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ALGORITHMETICAL ASPECTS OF THE PROBLEM OF CLASSIFYING MULTI-PROJECTIONS OF VERONESIAN VARIETIES

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It is shown that in order to check the arithmetical Cohen-Macaulayness or Buchsbaumness of projections of Veronesian varieties one only needs finitely many operations. A practical criterion for a class of projections of one-dimensional Veronesian varieties to have these properties is given. As a consequence, we obtain an upper bound for the difference between the Buchsbaum invariant and the length of the semigroup ideal for the Buchsbaum projections.

1. Introduction

This paper is a case study of a problem on the boundary between commutative algebra and computer algebra. The question is that of deciding the Cohen-Macaulayness or Buchsbaumness of multi-projections of Veronesian varieties.

An r-dimensional Veronesian variety V is a projective variety over a field k given parametrically by the set $M_{r,d}$ of all monomials of degree d in r indeterminates (say, $t_1,\dots,t_r$) for some integers $r \geq 2$, $d \geq 1$. By a projection W of V we understand a projective variety over k given parametrically by a subset M_W of $M_{r,d}$.

In his fundamental paper [4] Gröbner showed that the defining prime ideal of V is always perfect, i.e. the homogeneous coordinate ring of V is a Cohen-Macaulay ring, but certain projections of V have imperfect defining prime ideals. In view of this phenomenon he then posed the problem of classifying projections of Veronesian varieties.

For simple, double or triple projections of V whose sets of parameters are obtained from $M_{r,d}$ by deleting respectively one, two or three monomials, Gröbner's problem found a satisfactory solution in [8], [9] and [5] where one showed, in terms of the deleted monomials, which ones of these projections are arithmetically Cohen-Macaulay or arithmetically Buchsbaum, i.e. the local ring of the vertex of its affine cone is Cohen-Macaulay or Buchsbaum, respectively. For one-dimensional Veronesian varieties there is some effort to solve this problem in [1] and [10].

The aim of this paper is to solve this problem from the algorithmetical viewpoint for the simplicial case, i.e. for all projections whose sets of parameters contain all monomials $t_1^d, t_2^d, \dots, t_r^d$ (see Section 2). Namely, we shall show that to check whether such a projection W is arithmetically Cohen-Macaulay or arithmetically Buchsbaum or not one only needs finitely many operations (Theorem 2.2). This one is able to do, because in this case a general combinatorial topological criterion of [11] for an affine semigroup ring to be Cohen-Macaulay turns out only a combinatorial criterion. This result generalizes a result of [1] and [10] for one-dimensional Veronesian varieties. But the number of operations is still too large. Even for the one-dimensional case the arithmetical conditions in [1] or in [10] are very complicated, because following Bresinsky one must define the Apery sequence of a numerical semigroup. In Section 3 we will give a practical criterion for projections of one-dimensional Veronesian varieties whose sets of parameters contain $t_1^{d-1}t_2$ to be arithmetically Cohen-Macaulay or arithmetically Buchsbaum. As a consequence, for such a arithmetically Buchsbaum projection we can give some bound for the difference between the Buchsbaum invariant and the so-called length of its associated semigroup ideal.

The following notation will be used throughout. Let Z be the set of integers. If p, q are integers and $p \leq q$, $[p,q]$ denotes the set of integers n with $p \leq n \leq q$. If A, B are subsets of Z^r we denote by $A \pm B$ the sets $\{a \pm b,\ a \in A \text{ and } b \in B\}$.

2. Simplicial case

Let W be an arbitrary projection of a Veronesian variety V. Let M_W be its associated set of parameters. Put

$$I_W = \{(\alpha_1,\ldots,\alpha_r) \in N^r \ ;\ t_1^{\alpha_1}\ldots t_r^{\alpha_r} \in M_W\},$$

where N denotes the set of non-negative integers. Then one can identify the homogeneous coordinate ring of W with the semigroup ring of the affine semigroup H_W generated by I_W in N^r. We call H_W the associated affine semigroup of W. If no confusion can arise we shall drop the index W in the notation M_W, I_W etc.

