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SOME HOMOLOGICAL PROPERTIES OF COMMUTATIVE SEMITRIVIAL RING EXTENSIONS

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Let R be a commutative ring with 1 and M an R-module. If $\phi : M \otimes_R M \to R$ is an R-module homomorphism satisfying $\phi(m \otimes m') = \phi(m' \otimes m)$ and $\phi(m \otimes m')m'' = m\phi(m' \otimes m'')$, the additive abelian group $R \oplus M$ becomes a commutative ring, if multiplication is defined by $(r,m)(r',m') = (rr' + \phi(m \otimes m'), rm' + r'm)$. This ring is called the *semitrivial extension* of R by M and ϕ and it is denoted by $R\alpha_\phi M$. This generalizes the notion of a trivial extension and leads to a more interesting variety of examples. The purpose of this paper is to study $R\alpha_\phi M$; in particular, we are interested in some homological properties of $R\alpha_\phi M$ as that of being Cohen-Macaulay, Gorenstein or regular. A sample result: Let $(R, \mathbf{m})$ be a local Noetherian ring, M a finitely generated R-module and $\mathrm{Im}(\phi) \subseteq \mathbf{m}$. Then $R\alpha_\phi M$ is Gorenstein if and only if either $R\alpha M$ is Gorenstein or R is Gorenstein, M is a maximal Cohen-Macaulay module and $M \cong M^*$, where the isomorphism is given by the adjoint of ϕ.

1. INTRODUCTION

Recall that the trivial extension $R\alpha M$ of a commutative ring R by an R-module M is the additive abelian group $R \oplus M$ endowed with the multiplication $(r,m)(r',m') = (rr', r'm + rm')$ $(r, r' \in R, m, m' \in M)$. This definition is due to Nagata who introduced trivial extensions (under the name 'idealization') in order to be able to handle primary decomposition of both ideals and submodules in an unified way ([**Na**]). Since then trivial extensions have turned out to be quite useful tools in commutative algebra and they have been a succesful ingredient in many constructions. To mention a few examples, the Gorenstein rings with transcendental Poincaré series given by Bøgvad ([**Bø**]) are trivial extensions as well as the rings with finite λ-dimension n constructed by Roos ([**Ro**]); an other application is the construction of Buchsbaum rings with prescribed local cohomology ([**SV**]). The popularity of trivial extensions as a tool is partly explained by the precise and computable relation that there is between the homological properties of R and M and of $R\alpha M$. For example the Poincaré series of $R\alpha M$, $P_{R\alpha M}$, is a simple function of P_R and P_M.

The purpose of this paper is to study semitrivial extensions which are a natural generalization of trivial extensions (probably due to Reiten and Roos independently). The idea is to allow a nontrivial multiplication of the elements of the module M. This leads to the following definition:

1.1 **Definition:** Let R be a commutative ring with 1, M an R-module and let $\phi : M \otimes_R M \to R$ be an R-module homomorphism which is *symmetric* and *associative*, that is, which satisfies

$$\phi(m \otimes m') = \phi(m' \otimes m)$$

and (1.1)

$$m\,\phi(m' \otimes m'') = \phi(m \otimes m')\,m''$$

for all $m, m', m'' \in M$. Then the ring obtained from the (additive) abelian group $R \oplus M$ by defining the multiplication by

$$(r, m)(r', m') = (rr' + \phi(m \otimes m'), rm' + r'm)$$

$(r, r' \in R, m, m' \in M)$ is called the *semitrivial extension* of R by M and ϕ, and we shall denote it by $R\alpha_\phi M$.

It is clear that $R\alpha_\phi M$ is a commutative ring with unit-element (1,0). Trivial extensions correspond to the special case $\phi = 0$.

Not surprisingly the mixture of the module part and ring part that ϕ causes makes the relation between $R\alpha_\phi M$ and between R and M more subtle and complicated than in the case of a trivial extension. This makes semitrivial extensions more difficult to deal with but, on the other hand, leads to a more interesting variety of examples; for example, a semitrivial extension can be a domain or even regular.

The basic difficulty in constructing examples of semitrivial extensions is to find modules with associative homomorphisms. It is clear that the existence of such homomorphisms implies restrictions on the module (see lemma 2.1 and remark

2.2). Nevertheless, there are natural examples of semitrivial extensions. Let us give a few simple ones. First of all, we mention the obvious fact that a semitrivial extension is essentially the same thing as a $\mathbf{Z}/2\mathbf{Z}$-graded ring. Perhaps the simplest example of a (non-trivial) semitrivial extension is obtained by taking $M = R$ and an element $s \in R$ and putting $\phi = s\mu : r \otimes r' \mapsto srr'$. (Here as always in this paper μ denotes a multiplication which is obvious from the context.) It is easy to see that $R\alpha_\phi R$ is then isomorphic to $R[X]/(X^2 - s)$. Another example is provided by a graded ring: $R = \bigoplus_{n=0}^{\infty} R_n$ can be written as a semitrivial extension of its second Veronese subring,

$$R \cong R^{(2)}\alpha_\mu R^{(\mathrm{odd})} := (\bigoplus_{n=0}^{\infty} R_{2n})\,\alpha_\mu\,(\bigoplus_{n=0}^{\infty} R_{2n+1})\,. \tag{1.2}$$

Especially polynomial rings are semitrivial extensions in this way. Also powerseries rings can be decomposed analogously. Later we shall encounter less immediate examples (cf. also [**Va**]).

The definition of a semitrivial extension makes of course sense also in a non-commutative context and, indeed, most papers on them have been in this greater generality. We mention Palmér [**Pa**] and Garcia-Herreros Mantilla [**GHM**]. We shall however restrict our attention to the commutative case and so we make the convention that our rings are commutative and with a unit-element. All modules will be unitary.

The content of this paper is as follows: In chapter 2 we study some general properties of $R\alpha_\phi M$ (as that of being a domain or reduced). In chapter 3 follows a discussion on some, mostly well-known facts on trivial extensions. The main-part of this paper is devoted to the study of some homological properties of $R\alpha_\phi M$: we begin by studying when an Artinian semitrivial extension is Gorenstein (chapter 4). Then we move on to the local Noetherian case in chapter 5 and study the Cohen-Macaulay and Gorenstein properties — the main-tool here will be local cohomology. In chapter 6 we discuss regularity and we end with a few remarks on complete intersections and some open problems in chapter 7.

2. SOME GENERAL PROPERTIES

Let R be a ring (commutative and with 1) and M an R-module (unitary). Denote

$$\Phi(M) := \{\phi \in \mathrm{Hom}_R(M \otimes_R M, R) \mid \phi \text{ satisfies (1.1) } \}$$

It is obvious that $\Phi(M)$ is then an R-submodule of $(M\otimes M)^* := \mathrm{Hom}_R(M\otimes M, R)$.

The following observation is important:

2.1 **Lemma**: If $\Phi(M) \neq \{0\}$ and if M is free or torsion-free, then M is indecomposable.

Proof: Suppose that M is torsion-free and that $M = M' \oplus M''$ is a nontrivial decomposition. Let $m' \in M', m'' \in M''$ be nonzero elements. Then $\phi(m' \otimes m'')\, m' = \phi(m' \otimes m')\, m''$ and so $\phi = 0$. The case where M is free is similar. ∎

2.2 **Remarks**: (a) The fact that $\Phi(M) \neq \{0\}$ does not alone guarantee indecomposability of M. To see this let $R = \mathbf{Z}/4\mathbf{Z}$ and $M = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, where $\mathbf{Z}/2\mathbf{Z}$ is considered as an R-module in the natural way. It is easily seen that $\Phi(M) \neq \{0\}$ (cf. also (c) below).

(b) The existence of a nonzero $\phi \in \Phi(M)$ implies of course also other restrictions on M than (possibly) indecomposability: Let $\{m_i | i \in I\}$ be a system of generators of M and let $\phi \in \Phi(M)$. Write $r_{ij} = \phi(m_i \otimes m_j)$. Then we have by (1.1) that $r_{ij} m_k = r_{jk} m_i$ for all $i, j, k \in I$, so there should be 'many relations' between the generators of M. On the other hand, this gives also a way to construct modules M with $\Phi(M) \neq \{0\}$. Let $F = \bigoplus_{i \in I} Re_i$ be a free R-module with basis $(e_i)_{i\in I}$ and let $(r_{ij})_{i,j\in I}$ be a symmetric matrix where $r_{ij} \in R$ satisfy $r_{ij} r_{kl} = r_{ik} r_{jl}$ for all $i, j, k, l \in I$, i.e. all 2×2-minors should be 0 (this corresponds to associativity in (1.1)). Let $K \subseteq F$ be the submodule generated by all elements of the form $r_{ij} e_k - r_{jk} e_i$ and let $M = F/K$. Then we can define a homomorphism $\phi : M \otimes_R M \to R$ satisfying (1.1) by setting $\phi(m_i \otimes m_j) = r_{ij}$, where $m_i = e_i \bmod K$. If e.g. R is a local ring with maximal ideal $\mathbf{m}$ and $r_{ij} \in \mathbf{m}$ for all i, j, then M will certainly be nonzero by Nakayama's lemma.

