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ON THE NUMBER OF NON-EQUIVALENT DIFFERENTIABLE STRUCTURES ON 4-MANIFOLDS

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By using results from [7], [8] we show that for any positive integer k there exist k simply-conected algebraic surfaces of general type which are pairwise homeomorphic but not diffeomorphic.

Introduction. By using ramified coverings of $\mathbf{CP}^1 \times \mathbf{CP}^1$, in [7] infinitely many pairs of homeomorphic, non-diffeomorphic simply connected algebraic surfaces of general type were found.

In this paper we obtain a stronger result: namely, we prove that for any given positive integer k there are k surfaces $Y^{(1)}, \ldots, Y^{(k)}$ pairwise homeomorphic (so they have same invariants K^2, χ) but not diffeomorphic. In particular, they must lie in k different connected components of the coarse moduli space $M(K^2, \chi)$ of complex structures on the oriented 4-manifold underlying $Y^{(i)}$, showing that the number of connected components of $M(K^2, \chi)$ can be arbitrarely large. This last result was recently proved in [4] by using different methods (see also [2], [3]).

Our surfaces are still obtained as ramified coverings, but we consider $\mathbb{CP}^2$ instead of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let $n_1, \ldots, n_r$, $d_1, \ldots, d_r$ be sequences of positive integers such that $d_i | n_i$, $i=1,\ldots,r$. Let C_i be a smooth curve in $\mathbb{CP}^2$ of degree n_i, $i=1,\ldots,r$, such that $C = C_1 \cup \ldots \cup C_r$ has normal crossing singularities. Construct a sequence $\beta_i : Y_i \to Y_{i-1}$, $Y_0 = \mathbb{CP}^2$, where β_i is the cyclic covering of degree d_i ramified over $\alpha_{i-1}{}^*(C_i)$, $\alpha_i = \beta_i \circ \ldots \circ \beta_1$ $(i=1,\ldots,r)$. The surfaces $Y_r = Y_r(n_1,\ldots,n_r;d_1,\ldots,d_r)$ are the ones we consider. The problem of determining if two surfaces corresponding to different sequences are homeomorphic and not diffeomorphic is a numerical one. In fact, one checks if two such surfaces are homeomorphic by looking at their intersection forms ([6]). To see that two (homeomorphic) surfaces are not diffeomorphic result from [7] is employed: a new Donaldson invariant is used to proof that an orientation preserving diffeomorphism between two simply connected algebraic surfaces of general type, which have even p_g and *big monodromy groups*, must preserve the divisibility of the canonical class.

Following [8], some results from [1], [5] are used to deduce that the surfaces have big monodromy groups (and for our examples the proofs are easier than in [7], [8]). Moreover, the proof of the main theorem uses a little bit of number theory.

We recall the main points which will be used in the following.

Let X be a compact complex manifold, $\mathfrak{D}$ a linear system of dimension N on X. Set $W = \{(x,t) \in X \times \mathbf{CP}^N \mid x \in E_t\}$ ($E_t \in \mathfrak{D}$ corresponding to $t \in \mathbf{CP}^N$) and let $f: W \to \mathbf{CP}^N$, $\pi: W \to X$ be the projections. Set also $S(\mathfrak{D}) = \{p \in W \mid p$ is a singular point of $f^{-1}(f(p))\}$, $q(\mathfrak{D}) = \{p \in W \mid p$ is an isolated singular point of $f^{-1}(f(p))\}$, $S_b(\mathfrak{D}) = f(S(\mathfrak{D}))$.

Definition . $\mathfrak{D}$ is called a linear system of Lefschetz type if: (a) $\#\mathfrak{D} = \infty$ and $\mathfrak{D}$ has no basepoints; (b) $f(\ S(\mathfrak{D}) \setminus q(\mathfrak{D})\ )$ has codimension ≥ 2 in $\mathbf{CP}^N$; (c) if $\dim_{\mathbf{C}} X \geq 2$, then for any generic $E_t \in \mathfrak{D}$ the linear system $\mathfrak{D}|E_t$ is of Lefschetz type.