Recall that an affine semigroup H in N^r (i.e. a finitely generated semigroup) is called simplicial if there exist elements $f_1,\ldots,f_n$ in H such that

(i) $f_1,\ldots,f_n$ are linearly independent over Q,

(ii) For all $f \in H$ there exist non-negative integers $m > 0$ and $m_1,\ldots,m_n$ such that $mf = m_1f_1 + \ldots + m_nf_n$. (See [3]).

In this paper we consider only the projections whose associated affine semigroups are simplicial. For $i = 1,\ldots,r$, let

$$e_i = (\ \ldots,d,\ldots),$$

where d is in the i-th place and the lined points denote the zero components. Then, by using a suitable Hochster-transformation as in [6], we can always reduce to the case $e_i \in I_W$ for every $i = 1,\ldots,r$. Without loss of generality we may therefore assume that $e_1,\ldots,e_r \in I_W$. Let

$$\bar{H} = \{e \in Z(H) \ ;\ me \in H \text{ for some integer } m > 0\},$$

where Z(H) denotes the additive group generated by H in Z^r. For a positive integer p, put

$$H_p = \{e \in \bar{H} \; ; \; e + pe_i \in H \text{ and } e + pe_j \in H \text{ for some } 1 \leq i,j \leq r\}.$$

If $e = ([e]_1,\ldots,[e]_r) \in \bar{H}$, then $\delta(e) = ([e]_1+\ldots+[e]_r)/($ will be called the <u>degree</u> of e. Note that $\delta(e) = 1$ if $e \in I$. Let c denote the number of elements of I. Then we have

<u>LEMMA 2.1</u>. <u>Let</u> e <u>be</u> <u>an</u> <u>arbitrary</u> <u>element</u> <u>of</u> H_p. <u>Then</u>

$$e = h + e' ,$$

<u>for</u> <u>some</u> $h \in H$ <u>and</u> $e' \in H_p$ <u>with</u> $\delta(e') < 2cd-1$.

<u>PROOF</u>. We prove this lemma by induction on the degree of $e \in H_p$. If $\delta(e) < 2cd-1$ there is nothing to prove. Assume $e \in H_p$ and $\delta(e) \geq 2cd-1$. By the definition of H_p there exist $i \neq j$, $1 \leq i,j \leq r$ such that

$$e + pe_i = \sum_{a \in I} m_a a , \quad (1)$$

and

$$e + pe_j = \sum_{a \in I} n_a a , \quad (2)$$

where m_a, $n_a \in N$. Put

$$L_1 = \{l \; ; \; [a]_l = 0 \text{ for all } a \in I \text{ with } m_a \geq d \text{ in } (1)\},$$

$$L_2 = \{l \; ; \; [a]_l = 0 \text{ for all } a \in I \text{ with } n_a \geq d \text{ in } (2)\}.$$

Since $m_a[a]_l \leq (d-1)[a]_l$ for $l \in L_1$, from (1) we have

$$\sum_{l \in L_1} [e]_l \leq \sum_{l \in L_1} [e+pe_i]_l = \sum_{l \in L_1, a \in I} m_a[a]_l$$

$$\leq (d-1) \sum_{l \in L_1, a \in I} [a]_l$$

$$\leq (d-1) \sum_{a \in I} \sum_{l=1}^{r} [a]_l = cd(d-1) .$$

Similarly as above, we have

$$\sum_{l \in L_2} [e]_l \leq cd(d-1) .$$

Summing up, we then get

$$\sum_{l \in L_1 \cup L_2} [e]_l \leq 2cd(d-1) < d(2cd-1) \leq d\delta(e)$$
$$= \sum_{l=1}^{r} [e]_l \,.$$

Therefore $[1,r] \setminus (L_1 \cup L_2) \neq \emptyset$. Let l be an index of $[1,r] \setminus (L_1 \cup L_2)$. Then we can find elements $a', b' \in I$ with $[a']_l \neq 0$, $m_{a'} \geq d$ and $[b']_l \neq 0$ and $n_{b'} \geq d$. Substituting in (1) the expression

$$\begin{aligned} m_{a'}a' &= (m_{a'} - d)a' + da' \\ &= (m_{a'} - d)a' + [a']_1 e_1 + \ldots + [a']_r e_r \,, \end{aligned}$$

we get

$$\begin{aligned} (e - e_l) + pe_i = &\sum_{a \in I, a \neq a'} m_a a + (m_{a'} - d)a' + \\ &+ \sum_{k \neq l} [a']_k e_k + ([a']_l - 1)e_l \,. \end{aligned}$$

Hence $(e - e_l) + pe_i \in H$. Similarly $(e - e_l) + pe_j \in H$. Thus $e - e_l \in H_p$. By the inductive assumption the statement is now immediate.