(c) Let us give an example of the construction in (b). Let again $R = \mathbf{Z}/4\mathbf{Z}$ and let $F = \bigoplus_{i=1}^n Re_i$. Take (r_{ij}) to be the $n \times n$-matrix where $r_{ij} = 2\delta_{ij}$. Then the construction gives $M \cong (\mathbf{Z}/2\mathbf{Z})^n$ (and ϕ is given by $\phi(m_i \otimes m_j) = 2\delta_{ij}$).

2.3 **Definition**: A homomorphism $\phi \in \Phi(M)$ is called *diagonally squarefree* (*DSF*) if the equation $\phi(m \otimes m) = r^2$ $(m \in M, r \in R)$ implies that $m = 0$ and $r = 0$.

2.4 **Proposition**: Let $\phi \in \Phi(M)$. Then $R\alpha_\phi M$ is an integral domain if and only if the following conditions are satisfied:

(a) R is an integral domain;

(b) M is torsion-free;

(c) ϕ is DSF.

Proof: $(\Rightarrow)$: Suppose that $R\alpha_\phi M$ is a domain. Let $r, r' \in R, m \in M$.

(a) $r\,r' = 0 \;\Rightarrow\; (r,0)(r',0) = 0 \;\Rightarrow\; r = 0$ or $r' = 0$.

(b) $r\,m = 0 \;\Rightarrow\; (r,0)(0,m) = 0 \;\Rightarrow\; r = 0$ or $m = 0$.

(c) Suppose that $\phi(m \otimes m) = r^2$ for some $r \in R, m \in M$. Then $(r,m)\,(-r,m) = 0$, so that $r = 0, m = 0$.

$(\Leftarrow)$: Let $(r,m) \neq 0$ and suppose that $(r,m)\,(r',m') = 0$. Then

$$r\,r' + \phi(m \otimes m') = 0 \tag{2.1}$$

and

$$r\,m' + r'\,m = 0, \tag{2.2}$$

so that $r'\phi(m \otimes m) = \phi(r'\,m \otimes m) = -\phi(r\,m' \otimes m) = r'\,r^2$. If $r' \neq 0$, we see from (a) that $\phi(m \otimes m) = r^2$ and so $m = 0, r = 0$ by (c). Thus $r' = 0$. We claim that also $m' = 0$. Suppose not. Then, as $r\,m' = 0$ by (2.2) and M is torsion-free, $r = 0$. But by (2.1) $\phi(m \otimes m') = 0$, so

$$\begin{aligned} 0 &= \phi(m \otimes m')^2 \\ &= \phi(m \otimes \phi(m \otimes m')\,m') \\ &= \phi(m \otimes \phi(m' \otimes m')\,m) \\ &= \phi(m \otimes m)\,\phi(m' \otimes m'). \end{aligned}$$

As $\phi(m' \otimes m') \neq 0$ by (c), $\phi(m \otimes m) = 0$ and again by (c), $m = 0$. But this is impossible, so that $m' = 0$. ∎

2.5 **Proposition:** Suppose that $2r = 0 \;\Rightarrow\; r = 0$ in R. Then $R\alpha_\phi M$ is reduced if and only if R is reduced and $(\phi(m \otimes m) = 0 \;\Rightarrow\; m = 0)$.

Proof: $(\Rightarrow)$: obvious.

$(\Leftarrow)$: Suppose that $(r,m)^2 = 0$. Then $r^2 + \phi(m \otimes m) = 0$ and $2rm = 0$, so that $0 = 2\phi(rm \otimes m) = -2\,r^3$. Thus $2r = 0$ and by assumption also $r = 0$. Further, $\phi(m \otimes m) = 0$, so $m = 0$ also. ∎

2.6 **Remark:** The assumption $(2r = 0 \;\Rightarrow\; r = 0)$ cannot be dropped. Take $R = M = \mathbf{Z}/2\mathbf{Z}$ and $\phi =$ ordinary multiplication. Then $(1,1)^2 = 0$ in $R\alpha_\phi M$

but R is reduced and ϕ satisfies the requirement of proposition 2.5. Note that $R\alpha_\phi M \cong (\mathbf{Z}/2\mathbf{Z})[X]/(X^2-1)$; (1,1) corresponds to $\overline{X+1}$ in this isomorphism. A similar example is provided by $R = M = \mathbf{Z}/2p\mathbf{Z}$ where p is an odd prime; now $(p,p)^2 = 0$.

The following result is well-known:

2.7 **Proposition** ([**Pa**, prop. 1]): (a) $R\alpha_\phi M$ is Noetherian (resp. Artinian) if and only if R is Noetherian (resp. Artinian) and M is finitely generated (f.g. for short).

(b) $\dim(R\alpha_\phi M) = \dim(R)$ (where dim stands for Krull-dimension both in classical and in Gabriel-Rentschler sense). ∎

2.8 **Corollary**: $R\alpha_\phi M$ is a field if and only if R is a field, $M \cong R$ and $\phi(1\otimes 1) \neq r^2$ for all $r \in R$ (where we think of ϕ as a homomorphism $R \otimes R \to R$).

Proof: Fields are precisely Artinian integral domains. ∎

2.9 **Example**: Let $\phi : R \otimes_R R \to R$ be $r \otimes r' \mapsto 2rr'$, where $R = \mathbf{Q}$ or $R = \mathbf{R}$. Then

$$\mathbf{Q}\alpha_\phi\mathbf{Q} \cong \mathbf{Q}[\mathbf{X}]/(X^2-2) \cong \mathbf{Q}[\sqrt{2}]$$

is a field while $\mathbf{R}\alpha_\phi\mathbf{R}$ is not. Note also that $\mathbf{C} \cong \mathbf{R}\alpha_{-\mu}\mathbf{R}$ (where μ is the ordinary multiplication).

Suppose then that $(R, \mathbf{m})$ is a local ring, M a f.g. R-module and $\phi \in \Phi(M)$. If ϕ is an epimorphism an easy computation shows that it is in fact an isomorphism and that M is projective (cf. Bass, [**Bs**, theorem 3.4]). Hence M is free as R is local and by lemma 2.1 $M \cong R$. Then $R\alpha_\phi M \cong R[X]/(X^2 - r)$ for some (invertible) $r \in R$. Thus the interesting case occurs when ϕ is not onto, that is, when $\operatorname{Im}\phi \subseteq \mathbf{m}$. Clearly $R\alpha_\phi M$ is then a local ring with maximal ideal $\mathbf{m} \times M$ and $R\alpha_\phi M/\mathbf{m} \times M \cong R/\mathbf{m}$. For simplicity we introduce the notation

$$\Phi_{\mathbf{m}}(M) := \{\phi \in \Phi(M) \mid \operatorname{Im}(\phi) \subseteq \mathbf{m}\}$$

(for not necessarily f.g. M). If $\phi \in \Phi_{\mathbf{m}}(M)$, it is seen by induction on n that

$$\mathbf{m}^{2n} \times \mathbf{m}^{2n}M \subseteq (\mathbf{m} \times M)^{2n} \subseteq \mathbf{m}^n \times \mathbf{m}^n M \tag{2.3}$$

From this we see that the degrees of the Hilbert-Samuel polynomials of R and $R\alpha_\phi M$ coincide; thus we get an alternative proof for proposition 2.7(b) in the case when R is a local Noetherian ring and M is f.g.. (The original proof of Palmér uses Krull-dimension in Gabriel-Rentschler sense and works so for more general rings.)

The inclusions (2.3) give also bounds for the multiplicity of $R\alpha_\phi M$. Recall that if N is a f.g. module over a local Noetherian ring $(S, \mathbf{n})$, the multiplicity $\mathrm{e}(N)$ can be defined by $\mathrm{e}(N) := \lim_{n\to\infty} d!\, n^{-d}\mathrm{l}(N/\mathbf{n}^n N)$, where $d = \dim(S)$.

