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# ON THE KÄHLER GEOMETRY OF THE HILBERT-SCHMIDT GRASSMANNIAN

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*We compute the Riemann curvature tensor of the Hilbert-Schmidt Grassmannian with respect to its natural Kähler structure. The sectional curvature is shown to be non-negative. We also discuss the Kähler structure of the Hilbert-Schmidt space of almost complex structures whose sectional curvature is shown to be non-positive.*

**1. Introduction.** - Fischer and Tromba [4-6,13] have provided a new *a priori* Riemannian approach to the theory of Teichmüller space which allows a quick presentation of the basic facts of its global geometry. In this note, we point out that their formalism also applies to another geometrically meaningful space, namely to the Hilbert-Schmidt Grassmannian.

The Hilbert-Schmidt Grassmannian manifold (also known as the Sato Grassmannian [11]) has recently been amply studied both in the context of completely integrable systems [9] and in Connes' non-commutative differential geometry [3] where the analogous object is referred to as a 2-summable Fredholm module. The Grassmannian manifold also is one of the key notions of string theory [1]. We shall follow the conventions of [10] where the cohomology of the Grassmannian was computed.

Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a graded separable complex Hilbert space where the grading is defined by an involution $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with eigenspaces $\mathcal{H}^+$ and $\mathcal{H}^-$. One can think of $\mathcal{H}$ as the Hilbert space of square integrable complex functions on the circle and $\varepsilon$ as the Toeplitz involution, i.e., the sign of Fourier modes, say, with the convention $\varepsilon(1) = +1$.

Recall [12] that the Schatten class $\mathcal{L}^p(\mathcal{H})$ for $1 \leq p < \infty$ consists of all compact operators $A$ such that

$$\|A\|_p \stackrel{def}{=} \left(\operatorname{tr}(A^*A)^{p/2}\right)^{1/p} < \infty$$

where the trace is the sum of eigenvalues. The $\mathcal{L}^p(\mathcal{H})$ are then Banach spaces with norms $\|.\|_p$ and ideals in $\mathcal{L}(\mathcal{H})$, the von Neumann algebra of bounded operators of $\mathcal{H}$. For $p = 1$, $\mathcal{L}^p(\mathcal{H})$ consists of trace class operators, and one has the trace map tr : $\mathcal{L}(\mathcal{H}) \longrightarrow \mathbf{C}$ which is con-

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tinuous and linear, hence smooth. For $p = 2$, it is the Hilbert space of Hilbert-Schmidt operators with the inner product $\mathrm{tr}(AB^*)$.

<u>DEFINITION 1.1</u>. *The Grassmannian* $\mathcal{G}r^p = \mathcal{G}r^p(\mathcal{H}, \varepsilon)$ *is the set of bounded seld-adjoint involutions of the Hilbert space* $\mathcal{H}$ *congruent to the grading* $\varepsilon$ *modulo the Schatten ideal* $\mathcal{L}^p(\mathcal{H})$, $1 \leq p < \infty$.

We emphasize that $\mathcal{G}r^p$ depends on the choice of $\varepsilon$.

In [10] it was shown that $\mathcal{G}r^p$ is a Banach manifold whose tangent space at a generic involution $F \in \mathcal{G}r^p$ is isomorphic to the Banach space of self-adjoint operators in $\mathcal{L}^p(\mathcal{H})$ that anticommute with $F$. The anticommutativity $F\dot{F} + \dot{F}F = 0$ results, of course, from formally differentiating the involution property $F^2 = 1$. Algebraically, anticommutativity is simpler to handle than the corresponding identity obtained by differentiating a parametrized family of projectors. Therefore the involution point of view of the Grassmannian is best adapted for our purposes.

To make contact with [3], note that the non-commutative differential of $F$

$$F \mapsto i\{\varepsilon, F\} = i(\varepsilon F - F\varepsilon) \tag{1.2}$$

defines a smooth vector field on $\mathcal{G}r^p$. Indeed, to see that $i\{\varepsilon, F\} \in \mathcal{L}^p(\mathcal{H})$ write

$$\varepsilon F - F\varepsilon = (\varepsilon - F)(\varepsilon + F).$$

Then check for anticommutativity. Conceptually, (1.2) can be understood as a non-commutative differentiation of the involution property.