We can now prove the main result of this section.

<u>THEOREM 2.2</u>. <u>To check if W is arithmetically Cohen-Macaulay or arithmetically Buchsbaum or not we only need finitely many operations</u>.

<u>PROOF</u>. Set $\underline{m}_H = k[H \setminus (0)]$ the maximal graded ideal of $k[H]$. Then, as we have remarked in the introduction, W is arithmetically Cohen-Macaulay (resp. Buchsbaum) iff $k[H]_{\underline{m}_H}$ is a Cohen-Macaulay (resp. Buchsbaum) ring, since $k[H]_{\underline{m}_H}$ is isomorphic to the local ring of the vertex of the affine cone of W. By [3] and [9] (see also [11]) that is equivalent to $H_1 = H$ or $(H \setminus (0)) + H_2 \subseteq H$, respectively. Put

$$E_p = \{e \;;\; \delta(e) < 2cd - 1 \text{ and } e \in H_p \setminus H\}.$$

Since H is generated by I, from the above statement and from Lemma 2.1 we can conclude that W is arithmetically Cohen-Macaulay iff $E_1 = \emptyset$ and W is arithmetically Buchsbaum iff $I + E_2 \subset H$. To compute E_1 and E_2 and to check whether $I + E_2 \subset H$ we need obviously only finitely many operations, q.e.d.

From the finiteness of Apery sequences one can deduce the above theorem from [1, Theorem 3.1] for projections of one-dimensional Veronesian varieties. Independently, for a class of such projections Trung [10] has obtained the same result. Obviously, the associated affine semigroups of projections of one-dimensional Veronesian varieties are simplicial. Also, our theorem is a generalization of Bresinsky's and Trung's result.

3. Projections of one-dimensional Veronesian varieties

In this section we consider the case r = 2, i.e. we consider projections of one-dimensional Veronesian varieties. Such projections are also called projective monomial curves. Let C be such a curve. Since the associated affine semigroup of C is simplicial, we may assume as usual that $e_1, e_2 \in I_C$. Let

$$J_C = \{\alpha \in N \;;\; (d-\alpha, \alpha) \in I_C\}.$$

Then we can find a non-decreasing sequence of integers $a_0, \ldots, a_{2s+1}$ with

$$0 = a_0 \leq a_1 < a_1+1 < a_2 \leq a_3 < \ldots < a_{2s} \leq a_{2s+1} = d\ ,$$

such that $J = \bigcup_{i=0}^{s} [a_{2i}, a_{2i+1}]$. We shall only consider the case $a_1 > 0$, i.e. $1 \in J$. (For the case $a_1 = 0$, $a_{2s} = d$, i.e. $1 \notin J$ and $d-1 \notin J$, there are only a few particular results in [1] and [10]).

Let t be any of 1,...,s. Denote by K^t the set of all elements of the form $(d-\alpha, \alpha) + ne_1$, $\alpha \in [a_{2t-1}+1, a_{2t}-1]$

and $n \geq 0$, and by H^t, $\underline{H}^t$ the additive semigroups in N^2 generated by the elements $(d-\alpha,\alpha)$ with $\alpha \in \bigcup_{i=t}^{s}[a_{2i},a_{2i+1}]$ or $\alpha \in \bigcup_{i=0}^{t-1} [a_{2i},a_{2i+1}]$, respectively. Set

$$\widetilde{E} = \bigcup_{t=1}^{s} (H^t - e_2) \cap (K^t \setminus \underline{H}^t) .$$

Then we have

<u>LEMMA 3.1</u>[10, Lemma 3.1]. <u>Suppose that</u> $a_1 > 0$. <u>Then</u> C <u>is arithmetically Buchsbaum iff</u> $\widetilde{E} + I \subseteq H$. <u>In that case</u> $i(C) = \#\widetilde{E}$, <u>where</u> $i(C)$ <u>denotes the Buchsbaum invariant of</u> C <u>and</u> # <u>denotes the cardinality</u>.