2.10 Proposition: Let $(R, \mathbf{m})$ be a d-dimensional local Noetherian ring, M a f.g. R-module and $\phi \in \Phi_{\mathbf{m}}(M)$. Then

$$\frac{1}{2^d}(\mathrm{e}(R) + \mathrm{e}(M)) \le \mathrm{e}(R\alpha_\phi M) \le \mathrm{e}(R) + \mathrm{e}(M).$$

Proof: From (2.3) we see that

$$\mathrm{l}(R/\mathbf{m}^{2n}) + \mathrm{l}(M/\mathbf{m}^{2n}M) \ge \mathrm{l}(R\alpha_\phi M/(\mathbf{m}\times M)^{2n}) \ge \mathrm{l}(R/\mathbf{m}^n) + \mathrm{l}(M/\mathbf{m}^n M).$$

This yields immediately the claim. ∎

Note that if $\mathrm{Im}(\phi) = \mathbf{m}$ (resp. $\phi = 0$) the lower (resp. the upper) bound is attained.

2.11 Proposition: Let $(R, \mathbf{m})$ be a local ring, M an R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then

$$\widehat{R\alpha_\phi M} \;\cong\; \widehat{R}\alpha_{\widehat{\phi}}\widehat{M}$$

where $\widehat{(\ )}$ denotes the respective maximal ideal adic completions and $\widehat{\phi}$ is induced by ϕ.

Proof: Let $\phi_k : M/\mathbf{m}^k M \otimes M/\mathbf{m}^k M \longrightarrow R/\mathbf{m}^k$, $(k \ge 1)$ be induced by ϕ, $\widehat{\phi}$ is then induced by the ϕ_k's. From (2.3) we get a sequence of epimorphisms

$$R\alpha_\phi M/\mathbf{m}^{2n}\times\mathbf{m}^{2n}M \twoheadrightarrow R\alpha_\phi M/(\mathbf{m}\times M)^{2n} \twoheadrightarrow R\alpha_\phi M/\mathbf{m}^n \times \mathbf{m}^n M\,.$$

As $R\alpha_\phi M/\mathbf{m}^k \times \mathbf{m}^k M \cong R/\mathbf{m}^k\alpha_{\phi_k}M/\mathbf{m}^k M$ ([**GHM**, p.11]), the inverse limit of left- and righthand terms equals $\widehat{R}\alpha_{\widehat{\phi}}\widehat{M}$. The claim follows as $\varprojlim R\alpha_\phi M/(\mathbf{m}\times M)^k \cong \widehat{R\alpha_\phi M}$. ∎

2.12 Corollary: Let R be a local Noetherian ring, M a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then $R\alpha_\phi M$ is complete if and only if R is complete.

Proof: ($\Rightarrow$): As $R\alpha_\phi M \cong \widehat{R\alpha_\phi M} \cong \widehat{R}\alpha_{\widehat{\phi}}\widehat{M}$ and $R \hookrightarrow \widehat{R}$, $M \hookrightarrow \widehat{M}$, R equals $\widehat{R}$.

($\Leftarrow$): As M is f.g., M is complete and so the claim follows from proposition 2.11. ∎

2.13 **Remark:** There is an isomorphism

$$(R\alpha_\phi M)[X] \cong R[X]\alpha_{\phi[X]}M[X]\,,$$

where $M[X] := M \otimes_R R[X]$ and $\phi[X] : M[X] \otimes_{R[X]} M[X] \to R[X]$ is defined by

$$\sum_{i=1}^{s} m_i x^i \otimes \sum_{j=1}^{t} m'_j x^j \mapsto \sum_{k=1}^{s+t}\Big(\sum_{i+j=k} \phi(m_i \otimes m'_j)\Big)x^k\,.$$

(It is clear that $\phi[X]$ is then associative.) A similar assertion is true also for powerseries rings.

3. TRIVIAL EXTENSIONS

The content of this chapter is mostly well-known. We shall recall some results of homological nature. Let R be a local Noetherian ring and M a f.g. R-module.

Here is a summary of some known facts:

3.1 **Facts:** (a) $R\alpha M$ is never regular.

(b) $R\alpha M$ is a hypersurface $\Longleftrightarrow$ R is regular and $M \cong R$.

(c) $R\alpha M$ is a complete intersection (CI) $\Longleftrightarrow$ R is a CI and $M \cong R$.

(d) $R\alpha M$ is Gorenstein $\Longleftrightarrow$ R is a Cohen-Macaulay ring which is a factor of a Gorenstein ring and M is a canonical module.

(e) $R\alpha M$ is CM $\Longleftrightarrow$ R is CM and M is a maximal CM-module.

Here (a) and (e) are obvious and (d) can be found in [**FGR**, theorem 5.6]; only (b) and (c) might deserve a comment.

Suppose first that R is a CI and $M \cong R$. Then $R\alpha M \cong R[X]/(X^2)$ is a CI. Suppose then that $R\alpha M$ is a CI. Let $p : R\alpha M \to R$, $(r,m) \mapsto r$ be the natural projection. It is obvious that it is *large*, that is, $p^* : \mathrm{Ext}^*_R(k,k) \to \mathrm{Ext}^*_{R\alpha M}(k,k)$ is injective (cf. [**Lev**, theorem 2.3]) As $R\alpha M$ is a CI, $\mathrm{Ext}^*_{R\alpha M}(k,k)$ is Noetherian and so its sub-Hopf-algebra $\mathrm{Ext}^*_R(k,k)$ is Noetherian too. Thus R is a CI ([**BH**, theorem A]). As M is canonical over R, $M \cong R$.

Clearly $\nu(R\alpha M) = \nu(R) + [M/\mathbf{m}M : k]$ (where ν stands for embeddingdimension). This means that if R is a CI, $\nu(R\alpha M) = \nu(R) + 1$. This gives (b).

3.2 **Remarks**: (a) Yamagishi has characterized those trivial extensions that are Buchsbaum or quasi-Buchsbaum (cf. [**Ya**, theorem (1.2)]).

(b) Lescot ([**Les**, chapter VII] has proved that if R is a (local Noetherian) Golod ring and M is a Golod module (an inert module in Lescot's terminology, [**Les**, def. 3.4]), then $R\alpha M$ is Golod. Conversely, if $R\alpha M$ is Golod, it has has been proved by Roos (unpublished) that R is Golod too. The question whether M must be Golod seems still to be open.

(c) $R\alpha_\phi M$ can be CI (resp. Golod) without R being CI (resp. Golod). Take e.g. $R = k[\![X,Y,Z]\!]^{(2)}, M = k[\![X,Y,Z]\!]^{(\mathrm{odd})}$ and ϕ = ordinary multiplication. Then obviously $R\alpha_\phi M \cong k[\![X,Y,Z]\!]$ (compare (1.2)), but R is neither CI nor Golod. (But in this example R is *Golod-attached* by a recent result of Backelin, see [**Bc**], where also terminology is explained.)

3.3 **Corollary**: Let R be a local CM-ring. Then the following conditions are equivalent:

(a) R is regular

(b) R is Golod and there exists a canonical R-module which is also Golod.

Proof: (a $\Rightarrow$ b): clear.

(b $\Leftarrow$ a): Suppose that the R-module Ω is both canonical and Golod. Then $R\alpha\Omega$ is Golod (by 3.2(b)) and Gorenstein, thus a hypersurface. But then R is regular. ∎

3.4 **Proposition**: Let R be a local CM-ring and M a (f.g.) maximal CM-module. Then $R\alpha M$ has a canonical module if and only if R has a canonical module.

Proof: Recall that a local CM-ring has a canonical module if and only if it is a factorring of a Gorenstein ring (cf. [**FGR**, ch. 5]). The implication from left to right follows from this.

($\Leftarrow$): Recall the following result which is a special case of [**HK**, Satz 5.12]. Let $f : A \to A'$ be a local homomorphism of local Noetherian rings. Suppose that A is CM, A' is finite over A and that $\dim(A) = \dim(A')$. Then, if Ω is a canonical A-module, $\mathrm{Hom}_A(A', \Omega)$ is canonical over A'. The claim follows by applying this to the inclusion $R \to R\alpha M$. ▮

We note that ($\Leftarrow$) in the above proof works also for $R\alpha_\phi M$ if $\mathrm{Im}(\phi) \subseteq \mathbf{m}$.

4. ARTINIAN GORENSTEIN RINGS

Let $(R, \mathbf{m}, k)$ be a local Artinian ring, M a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then $R\alpha_\phi M$ is a local Artinian ring too (see ch. 2). We shall now determine when it is Gorenstein. As $I_R(k)$, the injective envelope of k, is a canonical R-module, it follows from 3.1(d) that $R\alpha M$ is Gorenstein if and only if $M \cong I_R(k)$. Recall that a local Artinian ring is Gorenstein if and only if its socle is a onedimensional vectorspace over the residue class field. Now, an easy computation shows that

4.1 Lemma: $\mathrm{soc}(R\alpha_\phi M) = (\mathrm{soc}(R) \cap \mathrm{Ann}_R M) \times (\mathrm{soc}(M) \cap M^{\perp,\phi})$, where $M^{\perp,\phi} = \{m \in M | \phi(m \otimes M) = 0\}$. In particular, $\mathrm{soc}(R\alpha M) = (\mathrm{soc}(R) \cap \mathrm{Ann}_R M) \times \mathrm{soc}(M)$. ▮

(As usual, ϕ is said to be *non-degenerate* if $M^{\perp,\phi} = 0$.)

