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SOME REMARKS ON BOUNDED AND UNBOUNDED WEAK SOLUTIONS OF ELLIPTIC SYSTEMS

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The result of Ladyženskaya on global bounds for weak solutions of elliptic equations is carried over to certain quasilinear elliptic systems in diagonal form. A generalization of De Giorgi's example shows that there are unbounded weak solutions for systems with only "small" deviation from the diagonal form.

INTRODUCTION

On a bounded domain $\Omega \subset \mathbb{R}^N$ we consider the quasilinear elliptic system

$$(E) \qquad A(u) + B(u) = 0,$$

where the leading operator $A(u)$ is of the kind $\sum_{\ell=1}^{M} \sum_{i,j=1}^{N} D^i a_{ij}^{k\ell}(x, u, D(u)) D^j u_\ell$ and the perturbation $B(u)$ is given by $b^k(x, u, D(u))$, $k = 1, \cdots, M$.

As it is well known, a weak solution of a single elliptic equation is bounded in $L^\infty(\Omega)$ provided the coefficient functions are subject to a set of "natural" conditions, cf. [4,10]. On the other hand, in 1968 De Giorgi presented an example of an unbounded weak solution of a linear elliptic system satisfying the set of natural conditions when transferred in a straightforward manner from elliptic equations to elliptic systems cf. [1,2,3].

The system of this example is of the form

$$\Delta u_k + \sum_{\ell,i,j=1}^{N} D^i(b_{ij}^{k\ell}(x) D^j(u_\ell)) = 0, \; k = 1, \cdots, N.$$

As we shall point out in the last section of this note, there are unbounded weak solutions for systems of this kind even if the coefficient functions $b_{ij}^{k\ell}$ become arbitrarily small in the $L^\infty(\Omega)$-norm. In our examples the L^∞-bound on $b_{ij}^{k\ell}$ is linked with the dimension of the space. We have to admit high space dimensions to get the bound small. However, in addition to the L^∞-bound the ratio $|b_{ij}^{k\ell}(x)|(|\delta_{ij}\delta_{k\ell}|)^{-1}$ also becomes small uniformly in x. Here $|.|$ is the euclidean norm in $\mathbb{R}^{N^4}$, and $\delta_{ij}\delta_{k\ell}$ are the coefficients of the Laplacian.

These examples indicate that in general we can expect L^∞-bounds only for systems in diagonal form. On the other hand, we shall show that Ladyženskaya's result on global bounds can be transferred to certain quasilinear systems in diagonal form. Our proof is based on the estimates of $|D(|u|)|^p$ rather than $|D(u)|$ and is restricted to the Sobolev spaces $H^{1,p}(\Omega)$ with $p \geq 2$. A different approach has been used by Meier in [8]. However, despite greater technical efforts the bounds from below on the perturbation are not more general than those given here. For $p < 2$ the problem seems to be still open. The theory of [8], too, yielded *global* bounds only for differential operators in $H^{1,p}(\Omega)$, $p \geq 2$.

Finally we remark that our examples also show that a somewhat more general notion of a solution than the weak solution does not provide uniqueness even for linear uniformly elliptic systems; cf. [9], too.

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1. BOUNDED SOLUTIONS

The following hypotheses are similar to [4], p. 286. For sake of a less technical presentation, we do not assume another integrability property for the weak solution as the one provided by the Sobolev imbedding Theorem.

For any $\eta \in \mathbb{R}^M$, $\xi \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^{MN}$ and almost all $x \in \Omega$ we assume

$$(\mathcal{A}) \quad \sum_{i,j=1}^{N} a_{ij}(x,\eta,\zeta)\xi^j\xi^i \geq \nu|\zeta|^{p-2}|\xi|^2 - (1+|\eta|^{\beta_1})\varphi_1(x),$$

$$(\mathcal{B}) \quad \sum_{k=1}^{M} b^k(x,\eta,\zeta)\frac{\eta_k}{|\eta|} \geq -[(1+|\eta|^{\beta_2})\varphi_2(x) + (1+|\eta|^{\beta_3})\varphi_3(x)(|\zeta|^{p-2})^{\varepsilon}],$$

where $\nu > 0$ and φ_ι are functions in $L^{r_\iota}(\Omega)$, $\iota = 1,2,3$. The exponents ε, r_ι and β_ι, $\iota = 1,2,3$, are subject to the following inequalities:

(i) $0 \leq \varepsilon < \frac{N(p-1)+p}{Np}$ (or equivalently $(1-\varepsilon) > \frac{N-p}{Np}$).