Let now $\dim_{\mathbf{C}} X = 3$, $E_t \in \mathfrak{D}$ generic. The group $\mathrm{Diff}^+(E_t)$ of the orientation preserving diffeomorphisms acts on $H_2(E_t)$.

Theorem 1 (see [8;prop.5 and p.44] and [1],[5]). Let $\mathfrak{D}$ be of Lefschetz type on X and let $q(\mathfrak{D})$ be connected. If there exist $E_s \in \mathfrak{D}$ with only isolated singularities, one of which having in its universal deformation the singulariry U_{12} (= $\{z^3+y^3+x^4=0\}$) then $\forall t \notin S_b(\mathfrak{D})$ the image of $\mathrm{Diff}^+(E_t)$ in $\mathrm{Aut}(H_2(E_t), <\ >, (K_{E_t})$ has finite index.

Here $\mathrm{Aut}(H_2(E_t), <\ >, (K_{E_t})$ denotes the group of automorphisms which preserve the intersection form and the canonical class.

Theorem 2 ([7,§3],[8,p.48]). Let Y_1, Y_2 be simply connected algebraic surfaces of general type and suppose: (i) the image of $\mathrm{Diff}^+(Y_i)$ in $\mathrm{Aut}(H_2(Y_i),(K_Y))$ has finite index; (ii) $p_g(Y_i)\equiv 0 \pmod 2$; (iii) $K_Y = n_i\, k_i$, $n_i \in \mathbf{Z}^+$, k_i primitive in $H_2(Y_i;\mathbf{Z})$, and $n_1 \neq n_2$. Then Y_1 is not diffeomorphic to Y_2 (by an orientation preserving diffeomorphism).

Let now $(n_1,\ldots,n_r)$, $(d_1,\ldots,d_r)$ be sequences of positive integers such that $d_i \mid n_i$, $i=1,\ldots,r$, and for each i let $C_i \subset \mathbf{CP}^2$ be a nonsingular curve of degree n_i. Suppose that $C = C_1 \cup \ldots \cup C_r$ has only normal singularities. We recursively construct a finite covering α_i: $Y_i \to \mathbf{CP}^2$, $i=1,\ldots,r$. Given $Y_0 = \mathbf{CP}^2$, $\alpha_0 = \mathrm{id}$, let β_i: $Y_i \to Y_{i-1}$ be the cyclic covering of degree d_i ramified over $\alpha_{i-1}{}^*(C_i)$, and $\alpha_i = \beta_i \circ \alpha_{i-1}$. So $\alpha_r : Y_r = Y_r(n_1,\ldots,n_r;\, d_1,\ldots,d_r) \to \mathbf{CP}^2$ is finite of degree $d_1\ldots d_r$. Let us determine the invariants of Y_r.

K_{Y_r}. One has:

$$K_{Y_r} = \beta_r{}^*(K_{Y_{r-1}}) + [(d_r-1)/d_r]\, \beta_r{}^*(\alpha_{r-1}{}^*(C_r))=$$
$$= \ldots = \alpha_r{}^*(K_{P^2}) + \Sigma_{i=1}^{r} (d_i-1)/d_i\, \alpha_r{}^*(C_i) =$$
$$= [-3 + \Sigma ((d_i-1)/d_i)\, n_i]\, \alpha_r{}^*(L) =$$
$$= (-3 - \Sigma\, n_i/d_i + \Sigma\, n_i)\, \alpha_r{}^*(L).$$

$c_1{}^2(Y_r)$. $\quad c_1{}^2(Y_r) = (K_Y)^2 = d_1\ldots d_r\, (-3 - \Sigma\, n_i/d_i + \Sigma\, n_i)^2.$

$c_2(Y_r)$. $c_2(Y_r) = d_r\, c_2(Y_{r-1}) - (d_r - 1)\, e(\alpha_{r-1}{}^*(C_r)) =$

$= d_r\, c_2(Y_{r-1}) + (d_r - 1)\, (K_{Y_{r-1}} + \alpha_{r-1}{}^*(C_r))\ .\ \alpha_{r-1}{}^*(C_r) =$