Therefore $|\mathrm{soc}(R\alpha_\phi M)| = 1$ if and only if either

$$\mathrm{soc}(R) \cap \mathrm{Ann}_R M = 0 \quad \text{and} \quad |\mathrm{soc}(M) \cap M^{\perp,\phi}| = 1 \tag{4.1}$$

or

$$\mathrm{soc}(M) \cap M^{\perp,\phi} = 0 \quad \text{and} \quad |\mathrm{soc}(R) \cap \mathrm{Ann}_R M| = 1. \tag{4.2}$$

As $\mathbf{m}$ is nilpotent, $\mathrm{soc}(N)$ is an essential submodule of N for any R-module N. Therefore the first equation in (4.1) (resp.(4.2)) is equivalent to '$\mathrm{Ann}_R M = 0$' (resp. '$M^{\perp,\phi} = 0$').

4.2 Lemma: (a) If $\mathrm{Ann}_R M = 0$, then $\mathrm{soc}(M) \subseteq M^{\perp,\phi}$.

(b) If $M^{\perp,\phi} = 0$, then $\mathrm{soc}(R) \subseteq \mathrm{Ann}_R M$.

Proof: (a) Suppose that $\mathrm{Ann}_R M = 0$ and let $m \in \mathrm{soc}(M)$. Then

$\phi(m \otimes M)M = \mathrm{Im}\phi \cdot m \subseteq \mathbf{m} \cdot m = 0$, so that $\phi(m \otimes M) = 0$.

(b) Suppose that $M^{\perp,\phi} = 0$. As $\mathrm{Im}\phi \subseteq \mathbf{m}$, we have that $0 =$

$\phi(M \otimes M)\mathrm{soc}(R) = \phi(M \otimes \mathrm{soc}(R)M)$. Hence $\mathrm{soc}(R)M = 0$. ▮

4.3 Theorem: Let $(R, \mathbf{m})$ be a local Artinian ring, M a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then $R\alpha_\phi M$ is Gorenstein if and only if either

(a) $M \cong I_R(k)$

or

(b) R is Gorenstein and ϕ is nondegenerate.

Proof: We have seen that $R\alpha_\phi M$ is Gorenstein if and only if either $\mathrm{Ann}_R M = 0$ and $|\mathrm{soc}(M)| = 1$ or $M^{\perp,\phi} = 0$ and $|\mathrm{soc}(R)| = 1$. The first of these possibilities occur precisely when $|\mathrm{soc}(R\alpha M)| = 1$, which gives (a). The second case corresponds to (b). ∎

4.4 **Examples:** (a) Let $R = k\alpha k^2$ where k is a field. Then R is a local Artinian ring which is not Gorenstein. It is easily seen that $R \cong k[x,y]/(x^2, xy, y^2)$. By e.g. [**GHM**, Satz 3.18] $I_R(k) \cong \mathrm{Hom}_k(R, I_k(k)) = \mathrm{Hom}_k(R,k)$. Thus $I_R(k) \cong kf \oplus kg \oplus kh$ where $f, g, h : R \to k$ are defined by $f(a+bx+cy) = a$, $g(a+bx+cy) = b$ and $h(a+bx+cy) = c$ $(a,b,c \in k)$, and x and y operate on $I_R(k)$ by $xg = f$, $yh = f$ and by zero otherwise. Then $I_R(k) \otimes_R I_R(k) = k^4$ (with trivial operations of x and y) with basis $\{g\otimes g, g\otimes h, h\otimes g, h\otimes h\}$. Therefore

$$\mathrm{Hom}_R(I_R(k) \otimes_R I_R(k), R) \cong (\mathrm{soc}(R))^4 = k^8.$$

It is easy to find homomorphisms that are associative and symmetric, for example ϕ defined by $\phi(g\otimes g) = x, \phi(g\otimes h) = \phi(h\otimes g) = \phi(h\otimes h) = 0$ is a member of $\Phi_{\mathbf{m}}(I_R(k))$. By 4.3(a) $R\alpha_\phi I_R(k)$ is then Gorenstein. Obviously the same construction works also for $k\alpha k^n$ for any $n \geq 2$.

(b) Let $(R, \mathbf{m}, k)$ be a local Artinian Gorenstein ring which is not a field (e.g. $R = k\alpha k$). Then there exists a monomorphism $j : k \hookrightarrow R$. Define ϕ to be the composition $k \otimes_R k \cong k \xrightarrow{j} R$. Then ϕ is associative and furthermore, as j is a monomorphism, ϕ is nondegenerate. By 4.3(b) $R\alpha_\phi k$ is Gorenstein.

(c) The rings constructed in 2.2(c) are Gorenstein.

4.5 **Remark:** Let R now be an arbitrary (not necessarily local) Artinian ring, M a f.g. R-module and $\phi \in \Phi(M)$. Then R can be written as a finite product of local Artinian rings $(R_i, \mathbf{m}_i)$, $i = 1, \ldots, n$, and $M = M_1 \times \cdots \times M_n$, where M_i is an R_i-module for each i. Let $j_i : M_i \hookrightarrow M$ be the inclusion and define $\phi_i = \phi \circ (j_i \otimes j_i) : M_i \otimes_{R_i} M_i \to R$. As $\mathrm{Im}(\phi_i) = \phi(M_i \otimes M_i) \subseteq \phi(R_i M \otimes M) \subseteq R_i$, we can consider ϕ_i as an R_i-homomorphism $M_i \otimes_{R_i} M_i \to R_i$ which obviously is associative. One checks easily that there is a ring-isomorphism

$$R\alpha_\phi M \cong \prod_{i=1}^{n} R_i \alpha_{\phi_i} M_i \,. \tag{4.3}$$

Order now the indexes so that ϕ_i is an epimorphism for $i = 1, \dots, k$ and that $\mathrm{Im}(\phi_i) \subseteq \mathbf{m}_i$ for $i = k+1, \dots, n$. Then (4.3) can be rewritten as

$$R\alpha_\phi M \cong \prod_{i=1}^{k} R_i[X]/(X^2 - r_i) \times \prod_{i=k+1}^{n} R_i \alpha_{\phi_i} M_i \,,$$

(where $r_i \in R_i$ for $i = 1, \dots, k$) and $R\alpha_\phi M$ is Gorenstein if and only if R_i is Gorenstein for $i = 1, \dots, k$ and $R_i\alpha_{\phi_i}M_i$ satisfies either (a) or (b) in theorem 4.3 for $i = k+1, \dots, n$.

5. LOCAL COHOMOLOGY AND APPLICATIONS

Let $(R, \mathbf{m})$ be a local Noetherian ring, M a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. We shall discuss the local cohomology of $R\alpha_\phi M$ and as an application determine when $R\alpha_\phi M$ is Cohen-Macaulay or Gorenstein. For the definition and basic properties of local cohomology we refer the reader to the notes of Grothendiecks seminar, [Gr], or to [**HK**, §4].

Recall the following change-of-rings lemma due to Grothendieck (cf. e.g. [**HK**, Lemma 4.11]). Suppose that $f : R \to S$ is a homomorphism of local Noetherian rings. Let $I \subseteq R$ be an ideal and let $J \subseteq S$ be the ideal generated by $f(I)$. If N is an S-module, write ${}_fN$ for N regarded as an R-module via f.