(ii) $r_\iota > \frac{N}{p}$, for $\iota = 1,2$;
$r_3 > \frac{N}{p(1-\varepsilon)}$, if $\varepsilon \leq \frac{p-1}{p}$, and $r_3 > \frac{Np}{(1-\varepsilon)Np-(N-p)} > \frac{N}{p(1-\varepsilon)}$, if $\varepsilon > \frac{p-1}{p}$.

(iii) $0 \leq \beta_1 < q(1-\frac{1}{r_1})$; $0 \leq \beta_2 < q(1-\frac{1}{r_2}) - 1$; $0 \leq \beta_3 < q(1-\varepsilon-\frac{1}{r_3}) - 1$;
where $q = \frac{Np}{N-p}$.

Remark: We assume that the domain is smooth enough for the Sobolev imbedding theorem to hold. Hence we consider the case $2 \le p \le N$ only.

Since we are dealing with unbounded solutions, we have to specify the notion of a weak solution. In order to do so, let us first introduce the following notations. We write

$$(A(u),\varphi)_{\tilde\Omega} = \int_{\tilde\Omega} \sum_{k,\ell=1}^{M} \sum_{i,j=1}^{N} a_{ij}^{k\ell}(x,u,D(u))D^j u_\ell D^i \varphi_k dx,$$

and

$$(B(u),\varphi)_{\tilde\Omega} = \int_{\tilde\Omega} \sum_{k=1}^{M} b^k(x,u,D(u))\varphi_k dx,$$

for a measurable set $\tilde\Omega \subset \Omega$, whenever the integrals on the righthand sides are finite. We omit the index $\tilde\Omega$ in case $\tilde\Omega = \Omega$.

DEFINITION 1. A function u in $H^{1,1}(\Omega)$ is called a *distributional solution* of (E) if

(i) the functions $\sum_{\ell=1}^{M}\sum_{j=1}^{N} a_{ij}^{k\ell}(x,u,D(u))D^j u_\ell$ and $b^k(x,u,D(u))$ are in $L^1(\Omega)$ for all $i = 1,\cdots,N$; $k = 1,\cdots,M$;

(ii) $(A(u),\varphi) + (B(u),\varphi) = 0$, for all $\varphi \in C_0^\infty(\Omega)$.

A function u in $H^{1,p}(\Omega)$ is called a *weak solution* of (E) if

(iii) the functions $\sum_{\ell=1}^{M}\sum_{j=1}^{N} a_{ij}^{k\ell}(x,u,D(u))D^j u_\ell$ and $b^k(x,u,D(u))$ are in $L^{p'}(\Omega)$ and $L^1(\Omega)$ respectively, for all $i = 1,\cdots,N$; $k = 1,\cdots,M$; $p' = \frac{p}{p-1}$;

(iv) $(A(u),\varphi) + (B(u),\varphi) = 0$, for all φ in $H_0^{1,p}(\Omega) \cap L^\infty$.

Using $(\mathcal{B})$ an easy approximation argument shows that we can use a weak solution u as testfunction in iv). Hence we have

THEOREM 1. Suppose that $(\mathcal{A})$ and $(\mathcal{B})$ are satisfied for a number $p \ge 2$ and suppose that a weak solution u of (E) has finite norm in $L^\infty(\partial\Omega)$, then u is bounded in $L^\infty(\Omega)$. More precisely we have $\|u\|_{\infty,\Omega} \le C$, where C is depending on $p, N, M, |\Omega|, \|u\|_{\infty,\partial\Omega}, \|u\|_q, \varepsilon, \nu, \beta_\iota, \|\varphi_\iota\|_{r_\iota}$, $\iota = 1,2,3$.