$= d_r\, c_2(Y_{r-1}) + (d_r - 1)\, (-3 - \Sigma_{i=1}^{r-1} n_i/d_i + \Sigma_{i=1}^{r-1} n_i +$

$n_r)\ \alpha_{r-1}{}^*(L)\ .\ \alpha_{r-1}{}^*(n_r\, L) =$

$= d_r\, c_2(Y_{r-1}) + (d_r - 1)\, d_1 \ldots d_{r-1}\, n_r\, (-3 - \Sigma_{i=1}^{r-1} n_i/d_i +$

$\Sigma_{i=1}^{r} n_i) = \ldots =$

$= d_1 \ldots d_r\, c_2(\mathbf{CP}^2) + \Sigma_{i=1}^{r} d_1 \ldots \hat{d_i} \ldots d_r\, (d_i - 1)\, n_i\, (-3 -$

$\Sigma_{j=1}^{i-1} n_j/d_j + \Sigma_{j=1}^{i} n_j) =$

$= d_1 \ldots d_r\, [3 + \Sigma_{i=1}^{r} ((d_i - 1)/d_i)\, n_i\, (-3 - \Sigma_{j=1}^{i-1} n_j/d_j +$

$\Sigma_{j=1}^{i} n_j)] =$

$= d_1 \ldots d_r\, [3 + 3\, \Sigma n_i/d_i - 3\, \Sigma n_i + \Sigma_{i>j}\, (n_i/d_i)\, (n_j/d_j) -$

$\Sigma_{i>j}\, n_i\, (n_j/d_j) + \Sigma_{i \geq j}\, n_i\, n_j - \Sigma_{i \geq j}\, (n_i/d_i)\, n_j\,] =$

$= (1/2)\, d_1 \ldots d_r\, [6 + 6\, \Sigma n_i/d_i - 6\, \Sigma n_i +$

$2\, \Sigma_{i>j}\, (n_i/d_i)(n_j/d_j) + \Sigma (n_i/d_i)^2 - 2\, \Sigma_{i,j}\, n_i\, (n_j/d_j) +$

$2\, \Sigma_{i>j}\, n_i\, n_j + 2\, \Sigma\, n_i^2 - \Sigma (n_i/d_i)^2] =$

$= (1/2)\, d_1 \ldots d_r\, [(\Sigma\, n_i)^2 + (\Sigma n_i/d_i)^2 - 2\, \Sigma_{i,j}\, n_i (n_j/d_j)$

$+ 9 - 6\, \Sigma\, n_i + 6\, \Sigma n_i/d_i + \Sigma\, n_i^2 - \Sigma (n_i/d_i)^2 - 3] =$

$= (1/2)\, d_1 \ldots d_r\, [(\Sigma\, n_i - \Sigma (n_i/d_i) - 3)^2 + (\Sigma\, n_i^2 -$

$\Sigma (n_i/d_i)^2 - 3)]$.

(We used the genus formula)

$\tau(Y_r)$. From $\tau(Y_r) = (1/3)\, (c_1^2(Y_r) - 2\, c_2(Y_r))$

it immediately derives:

$\tau(Y_r) = (1/3)\, d_1 \ldots d_r\, (3 + \Sigma (n_i/d_i)^2 - \Sigma\, n_i^2)$.

$\pi_1(Y_r)$. Assume by recurrence that Y_i is simply connected $(i \geq 0)$. Then ([2; §1]) $\pi_1(Y_i \setminus \alpha_i^*(C_{i+1}))$ is cyclic of order n_{i+1}, and it easily follows that Y_{i+1} is simply connected.

$p_g(Y_r)$. Since $\pi_1(Y_r) = 0$ one has $1+p_g(Y_r) = \chi(Y_r) =$
$= (1/12)(c_1^2 + c_2) = (1/24)\, d_1 \ldots d_r \,[3\,(\,\Sigma\, n_i - \Sigma\, n_i/d_i - 3)^2$
$+ (\,\Sigma\, n_i^2 - \Sigma\, (n_i/d_i)^2 - 3\,)\,]$.