<u>REMARK</u>. This Lemma is formulated in [10] only for the case $a_1 > 0$ and $a_{2s} = d$, but the proof is suitable for the general case $a_1 > 0$.

Let $S = \langle v_1,\dots,v_n \rangle$ denote the additive subsemigroup of N generated by the elements $v_1,\dots,v_n \in N$. If $v \in S$, $v = m_1v_1 + \dots + m_nv_n$ and $m_1 + \dots + m_n$ minimal, then $m_1 + \dots + m_n$ will be called the <u>degree</u> of v with respect to S and denoted by $\delta_S(v)$ (or simply by $\delta(v)$).

Let now S be a semigroup of N generated by J. Further, we denote by E the set of all non-negative integers α with the following properties:

(i) $\alpha \in [a_{2t-1}+1, a_{2t}-1]$ for some $t = 1,\dots,s$,

(ii) $\alpha + d \in S_\alpha := \langle \bigcup_{i=t}^{s} [a_{2i},a_{2i+1}] \rangle$,

(iii) $\delta(\alpha) \geq \delta_{S_\alpha}(\alpha+d)$.

<u>REMARK</u>. From (iii) it follows that $\alpha + d$ must belong to the subsemigroup $\langle \bigcup_{i=t}^{s} [a_{2i},a_{2i+1}] \setminus \{d\} \rangle$. Therefore, in the case $a_1 > 0$, $a_{2s} = d$ the conditions (i) and (ii) are equivalent to the following conditions

(i') $\alpha \in [a_{2t-1}+1, a_{2t}-1]$ for some $t = 1,\dots,s-1$, (ii') $\alpha + d \in \langle \bigcup_{i=t}^{s-1} [a_{2i}, a_{2i+1}] \rangle$.

Now we can simplify Trung's result as follows

THEOREM 3.2. *Suppose that* $a_1 > 0$. *Then* C *is arithmetically Buchsbaum if and only if*

$$\delta_{S_\alpha}(\alpha + d) \geq \delta(\alpha + \beta) ,$$

for all $\alpha \in E$ *and* $\beta \in J \setminus \{d\}$. *In that case,* $i(C) = \# E$.

In particular, C *is an arithmetically Cohen-Macaulay curve if and only if* $E = \emptyset$.

To prove the theorem, we shall use the following easy result

LEMMA 3.3([1, Lemma 3.1], [7, Lemma 1]). *Let* $J^* \subseteq [0,d]$. H^* *denotes the subsemigroup in* N^2 *generated by the elements* $(d-\alpha, \alpha)$, $\alpha \in J^*$, *and* S^* *denotes the subsemigroup in* N *generated by* J^*. *Let* α, $\beta \in N$ *with* $\alpha + \beta \equiv 0 \bmod d$. *Then*

$$(\beta,\alpha) \in H^* \text{ iff } \alpha \in S^* \text{ and } \alpha + \beta \geq \delta_{S^*}(\alpha)d.$$

REMARK. The above lemma differs from [1,Lemma 3.1] and [7, Lemma 1] in that here we may have $0 \notin J^*$ or $d \notin J^*$.

PROOF OF THEOREM 3.2. We will prove that the condition of the theorem is equivalent to the condition of Lemma 3.1. Let α, $\beta \geq 0$ and $\alpha + \beta \equiv 0 \bmod d$. By definition of K^t we have

$$(\beta,\alpha) \in K^t \text{ iff } \alpha \in [a_{2t-1}+1, a_{2t}-1] .$$

By definition of H^t and of S_α and by Lemma 3.3

$$(\beta,\alpha) \in H^t - e_2 \text{ iff } \alpha+d \in S_\alpha \text{ and } \beta+(\alpha+d) \geq \delta_{S_\alpha}(\alpha+d)d.$$