5.1 **Lemma**: There is an R-isomorphism $H^j_I({}_fN) \cong {}_fH^j_J(N)$ for all $j \in \mathbf{N}$ (where we have written H^j_I for $H^j_{V(I)}$ etc.). ▮

We apply this lemma to $i : R \to R\alpha_\phi M, r \mapsto (r, 0)$. Then

$$\begin{aligned} {}_iH^j_{\mathbf{m}\times M}(R\alpha_\phi M) &\cong {}_iH^j_{\mathbf{m}\times \mathbf{m}M}(R\alpha_\phi M) \qquad (\mathbf{m}\times\mathbf{m}M \text{ is } \mathbf{m}\times M\text{-primary}) \\ &\cong H^j_{\mathbf{m}}({}_i(R\alpha_\phi M)) \\ &\cong H^j_{\mathbf{m}}(R) \oplus H^j_{\mathbf{m}}(M)\,. \end{aligned}$$

Using the characterization of depth as the index of the lowest nonvanishing local cohomology module we get the next corollaries:

5.2 **Corollary**: $\mathrm{depth}(R\alpha_\phi M) = \min(\mathrm{depth}(R), \mathrm{depth}_R(M))$. ▮

5.3 **Corollary**: $R\alpha_\phi M$ is a CM-ring if and only if R is CM and M is a maximal CM-module. ▮

Recall that the *type* of a maximal CM-module N over a local CM-ring S, $\mathrm{r}(N)$, is defined by $\mathrm{r}(N) := \dim_k \mathrm{Ext}_S^{\dim(S)}(k, N)$. If $\underline{x}$ is a maximal S-sequence, it is well-known (and in any case easy to prove) that $\mathrm{r}(N) = \dim_k \mathrm{soc}(N/\underline{x}N)$. Using this we prove

5.4 **Proposition**: Let R be a local CM-ring, M a maximal CM-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then

$$\mathrm{r}(R\alpha_\phi M) \leq \mathrm{r}(R) + \mathrm{r}(M).$$

Proof: Let $x_1, \ldots, x_n$ be a maximal R-sequence (whence it is also a maximal M-sequence) and let $I = \langle x_1, \ldots, x_n \rangle \subseteq R$ be the ideal generated by it. Then $(x_1, 0), \ldots, (x_n, 0)$ is a maximal $R\alpha_\phi M$-sequence; let $J = \langle (x_1, 0), \ldots, (x_n, 0) \rangle \subseteq R\alpha_\phi M$. Obviously $J = I \oplus IM$ and by [**GHM**, p.11] $R\alpha_\phi M/J \cong (R/I)\alpha_{\bar{\phi}}(M/IM)$, where $\bar{\phi}$ is induced by ϕ. Then

$$\begin{aligned} \mathrm{r}(R\alpha_\phi M) &= \dim_k \mathrm{soc}((R/I)\alpha_{\bar{\phi}}(M/IM)) \\ &\leq \dim_k \mathrm{soc}(R/I) + \dim_k \mathrm{soc}(M/IM) = \mathrm{r}(R) + \mathrm{r}(M), \end{aligned}$$

where the inequality follows from lemma 4.1. ∎

We can now generalize theorem 4.3 to the Noetherian case.

5.5 **Theorem**: Let R be a local Noetherian ring, M a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then $R\alpha_\phi M$ is Gorenstein if and only if

(a) $R\alpha M$ is Gorenstein

or

(b) R is Gorenstein, M is a maximal CM-module and ϕ is *regular*.

Here we define:

5.6 **Definition**: Let $I \subseteq R$ be an ideal. We say that $\phi : M \otimes M \to R$ is *I-regular* if $\phi(m \otimes M) \subseteq I$ implies that $m \in IM$; ϕ is said to be *regular* if it is I-regular for every ideal I generated by a maximal $R \oplus M$-sequence.

Proof: In the situation of the theorem R is CM and M is a maximal CM-module. Let I, J and $\bar{\phi}$ be as in the proof of proposition 5.4. As I is $\mathbf{m}$-primary, R/I is Artinian and we see from theorem 4.3 that $(R/I)\alpha_{\bar{\phi}}(M/IM)$ is Gorenstein if and only if either $(R/I)\alpha(M/IM)$ is Gorenstein or if R/I is Gorenstein and $\bar{\phi}$ is nondegenerate. But I and J are generated by regular sequences, so

$(R/I)\alpha_{\bar\phi}(M/IM)$ (resp. $(R/I)\alpha(M/IM)$, R/I) is Gorenstein if and only if $R\alpha_\phi M$ (resp. $R\alpha M$, R) is Gorenstein. Furthermore, $\bar\phi$ is nondegenerate if and only if ϕ is I-regular. ∎

From examples in 4.4 together with remark 2.13 we find examples of both (a) and (b) in the theorem.

5.7 **Remark**: Let R be a local Gorenstein ring and M a maximal CM-module over R. It is clear from the above proof that a $\phi \in \Phi(M)$ is then regular if and only if it is I-regular for *some* I generated by a maximal $R \oplus M$-sequence.

Let us then characterize regularity of a homomorphism $\phi : M \otimes M \to R$ in terms of $\phi^a : M \to M^*$, where ϕ^a (the adjoint of ϕ) is defined by $\phi^a(m)(m') = \phi(m \otimes m')$.

5.8 **Theorem**: Let R be a local Gorenstein ring and M a maximal CM-module over R. Then an R-homomorphism $\phi : M \otimes M \to R$ is regular if and only if ϕ^a is an isomorphism.

Proof: ($\Leftarrow$): Suppose that ϕ^a is an isomorphism. Let $I \subseteq R$ be an ideal generated by a maximal R-sequence. Denote $\overline{R} := R/I, \overline{M} := M/IM$ and let $\bar\phi : \overline{M} \otimes_{\overline{R}} \overline{M} \to \overline{R}$ and $\overline{\phi^a} : \overline{M} \to \overline{M^*}$ be the induced homomorphisms. As R is Gorenstein, M is a maximal CM-module and I is generated by a regular sequence, there is an isomorphism $\overline{M^*} \cong \mathrm{Hom}_{\overline{R}}(\overline{M}, \overline{R})$ (see [**HK**, Lemma 6.5]). Thus $\overline{\phi^a}$ can be identified with $(\bar\phi)^a$. As ϕ^a is an isomorphism, so is also $(\bar\phi)^a = \overline{\phi^a}$. Suppose now that $\phi(m \otimes M) \subseteq I$. Then $\bar\phi(\bar m \otimes \overline{M}) = 0$ so that $(\bar\phi)^a(\bar m) = 0$ and thus $\bar m = 0$, i.e. $m \in IM$.

($\Rightarrow$): Suppose that ϕ is regular.

(i): ϕ^a is mono: Suppose that $\phi^a(m) = 0$. Then $\phi(m \otimes M) \subseteq I$ for every ideal I and so $m \in IM$ for every I generated by a regular sequence. But for any n there exixts a regular sequence in $\mathbf{m}^n$. Thus $m = 0$ by Krull's intersection theorem.

(ii): ϕ^a is epi: Let $Q = \mathrm{Coker}(\phi^a)$ so that the sequence

$$0 \longrightarrow M \xrightarrow{\phi^a} M^* \xrightarrow{p} Q \longrightarrow 0 \tag{5.1}$$

is exact. Dualizing we get an exact sequence

$$0 \longrightarrow Q^* \longrightarrow M^{**} \xrightarrow{(\phi^a)^*} M^* \longrightarrow \mathrm{Ext}^1_R(Q, R) \longrightarrow \mathrm{Ext}^1_R(M, R) \longrightarrow \dots \tag{5.2}$$

As R is Gorenstein and M is a maximal CM-module, $\mathrm{Ext}_R^1(M, R) = 0$. Furthermore, the canonical homomorphism (evaluation) $\alpha : M \to M^{**}$ is an isomorphism ([**HK**, Kor. 6.8]).

5.9 **Lemma**: The following diagram is commutative.

$$\begin{tikzcd} & M \arrow[dl, "\alpha"'] \arrow[dr, "\phi^a"] & \\ M^{**} \arrow[rr, "(\phi^a)^*"] & & M^* \end{tikzcd}$$

Proof: Let $m, m' \in M$. Then $\alpha(m)$ is given by $g \mapsto g(m)$ $(g \in M^*)$ and $((\phi^a)^* \circ \alpha)(m) = \alpha(m) \circ \phi^a : M \to R$. Thus $((\phi^a)^* \circ \alpha)(m)(m') = (\alpha(m)(\phi^a))(m') = \phi^a(m)(m')$. ∎

From (5.2) we see now that

$$0 \longrightarrow Q^* \longrightarrow M \xrightarrow{\phi^a} M^* \longrightarrow \mathrm{Ext}_R^1(Q, R) \longrightarrow 0$$

is exact. Thus by (5.1) $Q^* = 0$. We show next that $\dim Q \leq n - 1$ (where $n = \dim R$). Note that $\mathrm{Supp}(Q) \subseteq \mathrm{Supp}(M^*) = \mathrm{Supp}(M)$. Let $\mathbf{p} \in \mathrm{Supp}(M)$ be a minimal prime (then $\mathrm{ht}(\mathbf{p}) = 0$). Then $0 = (Q^*)_{\mathbf{p}} \cong \mathrm{Hom}_{R_{\mathbf{p}}}(Q_{\mathbf{p}}, R_{\mathbf{p}})$. As $R_{\mathbf{p}}$ is Artinian Gorenstein, $Q_{\mathbf{p}} \cong \mathrm{Hom}_{R_{\mathbf{p}}}(\mathrm{Hom}_{R_{\mathbf{p}}}(Q_{\mathbf{p}}, R_{\mathbf{p}}), R_{\mathbf{p}}) = 0$ by Matlis-duality. This shows that every $\mathbf{p} \in \mathrm{Supp}(Q)$ is of height ≥ 1 and the assertion on dimension is proved.