Proof: Let u^θ be defined by $u^\theta = u$, if $|u| < \theta$, and by $u^\theta = \theta\frac{u}{|u|}$, if $|u| \ge \theta$, then $D^i(u_k^\theta) = \frac{\theta}{|u|}[D^i u_k - |u|^{-2}u_k \sum_{\ell=1}^{M} u_\ell D^i u_\ell]$ for $|u| \ge \theta$, cf. [6]. For $p \ge 2$ we have

$$|D(|u|)|^p \le (\sum_{i=1}^{N} |D^i u|^2 \cos^2\gamma_i)|D(u)|^{p-2} \le |D(u)|^p,$$

with $\cos\gamma_i = \sum_{\ell=1}^{M} u_\ell D^i u_\ell (|u||D^i u|)^{-1}$. For $|u| \geq \theta$ this yields

$$|D(|u|)|^p \leq |D(u)|^{p-2}[(1-\frac{\theta}{|u|})|D(u)|^2 + \frac{\theta}{|u|}\sum_{i=1}^{N}|D^i u|^2\cos^2\gamma_i]$$

$$\leq \tfrac{1}{\nu}[\sum_{k=1}^{M}\sum_{i,j=1}^{N} a_{ij}(x,u,D(u))D^j u_k D^i(u_k - u_k^\theta) + M(1+|u|^{\beta_1})\varphi_1].$$

On the other hand, for $|u| \geq \max\{\theta, 1\}$ we have

$$-B(x,u,D(u))\cdot(u-u^\theta) \leq \tfrac{1}{2\nu}(1-\frac{\theta}{|u|})|D(u)|^p + C[\varphi_2|u|^{\beta_2+1} + \varphi_3|u|^{\beta_3+1}]^{\frac{1}{1-\epsilon}}.$$

Let $P^\theta = \{x \in \Omega|\ |u(x)| > \theta\}$ are the level sets of $|u|$; testing the differential equation with $u - u^\theta$ we obtain

$$\int_{P^\theta} |D(|u|)|^p dx \leq \sum_{\iota=1}^{3}\int_{P^\theta} |u|^{\widetilde{\beta}_\iota}\psi_\iota dx,$$

for $\theta \geq \max\{1, \|u\|_{\infty,\partial\Omega}\}$; where $\widetilde{\beta}_1 = \beta_1$, $\widetilde{\beta}_2 = \beta_2 + 1$, $\widetilde{\beta}_3 = (\beta_3+1)\frac{1}{1-\varepsilon}$, and $\psi_\iota \in L^{s_\iota}$ with $s_\iota = r_\iota$, for $\iota = 1,2$ and $s_3 = r_3(1-\varepsilon)$. Hence we have

$$\int_{P^\theta} |u|^{\widetilde{\beta}_\iota}\psi_\iota dx \leq C\|u\|_q^{\alpha_\iota}\|\psi_\iota\|_{s_\iota}(\int_{P^\theta}|u|^{\ell_\iota}dx)^{\frac{p}{\ell_\iota}} \leq C[(\int_{P^\theta}(|u|-\theta)^\ell dx)^{\frac{p}{\ell}} + \theta^p|P^\theta|^{\frac{p}{\ell_\iota}}],$$

where $\alpha_\iota = \max_{\iota=1,2,3}\{0, \widetilde{\beta}_\iota - p\}$, $\ell_\iota = p(1 - \frac{1}{s_\iota} - \frac{\alpha_\iota}{q})^{-1}$, $0 < \ell_\iota < \frac{Np}{N-p}$, and $\ell = \max_{\iota=1,2,3}\{\ell_\iota\}$. Invoking Theorem 5.1 p. 73 in [4] we conclude that u is a bounded function as stated.

