z_0 z_2^{tm} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{m(k+1)} + z_1 z_2^{k+1} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{mk+1} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mnk+m+n} + z_0 z_1^{\frac{mnk+m+n-1}{n}} + z_1 z_2^{nk+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{m(nk+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $n/(m-1)$ $\left\{\begin{array}{l} f = z_0^{mnk+m+n} + z_0 z_1^{\frac{mnk+m+n-1}{m}} + z_1 z_2^{mk+1} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{nk+1} + z_0 z_2^{m(nk+1)} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $m/(n-1)$

Type of f	Type of g	Conditions On f and g
V	V	$\left\{\begin{array}{l} f = z_0^{a'_0} z_1 + z_1^{a'_1} z_2 + z_0 z_2^{a'_2} \\ g = z_0^{a_0} z_1 + z_1^{a_1} z - 2 + z_0 z_2^{a_2} \end{array}\right\}$ with either (1) $a'_0 = a_{i_0}, a'_1 = a_{i_1}, a'_2 = a_{i_2}$ where $\{i_0, i_1, i_2\}$ is a even permutation of $\{0, 1, 2\}$ or (2) $a'_0 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1}$ $a'_1 = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}$ $a'_2 = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}$ or (3) $a'_0 = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}$ $a'_1 = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}$ $a'_2 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1}$ or (4) $a'_0 = \frac{a_2(a_0a_1 - a_1 + 1)}{a_0a_2 - a_0 + 1}$ $a'_1 = \frac{a_1(a_0a_2 - a_0 + 1)}{a_1a_2 - a_2 + 1}$ $a'_2 = \frac{a_0(a_1a_2 - a_2 + 1)}{a_0a_1 - a_1 + 1}$
V	VI	$\left\{\begin{array}{l} f = z_0^{mk} z_1 + z_1^{mk-m+1} z_2 + z_0 z_2^k \\ g = z_0^{mk+1} + z_0 z_1^{mk} + z_0 z_2^k + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk} z_1 + z_1^{mk-m+1} z_2 + z_0 z_2^k \\ g = z_0^{mk+1} + z_0 z_1^k + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^k z_1 + z_1^{mk} z_2 + z_0 z_2^{mk-m+1} \\ g = z_0^{mk+1} + z_0 z_1^k + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^k z_1 + z_1^{mk} z_2 + z_0 z_2^{mk-m+1} \\ g = z_0^{mk+1} + z_0 z_1^{mk} + z_0 z_2^k + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk-m+1} z_1 + z_1^k z_2 + z_0 z_2^{mk} \\ g = z_0^{mk+1} + z_0 z_1^k + z_0 z_2^{mk} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk-m+1} z_1 + z_1^k z_2 + z_0 z_2^{mk} \\ g = z_0^{mk+1} + z_0 z_1^{mk} + z_0 z_2^k + z_1^{b_1} z_2^{b_2} \end{array}\right\}$

Type of f	Type of g	Conditions On f and g
V	VII	$\left\{\begin{array}{l} f = z_0^{mk+1}z_1 + z_1^{a'_1}z_2 + z_0z_2^{tm} \\ g = z_0^{mk+1}z_1 + z_0z_1^{nk+1} + z_0z_2^{tm} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$ $a'_1 = (nk+1) + \frac{tn-nk-1}{tm}$ with $t/(nk+1), tm/(tn-nk-1)$
		$\left\{\begin{array}{l} f = z_0^{tm}z_1 + z_1^{mk+1}z_2 + z_0z_2^{a'_2} \\ g = z_0^{mk+1}z_1 + z_0z_1^{nk+1} + z_0z_2^{tm} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$ $a'_2 = (nk+1) + \frac{tn-nk-1}{tm}$ with $t/(nk+1), tm/(tn-nk-1)$
		$\left\{\begin{array}{l} f = z_0^{a'_0}z_1 + z_1^{tm}z_2 + z_0z_2^{mk+1} \\ g = z_0^{mk+1}z_1 + z_0z_1^{nk+1} + z_0z_2^{tm} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$ $a'_0 = (nk+1) + \frac{tn-nk-1}{tm}$ with $t/(nk+1), tm/(tn-nk-1)$
		$\left\{\begin{array}{l} f = z_0^{a'_0}z_1 + z_1^{a'_1}z_2 + z_0z_2^{a_1} \\ g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$ $a'_0 = \frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_1-1)} \in \mathbf{Z}$ $a'_1 = \frac{a_0a_2(a_1-1)}{a_1(a_0-1)} \in \mathbf{Z}$
		$\left\{\begin{array}{l} f = z_0^{a_1}z_1 + z_1^{a'_1}z_2 + z_0z_2^{a'_2} \\ g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$ $a'_1 = \frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_1-1)} \in \mathbf{Z}$ $a'_2 = \frac{a_0a_2(a_1-1)}{a_1(a_0-1)} \in \mathbf{Z}$
		$\left\{\begin{array}{l} f = z_0^{a'_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a'_2} \\ g = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2} \end{array}\right\}$ $a'_0 = \frac{a_0a_2(a_1-1)}{a_1(a_0-1)} \in \mathbf{Z}$ $a'_2 = \frac{a_2(a_0a_1-1)-a_1(a_0-1)}{a_2(a_1-1)} \in \mathbf{Z}$