Since $1 \in J$, we have always $\alpha \in \underline{S} := \langle \bigcup_{i=0}^{t-1} [a_{2i}, a_{2i+1}] \rangle$. Again by Lemma 3.3, we get

$$(\beta,\alpha) \notin \underline{H}^t \text{ iff } \alpha + \beta < \delta_{\underline{S}}(\alpha)d .$$

On the other hand, it is trivial that $\delta_{\underline{S}}(\alpha) = \delta(\alpha)$ if $\alpha \in [a_{2t-1}+1, a_{2t}-1]$. Combining the above conditions, we then get that $(\beta,\alpha) \in \widetilde{E}$ iff α satisfies the conditions (i), (ii) of the definition of E and

$$(\delta_{S_\alpha}(\alpha + d) - 1)d \leq \alpha + \beta \leq (\delta(\alpha) - 1)d.$$

From this it follows in particular that $\delta(\alpha) \geq \delta_{S_\alpha}(\alpha+d)$. Summing up, we have

$$\widetilde{E} = \{(\beta,\alpha) \; ; \; \alpha \in E \text{ and } (\delta_{S_\alpha}(\alpha+d)-1)d - \alpha \leq \beta \leq \\ \leq (\delta(\alpha)-1)d - \alpha\}.$$

Finally we set

$$\mathcal{E} = \{([\delta_{S_\alpha}(\alpha+d)-1]d - \alpha,\alpha) \; ; \; \alpha \in E\} \subseteq \widetilde{E}.$$

By Lemma 3.3, it is easily seen that the condition of the theorem is equivalent to the condition $\mathcal{E} + I \subseteq H$. From this it now follows that the implication ($\Leftarrow$) is trivial.

Conversely, assume that the condition of the theorem is satisfied. Then, putting $\beta = 0 \in J$, we have

$$\delta_{S_\alpha}(\alpha+d) \geq \delta(\alpha+0) = \delta(\alpha) ,$$

for all $\alpha \in E$. Combining this with the condition (iii) of the definition of E, we get that $\delta(\alpha) = \delta_{S_\alpha}(\alpha+d)$ for all $\alpha \in E$. Hence $\mathcal{E} = \widetilde{E}$. Since $\mathcal{E} + I \subseteq H$, we must have $\widetilde{E} + I \subseteq H$. As a consequence, we have

$$i(C) = \# \widetilde{E} = \# \mathcal{E} = \# E .$$

The proof of Theorem 3.2 is now complete.

The following corollary is an improvement of [1, Theorem 4.5](cf. also [10,Theorem 4.1]).

<u>COROLLARY 3.4</u>. <u>Suppose that</u> $a_1 > 0$. <u>If</u> $\delta(t) \leq 2$ <u>for every integer</u> $1 \leq t \leq a_{2(s-1)} + a_{2s-1} - 1$ <u>in case</u> $a_{2s} = d$ <u>and</u> $1 \leq t \leq a_{2s} + d - 2$ <u>in case</u> $a_{2s} < d$, <u>then</u> C <u>is an arithmetically Buchsbaum curve</u>. <u>If</u> $a_{2s} < d$, <u>this condition is also necessary</u>.

<u>PROOF</u>. <u>Sufficiency</u>. Since

$$\max_{\alpha \in E} \alpha \leq \begin{cases} a_{2(s-1)} - 1, & \text{if } a_{2s} = d, \\ a_{2s} - 1, & \text{if } a_{2s} < d, \end{cases}$$

it follows immediately from the condition of the theorem that

$$\delta(\alpha + \beta) \leq 2 \leq \delta_{S_\alpha}(\alpha + d) \ ,$$

for all $\beta \in J \setminus \{d\}$ and $\alpha \in E$.