Claim: $Q = 0$.

Proof: Suppose that $Q \neq 0$. Then Q is by the above argument and (5.1) an $(n-1)$-dimensional (CM-)module. Hence there exists an $x \in \mathrm{Ann}(Q)$ such that x is R-regular. Let $f \in M^*$. Then $p(xf) = 0 \in Q$ so that $xf = \phi^a(m)$ for some $m \in M$. Thus $\phi(m \otimes M) \subseteq \langle x \rangle$ and as ϕ is regular, $m \in xM$ (compare the proof of injectivity of ϕ), i.e. $m = x\hat{m}$ for some $\hat{m} \in M$. But then $f = \phi^a(\hat{m})$ and ϕ^a is onto. This contradiction shows that $Q = 0$. ∎

5.10 **Remark**: Let R be a local Gorenstein ring and let $\mathcal{MC}$ be the category of isomorphism classes of maximal CM-modules over R. Then $\mathbf{Z}/2\mathbf{Z} = \{1, g\}$ operates on $\mathcal{MC}$ by $gM = M^*$ (see [**HK**, Kor. 6.8]). Theorem 5.8 characterizes the fixed points of this operation (note that we did not assume that ϕ was associative in 5.8).

Theorems 5.5 and 5.8 combined give

5.11 Corollary: Let R be a local Noetherian ring, M a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then $R\alpha_\phi M$ is Gorenstein if and only if either

(a) $R\alpha M$ is Gorenstein

or

(b) R is Gorenstein, M is a maximal CM-module and $\phi^a : M \to M^*$ is an isomorphism. ∎

Next we shall examine more closely the $R\alpha_\phi M$-module structure of the local cohomology modules $H^j_{\mathbf{m}\times M}(R\alpha_\phi M)$.

5.12 Proposition: $H^0_{\mathbf{m}\times M}(R\alpha_\phi M) = H^0_{\mathbf{m}}(R)\alpha_\phi H^0_{\mathbf{m}}(M)$.
(where we have written ϕ also for $\phi|H^0_{\mathbf{m}}(M)$)

Proof: From (2.3) we get that

$$\begin{aligned} H^0_{\mathbf{m}\times M}(R\alpha_\phi M) &= \{(x,m) \in R\alpha_\phi M \mid \exists n \in \mathbf{N} \text{ s.t. } (\mathbf{m}\times M)^n(x,m) = 0\} \\ &= \{(x,m) \in R\alpha_\phi M \mid \exists n \in \mathbf{N} \text{ s.t. } \mathbf{m}^n \times \mathbf{m}^n M \cdot (x,m) = 0\} \\ &= \{(x,m) \in R\alpha_\phi M \mid \exists n \in \mathbf{N} \text{ s.t. } \mathbf{m}^n x = 0 \text{ and } \mathbf{m}^n m = 0\} \\ &= H^0_{\mathbf{m}}(R)\alpha_\phi H^0_{\mathbf{m}}(M). \ ∎ \end{aligned}$$

To study the H^i's for $i > 0$ we must recall how to compute them (see e.g. [**HK**, §4] for details). Let R be a local Noetherian ring and let $\underline{x} = (x_1, \ldots, x_s)$ be a sequence of elements of R. Write $\underline{x}^\nu = (x_1^\nu, \ldots, x_s^\nu)$. Let $K_*(\underline{x}; R)$ be the ordinary Koszul-complex associated to $\underline{x}$ and put

$$K^*(\underline{x}; M) := \mathrm{Hom}_R(K_*(\underline{x}; R), M)$$

and

$$H^i(\underline{x}; M) := H^i(K^*(\underline{x}; M)).$$

Finally define

$$H^i_{\underline{x}}(M) := \varinjlim H^i(\underline{x}^\nu; M).$$

(For details on this direct system we refer to [**HK**].) One can show that $H^i_{\underline{x}}(M)$ depends only on $V(\underline{x}) = \{\mathbf{p} \in \mathrm{Spec}(R) \mid x_i \in \mathbf{p} \text{ for all } i\}$ and so one writes $H^i_{V(\underline{x})}(M) := H^i_{\underline{x}}(M)$.

Let then $x_1, \ldots, x_s$ be a system of generators for $\mathbf{m}$. Then the $\mathbf{m}\times M$-primary ideal $\mathbf{m} \times \mathbf{m}M \subseteq R\alpha_\phi M$ is generated by $(\underline{x}, 0) := ((x_1, 0), \ldots, (x_s, 0))$. There is an isomorphism of complexes

$$0 \longrightarrow R\alpha_\phi M \xrightarrow{\mu_{(x_i,0)}} R\alpha_\phi M \longrightarrow 0$$

$$\cong (R\alpha_\phi M) \otimes_R (0 \longrightarrow R \xrightarrow{\mu_{x_i}} R \longrightarrow 0),$$

which implies that

$$K_*((\underline{x}, 0); R\alpha_\phi M) \cong (R\alpha_\phi M) \otimes K_*(\underline{x}; R)$$

and so

$$\begin{aligned}\mathrm{Hom}_{R\alpha_\phi M}(K_*((\underline{x},0); R\alpha_\phi M),\, R\alpha_\phi M) &\cong \mathrm{Hom}_{R\alpha_\phi M}(K_*(\underline{x}; R) \otimes_R R\alpha_\phi M,\, R\alpha_\phi M) \\ &\cong \mathrm{Hom}_R(K_*(\underline{x}; R),\, R\alpha_\phi M).\end{aligned}$$

As an R-module this is isomorphic to $K^*(\underline{x}; R) \oplus K^*(\underline{x}; M)$ and we recover our earlier result that there is an R-isomorphism

$$H^j_{\mathbf{m}\times M}(R\alpha_\phi M) \cong H^j_{\mathbf{m}}(R) \oplus H^j_{\mathbf{m}}(M). \tag{5.3}$$

Let $(x, y) \in H^j_{\mathbf{m}}(R) \oplus H^j_{\mathbf{m}}(M)$. Then

x is represented by a $\tilde{x} \in H^j(\underline{x}^\nu; R)$,

y is represented by a $\tilde{y} \in H^j(\underline{x}^\nu; M)$

(where $\nu \in \mathbf{N}$) and

$\tilde{x}$ is represented by a $f \in \mathrm{Hom}_R(K^\nu_j, R)$,

$\tilde{y}$ is represented by a $g \in \mathrm{Hom}_R(K^\nu_j, M)$,

where $K^\nu_j = K_j(\underline{x}^\nu; R)$.

Write $x = [f]$, $y = [g]$ and $(x, y) = [(f, g)]$. Then

$$\begin{aligned}(r, m)(x, y) &= [(r, m)(f, g)] \\ &= [(rf + \phi(m \otimes g), rg + f \cdot m)],\end{aligned} \tag{5.4}$$

where $\phi(m \otimes g) : K^\nu_j \to R$ is defined by $t \mapsto \phi(m \otimes g(t))$ $(t \in K^\nu_j)$ and $f \cdot m : K^\nu_j \to M$ by $t \mapsto f(t)m$. It is rather obvious that (5.4) is independent of the choise of representatives.

Suppose now that $\dim(R\alpha_\phi M) = n$. Then $\dim R = n$ also and as $H^n_{\mathbf{m}}(\cdot)$ is rightexact, there is an isomorphism

$$H^n_{\mathbf{m}}(M) \cong M \otimes H^n_{\mathbf{m}}(R)$$

so that

$$H^n_{\mathbf{m}}(R) \oplus H^n_{\mathbf{m}}(M) \cong (R \oplus M) \otimes_R H^n_{\mathbf{m}}(R)\,. \tag{5.5}$$

Let U be an R-module. Then the rule $T : U \mapsto (R\alpha_\phi M) \otimes_R U$ defines a functor from the category of R-modules to the category of $R\alpha_\phi M$-modules, $T : \mathbf{Mod}_R \to \mathbf{Mod}_{R\alpha_\phi M}$. As an R-module $T(U)$ is isomorphic to $U \oplus M \otimes_R U$; if we make this identification, $R\alpha_\phi M$-modulestructure of $T(U)$ is given by

$$(r, m)(u, m' \otimes u') = (ru + \phi(m \otimes m')u', rm' \otimes u + m \otimes u) \tag{5.6}$$

$(u, u' \in U,\ m' \in M)$. Formulas (5.3) and (5.5) suggest that there might be an isomorphism of $R\alpha_\phi M$-modules

$$H^n_{\mathbf{m}\times M}(R\alpha_\phi M) \cong T(H^n_{\mathbf{m}}(R))\,. \tag{5.7}$$

We claim that this is indeed the case. Make again the identification $T(U) = U \oplus M \otimes_R U$ and take $(x, m' \otimes z) \in H^n_{\mathbf{m}}(R) \oplus M \otimes_R H^n_{\mathbf{m}}(R)$. Let x and z be represented by $f, h \in \operatorname{Hom}_R(K^\nu_n, R)$ as above. Then (5.4) gives that

$$\begin{aligned}(r, m)(x, m' \otimes z) &= [(rf + \phi(m \otimes m'h), rm' \otimes h + m \cdot f)] \\ &= (rx + \phi(m \otimes m')z, rm' \otimes z + m \otimes x)\,,\end{aligned}$$

which is precisely (5.6). Thus (5.7) is valid.