COROLLARY 1. Suppose that u is a distributional solution of (E), and suppose that $\|u\|_{\infty,\partial\Omega}$ and $\|u\|_p$ are bounded. Then $(\mathcal{A})$ and $(\mathcal{B})$ imply that $\|u\|_{\infty,\Omega}$ is bounded provided u and u^θ can be used as test functions for all $\theta \geq \|u\|_{\infty,\partial\Omega}$.

Remarks: 1) The fact that u can be used as a test function is very often provided by the existence theory, even for strongly nonlinear problems, cf. [7]. Furthermore this property also is crucial in the uniqueness theory, cf. [5] and the discussion in the next section.

2) In our context the bound from below used in [8] for the perturbation reads as follows:

$$\eta\cdot B(x,\eta,\xi) \geq -\widetilde{\nu}^*[|\xi|^p + C|\eta|^p + f],\ \text{ with } \widetilde{\nu}^* < \nu \text{ and } f \in L^{r_1}(\Omega).$$

An easy computation gives

$$(A(u), u-u^\theta) + (B(u), u-u^\theta) \geq (\nu - \widetilde{\nu}^*)\int_\Omega |D(u)|^{p-2}[(1-\frac{\theta}{|u|})|D(u)|^2$$

$$+\frac{\theta}{|u|}\sum_{i=1}^{N}|D^i u|^2\cos^2\gamma_i] - ((1+|u|^{\beta_1})\varphi_1 + (1+|u|^{\beta_1^*})\varphi_1^*)dx.$$

Since $q > p$ we have that φ_1^* and β_1^* are subject to the above restrictions on the data. Consequently this bound can be incorporated in our theory.

Examples: 1) In the case of $A(u) = -\Delta u$, we are able to admit perturbations $B(u)$ of the type $B(u)_k = |D(u)|^p g(u) u_k + C|D(u)|^{\widetilde{p}}$, $\widetilde{p} < \frac{N(p-1)+p}{N}$, provided g is a real valued function satisfying $g(\eta)\eta \cdot \eta \geq \rho > -1$ for some real number ρ.

2) Let us consider the functional $H(u) = \int_\Omega h(u)|D(u)|^p dx, \quad p \geq 2,$
where $h : \mathbb{R}^M \to \mathbb{R}$ is a continuously differentiable function. If a critical value $u \in H^{1,p}(\Omega)$ of the functional is bounded at the boundary then u is bounded in the whole domain, provided there is a positive number τ such that

$$\eta \cdot \text{grad } h(\eta) \geq -\tau \quad \text{and} \quad \inf\{h(\eta) | \; \eta \in \mathbb{R}^M\} > \tau.$$

The system of Euler-Lagrange equations of the functional H is given by

$$p \text{ div}(h(u)|D(u)|^{p-2} \text{grad } u_k) + |D(u)|^p h_k(u) = 0, \quad k = 1, \cdots, M,$$

with h_k denoting the partial derivative of h with respect to the k-th variable.

3) For examples in differential geometry we refer the reader to [8].

2. UNBOUNDED WEAK SOLUTIONS

Let us consider De Giorgi's functional

$$J(u) = \int_{B(0,R)} |D(u)|^2 + [a \text{ div } u + b\frac{x}{|x|} D(u) \frac{x^t}{|x|}]^2 dx,$$

where u is a function from $\mathbb{R}^N$ into $\mathbb{R}^N$, $N \geq 3$, and a and b are positive numbers. $B(0, R)$ is the ball with center 0 and radius R. Actually De Giorgi is dealing with $a = N - 2$ and $b = N$ in [1].