Necessity. Suppose that $a_{2s} < d$. Then we have $d - 1 \in J$. We prove by descending induction on $t \leq d$ that either $t \in J$ or $t \in E$ and $\delta(t) = \delta(t + d) = 2$. Assume that $t < d$. If $t \in J$ there is nothing to prove. Hence we may assume $\delta(t) \geq 2$. If $t + 1 \in J$ then $\delta_{S_t}(t + d) = 2 = \delta(t)$, because $t + d = (t + 1) + (d - 1)$. Hence $t \in E$. If $t + 1 \notin J$ then by the inductive assumption $t + 1 \in E$. Since $d - 1 \in J$, by Theorem 3.2 and the inductive assumption we get

$$\delta(t + d) = \delta([t + 1] + [d - 1]) \leq \delta_{S_t}(t + 1 + d)$$
$$\leq \delta(t + 1) = 2.$$

Hence $\delta(t + d) = 2$. Note that $\delta_{S_t}(t + d) = 2$ iff $\delta(t + d) = 2$. Hence $\delta_{S_t}(t + d) = 2 \leq \delta(t)$. This shows again that $t \in E$. Thus, in both cases, $t \in E$ and $\delta(t + d) = 2$. Again by Theorem 3.2 we get

$$2 \leq \delta(t) = \delta(t + 0) \leq \delta_{S_t}(t + d) = 2.$$

Hence $\delta(t) = 2$, as required.

For the rest of this section we shall study the difference between the Buchsbaum invariant $i(C)$ and the so-called length of the associated semigroup ideal for an arithmetically Buchsbaum curve in the case $a_1 > 0$.

First let us recall the notation of the length of a semigroup ideal (see [1] and [7]). Let $K \subseteq N^r$ be an additive semigroup. $I \subseteq K$ is said to be a semigroup ideal of K if $I + K \subseteq I$. I is irreducible if for semigroup ideals I_1 and I_2 such that $I = I_1 \cap I_2$ either $I = I_1$ or

$I = I_2$. Then every semigroup ideal $I \subseteq K$ such that $\#(K \setminus I) < \infty$ has a unique irredudant representation as a finite intersection of $\ell(I)$ irreducible semigroup ideals. $\ell(I)$ is called the <u>length</u> of I. Moreover

$$\ell(I) = \#\{e \; ; \; e \in K \setminus I, \; e + (K \setminus (0)) \subseteq I\}.$$

Let $H \subseteq N^2$ be, as usual, the associated semigroup of C. Put

$$H' = \{e \in \bar{H} \; ; \; e + m_1e_1 \in H \text{ and } e + m_2e_2 \text{ for some } m_1, m_2 \in N\}.$$

Then we have

<u>LEMMA 3.5</u>([2],[9]). C <u>is arithmetically Buchsbaum if and only if</u> $(H \setminus (0)) + H' \subseteq H$, <u>i.e. iff</u> $H \setminus (0)$ <u>is a semigroup ideal of</u> H'. <u>In that case</u> $i(C) = \#(H' \setminus H)$.

Thus, if C is arithmetically Buchsbaum we can consider the length $\ell(H \setminus (0))$ of the semigroup ideal $H \setminus (0)$ of H'. In this case, we call $H \setminus (0)$ the <u>associated semigroup ideal of C</u>. Then

$$\begin{aligned}\ell(H \setminus (0)) &= \#\{e \in (0) \cup (H' \setminus H) \; ; \; e + (H' \setminus (0)) \subseteq H\} \\ &= \#\{e \in (0) \cup (H' \setminus H) \; ; \; e + (H' \setminus H) \subseteq H\}.\end{aligned}$$

If C is arithmetically Cohen-Macaulay then $H' = H$, hence $i(C) - \ell(H \setminus (0)) = -1$. If from now on we let C be arithmetically non-Cohen-Macaulay Buchsbaum, then we have $i(C) \geq \ell(H \setminus (0))$. Our aim is to give some bound for the difference $i(C) - \ell(H \setminus (0))$. For that we need the following lemma:

<u>LEMMA 3.6</u>. <u>Suppose that</u> $a_1 > 0$ <u>and</u> C <u>is arithmetically Buchsbaum</u>. <u>Then for every element</u> $\alpha \in E$

$$\delta(\alpha) = \delta_{S_\alpha}(\alpha + d) = \delta(\alpha + d).$$

<u>PROOF</u>. The equality $\delta(\alpha) = \delta_{S_\alpha}(\alpha + d)$ was shown in the proof of Theorem 3.2.