5.13 **Proposition:** Suppose that $\dim(R\alpha_\phi M) = n$. Then the following diagram is commutative upto isomorphism:

$$\begin{tikzcd} \mathbf{Mod}_R \arrow[r, "T"] \arrow[d, "H^n_{\mathbf{m}}(\cdot)"] & \mathbf{Mod}_{R\alpha_\phi M} \arrow[d, "H^n_{\mathbf{m}\times M}(\cdot)"] \\ \mathbf{Mod}_R \arrow[r, "T"] & \mathbf{Mod}_{R\alpha_\phi M} \end{tikzcd}$$

Proof: Both $T(H^n_{\mathbf{m}}(\cdot))$ and $H^n_{\mathbf{m}\times M}(T(\cdot))$ are rightexact functors that preserve direct sums. Thus by Watts' theorem it is enough to check that $T(H^n_{\mathbf{m}}(R)) \cong H^n_{\mathbf{m}\times M}(T(R))$. But we have already done this in (5.7). ∎

6. REGULARITY

Let R be a local Noetherian ring, M a f.g. R-module ($M \neq 0$) and let $\phi \in \Phi_{\mathbf{m}}(M)$. We shall now discuss the regularity of $R\alpha_\phi M$. The 0-dimensional case is already settled in corollary 2.8. The following result is a direct consequence of a theorem of Palmér:

6.1 **Proposition** : Suppose that $\dim(R) = 1$. Then $R\alpha_\phi M$ is regular if and only if R is regular, $M \cong R$ and ϕ (thought of as a homomorphism $R \otimes_R R \to R$) is of the form $r \otimes r' \mapsto xrr'$ for some $x \in \mathbf{m} \setminus \mathbf{m}^2$.

Proof: ($\Leftarrow$): Clear.

($\Rightarrow$): Suppose that $R\alpha_\phi M$ is 1-dimensional and regular. By [**Pa**, theorem 3] R is then regular, $R/\mathrm{Im}(\phi)$ is regular and M is projective. As R is local, M is free and by lemma 2.1 $M \cong R$. If we identify R and M we can write ϕ as $r \otimes r' \mapsto xrr'$ for some $x \in \mathbf{m}$. Then $R/\mathrm{Im}(\phi)$ is regular if and only if $x \notin \mathbf{m}^2$. ∎

For higher dimensions the situation is more complicated. For example, R need not to be Gorenstein even if $R\alpha_\phi M$ is regular. Here is a fundamental example.

6.2 **Example:** Let $S = k[\![x,y,z,w]\!]$, where k is a field. Then S can be decomposed as a semitrivial extension in four essentially different ways (where μ denotes always ordinary multiplication):

$$\begin{aligned} S &\cong k[\![x,y,z,w]\!]^{(2)}\alpha_\mu k[\![x,y,z,w]\!]^{(\mathrm{odd})} && (i)\\ &\cong k[\![x,y,z]\!]^{(2)}[\![w]\!]\,\alpha_\mu\, k[\![x,y,z]\!]^{(\mathrm{odd})}[\![w]\!] && (ii)\\ &\cong k[\![x,y]\!]^{(2)}[\![z,w]\!]\,\alpha_\mu\, k[\![x,y]\!]^{(\mathrm{odd})}[\![z,w]\!] && (iii)\\ &\cong k[\![x]\!]^{(2)}[\![y,z,w]\!]\,\alpha_\mu\, k[\![x]\!]^{(\mathrm{odd})}[\![y,z,w]\!]\,. && (iv)\end{aligned}$$

(Actually we should write $\mu[\![w]\!]$ for μ in (ii) etc., compare remark 2.13) Here R is (i) Gorenstein, but not CI; (ii) not Gorenstein (as 2 does not divide 3); (iii) a hypersurface and (iv) regular.

6.3 **Proposition:** Suppose that $\dim(R) \geq 1$. Then, if $\mathrm{Im}(\phi) \subseteq \mathbf{m}^2$, $R\alpha_\phi M$ is not regular.

Proof: Let $n = \dim(R) = \dim(R\alpha_\phi M)$. If $\mathrm{Im}(\phi) \subseteq \mathbf{m}^2$, we can evaluate the embeddingdimension of $R\alpha_\phi M$ as follows:

$$\begin{aligned}\nu(R\alpha_\phi M) &= |\mathbf{m} \times M/(\mathbf{m} \times M)^2| = |\mathbf{m}/\mathbf{m}^2 + \mathrm{Im}(\phi)| + |M/\mathbf{m}M|\\ &\geq |\mathbf{m}/\mathbf{m}^2| + 1 > \dim(R) = \dim(R\alpha_\phi M)\,. \ ∎\end{aligned}$$

6.4 **Proposition**: Suppose that $\dim(R) = n \geq 1$. If $R\alpha_\phi M$ is regular, then

$$\nu(R) \in \left\{n + \frac{j(j-1)}{2}\right\}_{j=1}^{n} .$$

Proof: (*i*): If $\mathrm{Im}(\phi) = \mathbf{m}$, then

$$\begin{aligned}\binom{n+1}{2} &= |(\mathbf{m} \times M)^2/(\mathbf{m} \times M)^3| \\ &= |\mathbf{m}^2 + \mathrm{Im}(\phi)/\mathbf{m}^3 + \mathbf{m}\,\mathrm{Im}(\phi)| + |\mathbf{m}M/(\mathbf{m}^2 + \mathrm{Im}(\phi))M| \\ &= |\mathbf{m}/\mathbf{m}^2| = \nu(R)\,.\end{aligned}$$

This gives the case $j = n$.

(*ii*): Suppose that $\mathrm{Im}(\phi) \neq \mathbf{m}$. Then there exists an $x \in \mathbf{m} \setminus (\mathbf{m}^2 + \mathrm{Im}(\phi))$, so that $(x, 0) \in (\mathbf{m} \times M) \setminus (\mathbf{m} \times M)^2$. Then $R\alpha_\phi M/\langle(x,0)\rangle \cong R/\langle x\rangle\alpha_{\bar{\phi}}M/xM$ is regular of dimension $n-1$. Induction gives that

$$\nu(R/\langle x\rangle) \in \left\{n - 1 + \frac{j(j-1)}{2}\right\}_{j=1}^{n-1} .$$

Thus $\nu(R) \in \left\{n + \frac{j(j-1)}{2}\right\}_{j=1}^{n-1}$. Finally note that the case $n = 1$ was proved in proposition 5.1. ∎

6.5 **Remark**: All possible values of $\nu(R)$ can be realized: write $k[\![x_1, \ldots, x_n]\!]$ as a semitrivial extension in n ways as we did above for $n = 4$. Then

$$\nu(k[\![x_1, \ldots, x_j]\!]^{(2)}[\![x_{j+1}, \ldots x_n]\!]) = n + \frac{j(j-1)}{2}\,.$$

Our next result shows that $R\alpha_\phi M$ is not regular so often.

6.6 **Proposition**: Let $(R, \mathbf{m}, k)$ be an n-dimensional ($n \geq 1$) local Noetherian ring which contains a field. Suppose that $\mathrm{char}(k) \neq 2$. Let M be a f.g. R-module and let $\phi \in \Phi_{\mathbf{m}}(M)$. Then, if $R\alpha_\phi M$ is regular, the $\mathbf{m}$-adic completions $\widehat{R}$ and $\widehat{M}$ occur (upto isomorphism) in the following list:

(1) $\widehat{R} = k[\![x_1, \ldots, x_n]\!]^{(2)}, \quad \widehat{M} = k[\![x_1, \ldots, x_n]\!]^{(\mathrm{odd})}$;

(2) $\widehat{R} = k[\![x_1, \ldots, x_{n-1}]\!]^{(2)}[\![x_n]\!], \quad \widehat{M} = k[\![x_1, \ldots, x_{n-1}]\!]^{(\mathrm{odd})}[\![x_n]\!]$;

$\vdots$

(n) $\widehat{R} = k[\![x_1]\!]^{(2)}[\![x_2, \ldots, x_n]\!], \quad \widehat{M} = k[\![x_1]\!]^{(\mathrm{odd})}[\![x_2, \ldots, x_n]\!]$.