Let now Q be a non singular surface of degree n_1 given in affine coordinates by $Q = \{x^4 + y^3 + P(x,y,z) = 0\}$, where $P(0,0,0) = 0$ and P contains x and y in degree at least 5 (for instance $P = x^{n_1} + y^{n_1} + z^{n_1}$, $n_1>4$). Let $E_s = \{z=0\}$, so that $E_s \cap Q = \{x^4 + y^3 + o(5) = 0\}$. We can suppose that E_s is tangent to Q only in $O = (0:0:0:1)$. Set $S_1 = Q$ and let $S_2,\ldots,S_r$ be non singular surfaces in $\mathbb{CP}^3$ non containing O and in general position with respect to E_s, such that $\deg S_i = n_i > 1$ $(i=1,\ldots,r)$ and $S = S_1 \cup \ldots \cup S_r$ has only normal singularities. Let us construct a sequence of cyclic coverings $\hat{\beta}_i : X_i \to X_{i-1}$, $i=1,\ldots,r$, with $X_0 = \mathbb{CP}^3$, $\hat{\alpha}_0 = id$, $\hat{\alpha}_i = \hat{\beta}_i \circ \hat{\alpha}_{i-1}$, and where (analog to the above construction) $\hat{\beta}_i$ is the cyclic covering of X_{i-1} of degree d_i ramified over $\hat{\alpha}_{i-1}^*(S_i)$.

The hyperplane linear system $\mathcal{D}$ in $\mathbb{CP}^3$ is clearly of Lefschetz type; indicate by $\mathcal{D}_i$ the linear system $\hat{\alpha}_i^*(\mathcal{D})$, which still is of Lefshetz type ([8; prop.9]). The divisor $\hat{\alpha}_1^*(E_s)$

has an isolated singularity of the kind $u^{d_1} = x^4 + y^3 + o(5)$, which is equivalent to $u^{d_1} = x^4 + y^3$ and if $d_1 \geq 3$ such singularity has the singularity U_{12} in its versal deformation. Therefore the same happens for the divisor $\alpha_r^*(E_s)$.

Proposition 1. $q(\mathfrak{D}_r)$ *is connected.*

Sketch of proof. Let W_r be the graphic of $\mathfrak{D}_r$ (constructed as above); indicate by $f_r : W_r \to \mathbf{CP}^3$, $\pi_r : W_r \to X_r$ the projections and set by brevity $\alpha = \alpha_r$. Clearly if $p = (x,t) \in W_r$ is such that $\alpha(x) \notin S$ then $\alpha^*(E_t)$ is nonsingular in x. For $y \in S$, denote by $TS(y) = \cap_{y \in S} (TS_j)_y$, where $(TS_j)_y$ is the tangent space to S_j in y. A point $p = (x,t) \in W_r$ lies in $S(\mathfrak{D}_r)$ if and only if $E_t \supset TS(\alpha(x))$. Moreover, p will be in $q(\mathfrak{D}_r)$ if and only if $E_t \supset TS(\alpha(x))$ but it does not contain $TS(y)$ for y in a punctured neighborhood of $\alpha(x)$.

If $(x,t), (x',t') \in q(\mathfrak{D}_r)$, $\alpha(x)$ and $\alpha(x')$ lie in the same S_j and t [t'] corresponds to the tangent space to S_j in $\alpha(x)$ [$\alpha(x')$] then take $\gamma : [0,1] \to \alpha^{-1}(S_j)$ connecting x and x', such that $\gamma((0,1)) \cap S_h = \emptyset$, $h \neq j$. We can lift γ to a path $\hat{\gamma}$ in $S(\mathfrak{D}_r)$ by setting $\hat{\gamma}(u) = (\gamma(u), t(u))$, $0 \leq u \leq 1$, where t(u) is the parameter of the divisor $(TS_j)_{\alpha(\gamma(u))}$. It follows that $\hat{\gamma} \subset q(\mathfrak{D}_r)$.