The system of Euler equations for this functional is given by

$$\begin{aligned} -L_{ab}(u)_k &= \Delta u_k + a \; D^k[a \text{ div } u + b\frac{x}{|x|} D(u) \frac{x^t}{|x|}] \\ &+ b \text{ div}[x_k \frac{x}{|x|^2}(a \text{ div } u + b\frac{x}{|x|} D(u) \frac{x^t}{|x|})] = 0, \quad k = 1, \cdots, N. \end{aligned} \tag{2.1}$$

Note that the coefficient functions are given by $a_{ij}^{k\ell} = \alpha_{ij}^{k\ell} + \beta_{ij}^{k\ell}$, where $\alpha_{ij}^{k\ell} = \delta_{ij}\delta_{k\ell}$, and $\beta_{ij}^{k\ell} = \chi_{ik}\chi_{j\ell}$, with $\chi_{j\ell} = a\delta_{j\ell} + b\frac{x_j x_\ell}{|x|^2}$, hence (2.1) can be written as

$$-L_{ab}(u)_k = \sum_{i,j,\ell=1}^N D^i(a_{ij}^{k\ell} D^j u_\ell) = 0, \; k = 1, \cdots, N.$$

We need the following generalization of the Lemma in De Giorgi's note.

LEMMA 2. Suppose that u is twice continuously differentiable and it is satisfying the Euler equation in $B(0,R)\backslash\{0\}$. If u is in $H^{1,\widetilde{p}}$ for $\widetilde{p} > \frac{N}{N-1}$, then u is a distributional solution of (2.1). If, in addition, u is in $H^{1,2}(\Omega)$, then u is a weak solution.

Proof: Let g be any test function in $C_0^\infty(B(0,R))$; as in [1] we consider the approximations $g_n(x) = (1-\rho(nx))g(x)$, where ρ is a smooth function identical one in $\mathbb{R}^N\backslash B(0,1)$ and identical zero in $B(0,\frac{1}{2})$ say. Since the sequence g_n is bounded in the spaces $H^{1,\widetilde{q}}(B(0,R))$ for any $\widetilde{q} < N$, the sequence $D(g_n)$ converges weakly to $D(g)$ in $L^{\widetilde{p}/\widetilde{p}-1}$, and hence u is a distributional solution. In case $u \in H^{1,2}(B(0,R))$ we obviously have that u is a weak solution.

THEOREM 2. Let $f:(0,R)\to\mathbb{R}$ be a twice differentiable function; then $u(x) = xf(|x|)$ is a solution of the Euler equation (2.1) in $B(0,R)\backslash\{0\}$ only if

$$f(\tau) = c_1\tau^{m_1} + c_2\tau^{m_2},$$

with $m_{1/2} = -\frac{1}{2}(N \mp \sqrt{N^2-4\beta})$, where $\beta = ((a+b)^2+1)^{-1}b(N-1)(b+Na)$ and c_1 and c_2 are constants in $\mathbb{R}$.

Remark: Note that we always have $N^2 - 4\beta > 0$, since the function $\mu(a) = N^2((a+b)^2+1) - 4\,b(N-1)(b+Na)$ has its minimum value N^2 at $a = b(1-\frac{2}{N})$.

Proof of Theorem 2. Straightforward differentiation yields

$$\begin{aligned}
&\Delta(x_k f(|x|)) = x_k[(N+1)f'(|x|)|x|^{-1} + f''(|x|)],\\
&\operatorname{div}(xf(|x|)) = Nf(|x|) + f'(|x|)|x|,\\
&\frac{x}{|x|}D(xf(|x|))\frac{x^t}{|x|} = \sum_{j,\ell=1}^{N}\frac{x_jx_\ell}{|x|^2}D^j(x_\ell f(|x|)) = f(|x|) + f'(|x|)|x|,\\
&D^k(\sigma f(|x|) + f'(|x|)|x|) = x_k[(\sigma+1)f'(|x|)|x|^{-1} + f''(|x|)],\\
&\operatorname{div}[x_k\frac{x}{|x|^2}(\sigma f(|x|) + f'(|x|)|x|)]\\
&= x_k[(N-1)\sigma f(|x|)|x|^{-2} + (N+\sigma)f'(|x|)|x|^{-1} + f''(|x|)].
\end{aligned}$$