For the second equality note that $\delta_{S_\alpha}(\alpha + d) \geq \delta(\alpha + d)$, because $S_\alpha \subseteq S$. Let $n = \delta(\alpha + d)$. Then there exist elements $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$, $\alpha_1, \dots, \alpha_n \in J$ such that $\alpha + d = \alpha_1 + \dots + \alpha_n$. Assume that $\alpha_1 < \alpha$. Then $\alpha - \alpha_1 > 0$ and $\delta(\alpha - \alpha_1 + d) \leq n - 1$(note that $n \geq 2$). By Lemma 3.3 it follows that

$$((n-2)d-(\alpha-\alpha_1)\ ,\ \alpha-\alpha_1) + e_2 \in H\ ,$$

and

$$((n-2)d-(\alpha-\alpha_1)\ ,\ \alpha-\alpha_1) + \delta(\alpha-\alpha_1)e_1 \in H.$$

Thus

$$((n-2)d-(\alpha-\alpha_1)\ ,\ \alpha-\alpha_1) \in H'.$$

Therefore, by Lemma 3.5

$$((n-1)d-\alpha,\alpha) = ((n-2)d-(\alpha-\alpha_1),\alpha-\alpha_1) + (d-\alpha_1,\alpha_1) \in H.$$

Again by Lemma 3.3 we must have $\delta(\alpha) \leq n - 1$. Hence $\delta_{S_\alpha}(\alpha + d) = \delta(\alpha) < \delta(\alpha + d)$, a contradition. Also $\alpha_1, \dots, \alpha_n \geqslant \alpha$. That means however $\delta_{S_\alpha}(\alpha + d) \leq n$. Hence $\delta_{S_\alpha}(\alpha + d) = \delta(\alpha + d)$, as required.

<u>COROLLARY 3.7</u>. <u>Suppose that</u> $a_1 > 0$ <u>and</u> C <u>is arithmetically non-Cohen-Macaulay Buchsbaum</u>. <u>Then</u>

$$i(C) - \ell(H \setminus (0)) = \#\{\alpha \in E\ ;\ \underline{\text{there exists}}\ \beta \in E\ \underline{\text{s.t.}}\ \alpha + \beta \in E\ \underline{\text{and}}\ \delta(\alpha + \beta) = \delta(\alpha) + \delta(\beta) - 1\}.$$

<u>PROOF</u>. As in the proof of Theorem 3.2 we set

$$\mathcal{E} = \{([\delta_{S_\alpha}(\alpha + d) - 1]d - \alpha,\alpha)\ ;\ \alpha \in E\}.$$

From the definition of H' and by Lemma 3.3 it is easy to show that $\mathcal{E} \subseteq H' \setminus H$. Since C is arithmetically Buchsbaum, $i(C) = \#(H' \setminus H)$. By Theorem 3.2, $i(C) = \#\mathcal{E}$. Hence $\mathcal{E} = H' \setminus H$. Moreover $H \neq H'$, because C is not arithmetically Cohen-Macaulay. Therefore

$$\ell(H \setminus (0)) = \#\{e \in \mathcal{E}\ ;\ e + \mathcal{E} \subseteq H\}.$$

From this it follows that

$$i(C) - \ell(H \setminus (0)) = \#\{e \in \mathcal{E}\ ;\ \text{there exists } f \in \mathcal{E}\ \text{s.t. } e + f \in \mathcal{E}\}.$$

By Lemma 3.6 this implies immediately the formula of the corollary, q.e.d.

In particular, we have

COROLLARY 3.8. *Assume that* $a_1 > 0$ *and* C *is arithmetically Buchsbaum. Moreover assume that* $\delta(\alpha) = 2k - 1$ *for some element* $\alpha \in E$ *and* $\delta(\beta) = k$ *for all other elements* $\beta \in E$ *and* $\beta \neq \alpha$, *where* k *is an integer* ≥ 2. *Then* $i(C) - \ell(H \setminus (0))$ *is equal to the number of representations of* α *as the sum of two elements of* E, *where* $a + b \neq b + a$.