If $\alpha(x) \in S_j$, $\alpha(x') \in S_h$, $j \neq h$, then choose $x'' \in \alpha^{-1}(S_j \cap S_h)$ such that, if $t^{(j)}$ $[t^{(h)}]$ is the parameter corresponding to $(TS_j)_{\alpha(x'')}$ $[(TS_h)_{\alpha(x'')}]$, then $(x'',t^{(j)})$, $(x'',t^{(h)}) \in q(\mathfrak{D}_r)$. Since $\{t' \in \mathbf{CP}^3 \mid E_{t'} \supset TS(\alpha(x''))\}$ is connected the thesis follows by using the preceding part. Q.E.D.

Proposition 2. *If* E_t *is generic with respect to* S *then* $\mathrm{Diff}^+(\hat{\alpha}_r{}^*(E_t))$ *induces a finite index subgroup in* $\mathrm{Aut}(H_2(\hat{\alpha}_r{}^*(E_t), \langle\ \rangle, (K_{E_t}))$.

Proof. The proof immediately follows by theorem 1, since by proposition 1 $q(\mathfrak{D}_r)$ is connected and $\hat{\alpha}_r{}^*(E_s)$ $(E_s=\{z=0\})$ has only one singular point which contains U_{12} in its versal deformation. Q.E.D.

Corollary. *For every surface* $Y_r = Y_r(n_1,\ldots,n_r;d_1,\ldots,d_r)$ *of the kind constructed before, where at least one* $n_i > 5$ *and the corresponding* $d_i \geq 3$, *the group* $\mathrm{Diff}^+(Y_r)$ *induces a finite index subgroup in* $\mathrm{Aut}(H_2(Y_r), (K_{Y_t}))$.

Proof. In fact, one can identify $Y_0 = \mathbf{CP}^2$ with the divisor E_t of proposition 2, C_i with $S \cap E_t$ and Y_r with $\alpha_r{}^{-1}(E_t)$.

Q.E.D.

Recall now from [6] that two simply connected surfaces Y, Y' are homeomorphic if they have isomorphic intersection forms (same rank, index and type). This is equivalent to the fact that Y and Y' have a same pair of invariants and intersection forms of the same type. In particular $Y = Y(n_1,\ldots,n_r;d_1,\ldots,d_r)$ and $Y' = Y'(n_1',\ldots,n_s';d_1',\ldots,d_s')$ will be homeomorphic if

$c_1^2(Y) = d_1\ldots d_r\ (-3 - \Sigma^r n_i/d_i + \Sigma^r n_i) = d_1'\ldots d_s'\ (-3 - \Sigma^s n_j'/d_j' + \Sigma^s n_j') = c_1^2(Y')$,

$-3\tau(Y) = d_1\ldots d_r\ (-3 - \Sigma^r (n_i/d_i)^2 + \Sigma^r n_i^2) = d_1'\ldots d_s'\ (-3 - \Sigma^s (n_j'/d_j')^2 + \Sigma^s (n_j')^2) = -3\tau(Y')$

and the intersection forms of Y and Y' have the same type. Y and Y' will be homeomorphic but not diffeomorphic if (by theorem 2) $p_g(Y) \equiv p_g(Y') \equiv 0 \pmod 2$ and $(-3 - \Sigma^r n_i/d_i + \Sigma^r n_i) \neq (-3 - \Sigma^s n_j'/d_j' + \Sigma^s n_j')$; in fact $\alpha_r^*(L)$ $[\alpha_s^*(L)]$ is primitive in Y [Y'].

We will see that the system

$$c_1^2(Y(n_1,\ldots,n_r;d_1,\ldots,d_r)) = M$$

$$-3\ \tau(Y(n_1,\ldots,n_r;d_1,\ldots,d_r)) = N$$

has many solutions even under the condition $d_1 = \ldots = d_r$. Integer solutions to such system will be parametrized by an integer i. So set $a_i = d_{i1} = \ldots = d_{ir}$, $n_{ij} = d_{ij}\ m_{ij} = a_i m_{ij}$, $j=1,\ldots,r$.

To simplify the proof we also assume $r \geq 16$, $r \equiv 0 \pmod 4$.