The curly bracket was used above for the algebraic commutator of operators. We shall henceforward observe this convention and reserve the hooked bracket for the bracket of vector fields $X, Y$ on $\mathcal{G}r^p$. Denoting the Fréchet derivative on $\mathcal{G}r^p$ by $D$ we define

$$[X, Y] = DY(X) - DX(Y).$$

By abuse of notation, we shall never distinguish between tangent vectors $X, Y, Z$ at a point $F \in \mathcal{G}r^p$ and their extensions to vector fields in a neighbourhood of $F$.

An almost complex structure $\mathcal{J}$ on $\mathcal{G}r^p$ for any $p$ is given by

$$\mathcal{J}_F(X) = iFX.$$

Indeed,

$$\mathcal{J}_F(X)F = iFXF = -iX = -F\mathcal{J}_F(X)$$

and

$$(iFX)^* = -iXF = iFX$$

so $\mathcal{J}_F$ is an endomorphism of the tangent space of $\mathcal{G}r^p$ at $F$ and, moreover, $(\mathcal{J}_F)^2 = -1$.

To prove that $\mathcal{J}$ is integrable we may appeal to an infinite-dimensional version of the Newlander-Nirenberg theorem [8]. As in finite dimensions, the integrability obstruction is the non-vanishing of the Nijenhuis tensor. The formal integrability of the Nijenhuis tensor can be established by a direct computation just like in [5]. Note carefully that the proof in [8] only works in the case of a real-analytic Banach manifold. However, the real-analyticity of $\mathcal{G}r^p$ can easily be checked using the explicit charts given in [10]. Thus, all the Grassmannians $\mathcal{G}r^p$ are complex Banach manifolds.

Moreover, $\mathcal{G}r^2$ is a complex Hilbert manifold with respect to the Hermitian metric

$$\langle X, Y\rangle_F = \operatorname{tr} XY.$$

The corresponding Kähler 2-form (symplectic form) $\omega$ on $\mathcal{G}r^2$ reads

$$\omega_F(X,Y) = \langle \mathcal{J}X, Y\rangle_F = i \operatorname{tr} FXY.$$

This is, of course, the same Kähler structure as that described in [9] and the complex structure is explicited by the Plücker coordinates [9]. To show directly that $\omega$ is closed, one may compute just like in [6].

**2. The Levi-Civita connection on $\mathcal{G}r^2$.** - In the sequel, we shall compute the curvature of the Hilbert-Schmidt Grassmannian $\mathcal{G}r^2$. Pioneering curvature computations of infinite-dimensional Kähler manifolds were recently undertaken by Freed [7] who studied the loop group and Bowick and Rajeev [2] who found the Ricci curvature of $\mathit{Diff}\, S^1/S^1$. Our computation is more elementary as no regularizations intervene. It was inspired by Tromba's computation of the Weil-Petersson curvature of Teichmüller space [13].

To construct the Levi-Civita connection on $\mathcal{G}r^2$ we start from the Fréchet derivative in $\mathcal{L}(\mathcal{H})$ which we project to the tangent space of the submanifold $\mathcal{G}r^2$. The intrinsic projector is given by the first (anticommuting) part of the decomposition

$$Z = \frac{1}{2}(Z - FZF) + \frac{1}{2}(Z + FZF).$$

Therefore, the Levi-Civita connection $\nabla$ on $\mathcal{G}r^2$ will equal

$$\nabla_Y X = \frac{1}{2}(DX(Y) - FDX(Y)F). \tag{2.1}$$

Differentiate the anticommutation relation

$$XF + FX = 0$$

in the direction $Y$ to find

$$DX(Y)F + XY + YX + FDX(Y) = 0. \tag{2.2}$$

Solving (2.2) for (2.1) we find

$$\nabla_Y X = DX(Y) + \frac{1}{2}F(XY + YX). \tag{2.3}$$

The latter term in (2.3) (extrinsic projection) is the second fundamental form.

The reader can now easily check that $\nabla$ is, indeed, torsion-free and annihilates both $\langle .,. \rangle$ and $\mathcal{J}$.

**3. The Riemann curvature tensor of $\mathcal{G}r^2$.** - By definition, the curvature is given by

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z.$$

One readily computes from (2.3)

$$\begin{aligned}\nabla_X \nabla_Y Z &= D^2 Z(X,Y) + DZ(DY(X)) \\ &+ \frac{1}{2}( FDZ(X)Y + FZDY(X) + FDY(X)Z \\ &+ FYDZ(X) + FXDZ(Y) + FDZ(Y)X ) \\ &+ \frac{1}{4}( XZY + XYZ + ZYX + YZX ).\end{aligned}$$

We obtain $\nabla_Y \nabla_X Z$ by changing $X$ and $Y$ while the bracket term yields

$$\nabla_{[X,Y]} Z = DZ([X,Y]) + \frac{1}{2}F(Z[X,Y] + [X,Y]Z).$$

Putting everything together we find:

THEOREM 3.1.

$$R(X,Y)Z = \frac{1}{4}\{\{X,Y\},Z\}$$

Thus the algebraic Bianchi identity is just the Jacobi identity for the operator bracket.