REMARK. Bresinsky has first in [1] examined the difference $i(C) - \ell(H \setminus (0))$ and questioned what non-negative values are assumed by this difference for arithmetically non-Cohen-Macaulay Buchsbaum curves. To answer this question, Kästner [7] has constructed a class of examples which shows that this difference can take any non-negative value. His examples satisfy all assumptions of the above corollary for $k = 2$. Thus, this corollary gives a method to solve Bresinsky's question.

We are now ready to give an estimate for $i(C) - \ell(H \setminus (0))$.

THEOREM 3.9. *Suppose that* $a_1 > 0$ *and* C *is arithmetically Buchsbaum. Let* $c = \# I_C = \# J_C$. *Then*

$$i(C) - \ell(H \setminus (0)) \leq \max \{0, d + 2 - 2c\}.$$

PROOF. If $E = \emptyset$ there is nothing to prove. Let $E \neq \emptyset$. We set

$$\mu = \max_{\alpha \in E} \delta(\alpha) .$$

If $\mu = 2$ then $\delta(\alpha) = 2$ for all $\alpha \in E$. From Corollary 3.7 it follows that $i(C) - \ell(H \setminus (0)) = 0$. Now we assume that $\mu \geq 3$. Let $\alpha_1 < \alpha_2 < \dots < \alpha_k$ be all elements of E with $\delta(\alpha_1) = \dots = \delta(\alpha_k) = \mu$. Denote by F the set

$$F = \{\alpha \in E \; ; \text{ there exists } \beta \in E \text{ s.t. } \alpha + \beta \in E \text{ and } \delta(\alpha + \beta) = \delta(\alpha) + \delta(\beta) - 1\} .$$

Then by Corollary 3.7 we must show that

$$\# F \leq \max (0, d + 2 - 2c).$$

Note that $\alpha_i \notin F$ for all $i = 1, \dots, k$. Let

$$J \cap [0,\alpha_k] = \left\{ \beta_0 = 0,\ \beta_1, \dots,\ \beta_m \right\}.$$

For $i = 0,1,\dots,m$ we have $\alpha_k = \beta_i + (\alpha_k - \beta_i)$. Since $\beta_i \in J$ and $\delta(\alpha_k) = \mu \geqq 3$, $\alpha_k - \beta_i \notin J$. In particular, $\alpha_k - \beta_i \neq \beta_j$ for all $i,\ j \leqq m$. If $\alpha_k - \beta_i \in E$, then by Lemma 3.6 and Theorem 3.2 we have

$$\delta(\alpha_k - \beta_i) \geqq \delta((\alpha_k - \beta_i) + \beta_i) = \delta(\alpha_k) = \mu.$$

Hence $\delta(\alpha_k - \beta_i) = \mu$. Thus, we can conclude that either $\alpha_k - \beta_i \notin E$ or $\alpha_k - \beta_i = \alpha_j$ for some $1 \leqq j \leqq k$. In both cases $\alpha_k - \beta_i \notin F$. Since $\beta_i \notin F$, we have

$$\#(F \cap [0,\alpha_k]) \leqq (\alpha_k+1) - 2(m+1) = \alpha_k - (2m+1). \quad (1)$$

Next, let

$$J \cap [\alpha_k + 1, d - 1] = \left\{ \gamma_1, \dots, \gamma_n \right\}.$$

(Note that we have always $\alpha_k + 1 \leqq d - 1$). In the representations

$$\alpha_k + d = \gamma_i + (\alpha_k + d - \gamma_i),$$

we have $\alpha_k + d - \gamma_i \in [\alpha_k + 1, d - 1]$ for $i = 1,\dots,n$. By Lemma 3.6 $\delta(\alpha_k + d) = \delta(\alpha_k) = \mu$. If $\alpha_k + d - \gamma_i \in E$, then arguing as above we get that $\delta(\alpha_k + d - \gamma_i) = \mu$, a contradition to the maximality of α_k. Hence $\alpha_k + d - \gamma_i \notin E$. Therefore

$$\#(E \cap [\alpha_k + 1, d - 1]) \leqq d - (\alpha_k + 1) - 2n. \quad (2)$$

Since $c = m + n + 2$ and $F \subseteq E$, combining (1) and (2) we have

$$\# F \leqq d - 2(m + n + 1) = d + 2 - 2c, \quad \text{q.e.d.}$$

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