Type of f	Type of g	Conditions On f and g
VI	VI	$\left\{\begin{array}{l} f = z_0^{a_0'} + z_0 z_1^{a_1'} + z_0 z_2^{a_2'} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2 a_2 + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $a_0' = a_0$, either $a_1 = a_1', a_2 = a_2'$ or $a_1' = a_2, a_2' = a_1$
VI	VII	$\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{k+1} + z_0 z_2^{a} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{k+1} + z_0 z_2^{\frac{ma(k+1)}{mk+1}} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $(mk+1)/a(k+1)$ $\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{a} + z_0 z_2^{(k+1)} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{mk+1} z_1 + z_0 z_1^{k+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $\left\{\begin{array}{l} f = z_0^{mk+m+1} + z_0 z_1^{\frac{ma(k+1)}{mk+1}} + z_0 z_2^{k+1} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{k+1} z_1 + z_0 z_1^{mk+1} + z_0 z_2^{a} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ $(mk+1)/a(k+1)$
VII	VII	$\left\{\begin{array}{l} f = z_0^{a_0'} z_1 + z_0 z_1^{a_1'} + z_0 z_2^{a_2'} + z_1^{b_1'} z_2^{b_2'} \\ g = z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2} \end{array}\right\}$ with either (1) $a_i' = a_i . i = 0, 1, 2$ or (2) $a_0' = a_1, a_1' = a_0, a_2' = \frac{a_0 a_2 (a_1 - 1)}{a_1 (a_0 - 1)}$ or (3) $a_0' = \frac{a_2 (a_0 a_1 - 1) - a_1 (a_0 - 1)}{a_2 (a_0 - 1)}$ $a_1' = \frac{a_0 a_2 (a_1 - 1)}{a_1 (a_0 - 1)}, a_2' = a_0$ or (4) $a_0' = a_2, a_1' = \frac{a_2 (a_0 a_1 - 1) - a_1 (a_0 - 1)}{a_2 (a_1 - 1)}$ $a_2' = \frac{a_2 (a_0 a_1 - 1) - a_1 (a_0 - 1)}{a_2 (a_0 - 1)}$ or (5) $a_0' = \frac{a_2 (a_0 a_1 - 1) - a_1 (a_0 - 1)}{a_2 (a_1 - 1)}$ $a_1' = a_2, a_2' = a_1$ or (6) $a_0' = \frac{a_0 a_2 (a_1 - 1)}{a_1 (a_0 - 1)}, a_1' = \frac{a_2 (a_0 a_1 - 1) - a_1 (a_0 - 1)}{a_2 (a_0 - 1)}$ $a_2' = \frac{a_2 (a_0 a_1 - 1) - a_1 (a_0 - 1)}{a_2 (a_1 - 1)}$

The end of the proof of Theorem and Corollaries 1-3 in section I

Although our proofs of Theorem 4.1 is by no means straight forward, we shall not give it here. They are consequence of Proposition 2.2 , tables 1 and 2 in section 2. The theorem in the introduction follows from the well-known fact that the first fundamental group $\pi_1(K_f)$ of the link and the characteristic polynomial $\Delta_f(z)$ of the Milnor fibration are invariants of topological type.

Observe also that from Theorem 4.1 if $\Delta_f(z) = \Delta_g(z)$ and $\pi_1(K_f) = \pi_1(K_g)$, then f and g have the same weights. Therefore Corollary1–3 in the introduction follows from Theorem 3.7 and Corollary 3.8. Q.E.D.

§5 Zariski Conjecture for Weighted Homogeneous Two–Dimensional Hypersurface Singularities

Lemma 5.1 Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial with weights (w_0, w_1, w_2). Suppose that f has an isolated singularity at origin. Denote themultiplicity

of $f(z_0, z_1, z_2)$ at the origin by m_f. Then $m_f \geq min\{w_0, w_1, w_2\}$. **proof** By definition of the multiplicity, there is a monomial $z_0^{\alpha_0} z_1^{\alpha_1} z_2^{\alpha_2}$ in f such that $m_f = \alpha_0 + \alpha_1 + \alpha_2$. Since f is weighted homogeneous with weights (w_0, w_1, w_2), we have $\frac{\alpha_0}{w_0} + \frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2} = 1$. It follows easily that $m_f \geq min\{w_0, w_1, w_2\}$. Q.E.D.

Proposition 5.2 . Suppose that f is one of the seven classes. Let $m = min\{\ n \in \mathbf{Z} \mid n \geq min\{w_0, w_1, w_2\}\}$. Then the multiplicity m_f of f is m.

proof Easy exercise.

Corollary Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial with weights (w_0, w_1, w_2) . Suppose f has an isolated singularity at origin. Then the multiplicity m_f of f at origin eqquals to $m = min\{m \in \mathbf{Z} \mid m \geq min\{w_0, w_1, w_2\}\}$. Hence the multiplicity of f is an invariant of topologicai type.

proof By Proposition 3.1, $f = g + h$ where g is one of the seven classes having the same weights $(w_0,\ w_1,\ w_2)$. It is clear that $m_g \geq m_f$. By Proposition 5.2 we have $m_g \leq min\{n \in \mathbf{Z} : n \geq min\{w_0, w_1, w_2\}\}$. Hence we have $m_f \leq min\{n \in \mathbf{Z} : n \geq min\{w_0, w_1, w_2\}\}$. Conversely by Lemma 5.1, we see that $m_f \geq min\{w_0, w_1, w_2\}$. Thus $m_f \geq min\{n \in \mathbf{Z} : n \geq min\{w_0, w_1, w_2\}\}$. Q.E.D.

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