Because of the Euler equation (2.1) we have
$0 = x_k[((a+b)^2+1)(f''(|x|) + (N+1)f'(|x|)|x|^{-1}) + b(N-1)(b+Na)f(|x|)|x|^{-2}]$, for $k = 1,\cdots,N$, and hence, as a necessary (and sufficient) condition on f, we obtain the Euler Differential Equation

$$f''(\tau) + \alpha\frac{f'(\tau)}{\tau} + \beta\frac{f(\tau)}{\tau^2} = 0,$$

with $\alpha = (N+1)$ and $\beta = ((a+b)^2+1)^{-1}b(N-1)(b+Na)$. Its characteristic equation is given by $m^2 + (\alpha-1)m + \beta = 0$, which proves Theorem 2.

As an immediate consequence of this proof and Lemma 2, we have

COROLLARY 2.

i) For $m_1 = -\frac{1}{2}(N - \sqrt{N^2 - 4\beta})$, the function $u(x) = x|x|^{m_1}$ is a weak solution of (3.1). If $\sqrt{N^2 - 4\beta} < N - 2$, then this weak solution is unbounded.

ii) For $m_2 = -\frac{1}{2}(N + \sqrt{N^2 - 4\beta})$, the function $u(x) = x|x|^{m_2}$ is a distributional solution of (2.1) if $\sqrt{N^2 - 4\beta} < N - 2$.

COROLLARY 3. Consider the Dirichlet problem

$$L_{ab}(u) = 0 \text{ in } B(0,R); \qquad u|_{\partial B(0,R)}(x) = x.$$

For $\sqrt{N^2 - 4\beta} < N - 2$, there is at least one unbounded weak solution given by

$$u(x) = x|x|^{m_1} R^{-m_1},$$

and there are infinitely many distributional solutions

$$v(x) = x(c_1|x|^{m_1} + c_2|x|^{m_2})(c_1 R^{m_1} + c_2 R^{m_2})^{-1}, \ c_1, c_2 \in \mathbb{R}.$$

Remarks: 1) We have $m_1 = -\frac{1}{2}(N - N(\sqrt{4(N-1)^2 + 1}\ )^{-1})$ for $a = N - 2$ and $b = N$. The function $u(x) = x|x|^{m_1}$ is the counter example presented by De Giorgi in [1].

2) Note that the nonuniqueness of the distributional solution is obtained for a linear operator satisfying $(L_{ab}(u) - L_{ab}(v), u - v) \geq \|D(u - v)\|_2^2$. Hence, it is crucial in the uniqueness theory to assume that the solution can be used as a test function. As the above example indicates, this assumption cannot be omitted without substitutional hypotheses.

Finally we point out that we can choose the nondiagonal part "small" compared to the Laplacian if the space dimension is large. Because of the equality $N^2 - 4\beta - (N-2)^2 = \frac{4(N-1)}{4b^2+1}[1 - b^2(N-3)]$, we have

LEMMA 3. Let $a = b$ and $N \geq 4$, then $\sqrt{N^2 - 4\beta} < N - 2$ for all b with $b > \frac{1}{\sqrt{N-3}}$.

Remarks. 1) Choosing $a = b = \frac{1}{\sqrt{N - \frac{7}{2}}}$, we have $|\beta_{ij}^{k\ell}(x)| \leq \frac{4}{N - \frac{7}{2}}$ for all $1 \leq k, \ell, i, j \leq N$.

2) Regarding $\Delta = (\delta_{ij}\delta_{k\ell})$ and $\beta(x) = (\beta_{ij}^{k\ell}(x))$ as vectors in $\mathbb{R}^{N^4}$ we have $|\Delta| = N$ and $|\beta(x)| \leq b^2(N+1)$, for $a = b$. Therefore the ratio $\frac{|\beta(x)|}{|\Delta|} \to 0$, as $N \to \infty$, if for example $a = b = \frac{1}{\sqrt{N - \frac{7}{2}}}$.

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