The sectional curvature $K$ is given by

$$K(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} = \frac{1}{2}\frac{\operatorname{tr}(X^2Y^2) - \operatorname{tr}(XY)^2}{\operatorname{tr} X^2 \operatorname{tr} Y^2 - (\operatorname{tr} XY)^2}. \tag{3.2}$$

In finite dimensions this convention gives $+1$ for the curvature of the unit sphere.

THEOREM 3.3. *The sectional curvature of the Hilbert-Schmidt Grassmannian is non-negative, but not strictly.*

PROOF: To see this we decompose $X, Y \in T_F \mathcal{G}r^2$ into block matrices with respect to the graduation determined by $F$. Using anticommutativity and self-adjointness, we find

$$X = \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & B \\ \overline{B} & 0 \end{pmatrix}$$

for some Hilbert-Schmidt operators $A, B$.

We then compute

$$K(X,Y) = \frac{1}{2} \frac{\operatorname{tr}(A\overline{B} - \overline{A}B)^2}{\operatorname{tr} X^2 \operatorname{tr} Y^2 - (\operatorname{tr} XY)^2} \geq 0.$$

If $X$ is a tangent vector at $F$, so are the odd powers of $X$ because of anticommutativity. Take for instance $Y = X^3$. From (3.2) we read off $K(X,Y) = 0$. More generally, the lower bound 0 is attained on any tangent 2-plane generated by algebraically commuting tangent vectors $X, Y$, i.e., when $\{X, Y\} = 0$.

For the holomorphic sectional curvature $H$ we find

$$H(X) = K(X, \mathcal{J}X) = \frac{\operatorname{tr} X^4}{(\operatorname{tr} X^2)^2}.$$

## 4. The Hilbert-Schmidt space of almost complex structures as a Kähler manifold.

- Let $\mathcal{H}$ be a separable complex Hilbert space as before. Besides the Grassmannians, the spaces of almost complex structures of $\mathcal{H}$ also have some importance and are discussed at some length in [9].

Let us fix an almost complex structure $\eta$ of $\mathcal{H}$, that is an endomorphism of $\mathcal{H}$ such that $\eta^2 = -1$. Thus, $\eta$ has the eigenvalues $\pm i$. The corresponding eigenspaces give a grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and $\eta$ can be decomposed as $\eta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

DEFINITION 4.1. *The $L^p$ space of almost complex structures $\mathcal{A}^p = \mathcal{A}^p(\mathcal{H}, \eta)$ is the set of bounded self-adjoint operators of the Hilbert space $\mathcal{H}$ congruent to $\eta$ modulo the Schatten ideal $\mathcal{L}^p(\mathcal{H})$, $1 \leq p < \infty$.*

Just like the Grassmannians the spaces of almost complex structures $\mathcal{A}^p$ turn out to be complex Banach manifolds whose tangent space at a generic almost complex structure $G \in \mathcal{A}^p$ is isomorphic to the Banach space of self-adjoint operators $X$ in $\mathcal{L}^p(\mathcal{H})$ that anticommute with $G$. The complex structure of $\mathcal{A}^p$ is then given by

$$\mathcal{J}_G(X) = GX.$$

In analogy with $\mathcal{G}r^2$, the Hilbert-Schmidt space of almost complex structures $\mathcal{A}^2$ is a Kähler manifold with the symplectic form

$$\omega_G(X,Y) = \mathrm{tr}\, GXY.$$

Indeed, the reader can easily check that switching from involutions to almost complex structures merely amounts to some changes of signs in the foregoing discussion. The Levi-Civita connection is found to equal

$$\nabla_Y X = DX(Y) - \frac{1}{2}G(XY + YX)$$

whence:

<u>THEOREM 4.2</u>. *The Riemann curvature tensor of $\mathcal{A}^2$ is*

$$R(X,Y)Z = -\frac{1}{4}\{\{X,Y\},Z\}.$$

The computation leading to this is formally the same as the one appearing in the first few pages of [13].

<u>THEOREM 4.3</u>. *The sectional curvature of the Hilbert-Schmidt space of almost complex structures is non-positive, but not strictly.*

The proof is similar to that of the theorem 3.3.

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