Proof: Suppose that $R\alpha_\phi M$ is regular. As $\widehat{R}\alpha_{\hat{\phi}}\widehat{M} \cong \widehat{R\alpha_\phi M} \cong k[\![x_1,\ldots,x_n]\!]$ by Cohen's structuretheorem, it suffices to study decompositions of $k[\![x_1,\ldots,x_n]\!]$ as a semitrivial extension. Let $G = \mathbf{Z}/2\mathbf{Z} = \{1,g\}$. Then G operates on $\widehat{R}\alpha_{\hat{\phi}}\widehat{M}$ by $g\,(r,m) = (r,-m)$. This operation is obviously k-linear and it is compatible with the multiplication in $\widehat{R}\alpha_{\hat{\phi}}\widehat{M}$. Clearly

$$\widehat{R} \cong (\widehat{R}\alpha_{\hat{\phi}}\widehat{M})^G$$

and

$$\widehat{M} \cong \{x \in \widehat{R}\alpha_{\hat{\phi}}\widehat{M} \mid x = -g\,x\}\,.$$

As the operation of g is determined by the $g(x_i)$'s ($i = 1,\ldots,n$), this operation corresponds to a unique k-linear homomorphism $L_g : V \to V$, where V is the k-vectorspace generated by $x_1,\ldots,x_n$. In a suitable basis of V L_g is given by a diagonal matrix M_g. As $(M_g)^2 = I_{n\times n}$ ($n \times n$-identity matrix), M_g must after a permutation of the basis-elements be of the form

$$M_g = \mathbf{diag}(\overbrace{-1,\ldots,-1}^{t\ \text{times}},\ \overbrace{1,\ldots,1}^{n-t\ \text{times}}).$$

Then $\widehat{R} \cong k[\![x_1,\ldots,x_t]\!]^{(2)}[\![x_{t+1},\ldots,x_n]\!]$ and $\widehat{M} \cong k[\![x_1,\ldots,x_t]\!]^{(\text{odd})}[\![x_{t+1},\ldots,x_n]\!]$. ∎

7. SOME CONCLUDING REMARKS

Let again R be a local Noetherian ring, M a f.g. R-module and let $0 \neq \phi \in \Phi_{\mathbf{m}}(M)$. We begin with a few simple observations.

7.1 **Proposition**: If $R\alpha_\phi M$ is a hypersurface and $\mathrm{Im}(\phi) \subseteq \mathbf{m}^2$, then R is regular and $M \cong R$.

Proof: Let $\dim(R) = n$. Then

$$\begin{aligned} n+1 = \nu(R\alpha_\phi M) &= |\mathbf{m}/\mathbf{m}^2 + \mathrm{Im}(\phi)| + |M/\mathbf{m}M| \\ &\geq |\mathbf{m}/\mathbf{m}^2| + 1\,. \end{aligned}$$

Thus $\nu(R) = |\mathbf{m}/\mathbf{m}^2| = n$ and R is regular. As M is a maximal CM-module over a regular ring M must be free, and thus by lemma 2.1 $M \cong R$. ∎

7.2 Proposition: There exists an $n = n(R) \in \mathbf{N}$ such that if $R\alpha_\phi M$ is a CI and $\mathrm{Im}(\phi) \subseteq \mathbf{m}^n$, then R is a CI too.

Proof: Recall that there exists an $n \in \mathbf{N}$ such that the projection $R \to R/\mathbf{m}^s$ is Golod for all $s \geq n$ (see [**Lev**$_2$]), and in particular small. If $\mathrm{Im}(\phi) \subseteq \mathbf{m}^n$, $R \twoheadrightarrow R/\mathrm{Im}(\phi)$ is small too. As $R \twoheadrightarrow R/\mathrm{Im}(\phi)$ can be decomposed as $R \hookrightarrow R\alpha_\phi M \overset{p}{\twoheadrightarrow} R/\mathrm{Im}(\phi)$ (where $p : (r,m) \mapsto r + \mathrm{Im}(\phi)$), $R \hookrightarrow R\alpha_\phi M$ is small. Thus by [**BH**, theorem A] $R\alpha_\phi M$ CI $\Longrightarrow$ $\mathrm{Ext}^*_{R\alpha_\phi M}(k,k)$ Noetherian $\Longrightarrow$ $\mathrm{Ext}^*_R(k,k)$ Noetherian $\Longrightarrow$ R CI. ▮

7.3 Example: (a) Let $s \geq 3$. As $k[\![x_1,\ldots,x_s]\!]^{(2)}$ is not a CI, we can conclude from proposition 7.2 that

$$k[\![x_1,\ldots,x_s]\!]^{(2)} \, \alpha_{x_1^{2n}\cdot\mu} k[\![x_1,\ldots,x_s]\!]^{(\mathrm{odd})}$$

is not a CI for $n \gg 0$.

(b) There are a lot of examples of semitrivial extensions that are CI's: let $\underline{f} := f_1,\ldots,f_t$ be a regular $k[\![x_1,\ldots,x_n]\!]$-sequence such that either $\{\underline{f}\} \subseteq k[\![\,\underline{x}\,]\!]^{(2)}$ or $\{\underline{f}\} \subseteq k[\![\,\underline{x}\,]\!]^{(\mathrm{odd})}$. Then $R := k[\![\,\underline{x}\,]\!]/\langle\underline{f}\rangle$ can be written as a semitrivial extension; for example in the first case

$$R \cong k[\![\,\underline{x}\,]\!]^{(2)}/\langle\underline{f}\rangle \; \alpha_\mu k[\![\,\underline{x}\,]\!]^{(\mathrm{odd})}/\langle\underline{f}\rangle \cdot k[\![\,\underline{x}\,]\!]^{(\mathrm{odd})},$$

where $\langle\underline{f}\rangle$ denotes the ideal that the f_i's generate in $k[\![\,\underline{x}\,]\!]^{(2)}$.

Proposition 6.6 suggests that there might a numerable list of possible R and M that can occur in a CI $R\alpha_\phi M$ in the case where R is complete, $\mathrm{char}(k) \neq 2$ and R contains a field.

We end with the following problems which seem to be quite difficult.

7.4 Problem: Characterize those semitrivial extensions $R\alpha_\phi M$ that are (say)

(a) hypersurfaces;

(b) complete intersections;

(c) Golod-rings or

(d) Buchsbaum.

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REFERENCES

[Bc] Backelin, J.: *Some homological properties of 'high' Veronese subrings*, Reports 12:1988, Department of mathematics, University of Stockholm

[Bs] Bass, H.: *Algebraic K-theory*, W.A.Benjamin, New York, 1968

[Bø] Bøgvad, R.: *Gorenstein rings with transcendental Poincaré series*, Math. Scand., 53 (1983), pp. 5 - 15

[BH] Bøgvad, R. and S. Halperin: *On a conjecture of Roos*, in SLN 1183 (1986), pp. 120 - 127

[FGR] Fossum, R., P. Griffith and I. Reiten: *Trivial extensions of abelian categories*, SLN 456, (1975)

[GHM] Garcia-Herreros Mantilla, E.: *Semitriviale Erweiterungen und generalisierte Matrizenringe*, Alg.Berichte, 54 (1986)

[Gr] Grothendieck, A.: *Local cohomology*, SLN 41 (1967)

[HK] Herzog, J. and E. Kunz: *Der Kanonische Modul eines Cohen-Macaulay Rings*, SLN 238, (1971)

[Les] Lescot, J.: *Sur la serie de Poincaré d'un module*, Reports 19:1983, Department of mathematics, University of Stockholm

[Lev] Levin, G.: *Large homomorphisms*, Math. Scand., 46, (1980), pp. 209 - 215

[Lev_2] Levin, G.: *Poincaré series of modules over local rings*, Proc.Am.Math.Soc, 72, (1978), pp. 6 - 10

[Na] Nagata, M.: *Local Rings*, Interscience, 1962

[Pa] Palmér, I.: *The global homological dimension of semitrivial extensions of rings*, Math.Scand. 37, (1975), pp. 223 - 256

[Ro] Roos, J.E.: *Finiteness conditions in commutative algebra and solution of a problem of Vasconcelos*, in Commutative Algebra, Durham 1981, ed. by R. Sharp, London Math. Soc. Lecture Notes Series, Vol. 72, 1982, pp. 179 - 203

[SV] Stückrad, J. and W. Vogel: *Buchsbaum rings and applications*, Springer, 1986

[Va] Valtonen, E.: *Gorenstein rings and tensorproducts of semitrivial ring extensions*, manuscript, 1988

[Ya] Yamagishi, K.: *Quasi-Buchsbaum rings obtained by idealization*, in Study of Buchsbaum rings and generalized Cohen-Macaulay rings, Research Inst. for Math. Sciences, Kyoto (1982), pp. 183 - 189

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