Theorem. *For each positive integer* k *there exist positive integers* a_i, $m_{i1}, \ldots, m_{ir}$, M, N, $i=1,\ldots,k$, *such that*

$$(*)\quad \begin{aligned} a_i^r\,[\,(a_i-1)\,\Sigma_{j=1}\, m_{ij} \; - \; 3\,]^2 &= M \\ a_i^r\,[\,(a_i^2-1)\,\Sigma_{j=1}\, (m_{ij})^2 \; - \; 3\,] &= N \end{aligned}$$

where the a_i*'s are distinct, and* $(3M + N)/24$ *is odd.*

We need the following lemma.

Lemma. *Set* $F_r(x) = \Sigma_{j=1}^{r} x_j^2$, $G_r(x) = \Sigma_{j=1}^{r} x_j$, *where* $x_j \in \mathbb{Z}$, $j = 1, \dots, r$.

(a) *The form* $F_r(x)$, *under the restriction* $G_r(x) = A \geq 0$, *represents all numbers* $n \equiv A \pmod 2$, $n \geq \mu_{r,A} := \min_{G_r(x)=A} F_r(x)$,

and it holds

$$A^2/r \leq \mu_{r,A} \leq (A^2/r) + \alpha_r$$

where α_r *is a constant depending only on* r.

(b) *Under the restrictions* $G_r(x) = A$, $x_j > 0$ *for* $j=1,\dots,r$, $F_r(x)$ *represents all* $n \equiv A \pmod 2$ *such that*

$$\mu_{r,A} \leq n < A^2/(r-1).$$

Proof of lemma. (a) It is easy to see that if $x_j - x_k \geq 2$, where for example $j < k$, then $F_r(\dots,x_j-1,\dots,x_k+1,\dots) < F_r(\dots,x_j,\dots,x_k,\dots)$. So the minimum value of $F_r(x)$ is attained at a point $(x_1,\dots,x_r)$ s.t. $|\, x_j - x_k \,| \leq 1$ for all j, k. If $\mathbf{A} = [A/r]$ and $A = r\,\mathbf{A} + s$, $0 \leq s < r$, then $\mu_{r,A} = F(\mathbf{x}_1,\dots,\mathbf{x}_r)$ where $\mathbf{x}_j = \mathbf{A} + 1$, $j=1,\dots,s$, $\mathbf{x}_j = \mathbf{A}$, $j=s+1,\dots,r$.

It follows that

$$\mu_{r,A} = r\,\Delta^2 + 2\,s\,\Delta + s = A^2/r + s(s-r)/r,$$

from which the inequalities for $\mu_{r,A}$ immediately follows with $\alpha_r = r/4$.

Setting now $x_j = \bar{x}_j + y_j$ we have to consider $F_r'(y_j) = F_r(\bar{x}_j + y_j)$, under the restriction $\Sigma_{j=1}^{r} y_j = 0$. Now $F_r'(y_j) = \Sigma_{j=1}^{r} (\bar{x}_j + y_j)^2 = \Sigma_{j=1}^{r} \bar{x}_j^2 + 2\,\Sigma_{j=1}^{r} \bar{x}_j\, y_j + \Sigma_{j=1}^{r} y_j^2 = \mu_{r,A} + \Sigma_{j=1}^{r} y_j^2 + 2\,(\Sigma_{j=1}^{s} (\Delta + 1)y_j + \Sigma_{j=s+1}^{r} \Delta\cdot y_j) = \mu_{r,A} + [\,\Sigma_{j=1}^{r} y_j^2 + 2\,\Sigma_{j=1}^{s} y_j\,]$

where the quantity in square brackets (say $F_r''(y_j)$) is not negative. We claim that $F_r''(y_j)$ represents all even numbers ≥ 0. Up to writing F_r'' as $\Sigma_{j=1}^{r} y_j^2 - 2\,\Sigma_{j=s+1}^{r} y_j$ we can assume that the linear part contains at most $r/2$ variables, i.e. $s \leq r/2$. Now set $F_r''' = F_r''|\{y_1 = \ldots = y_s = 0\}$, so F_r''' is a sum of ≥ 8 squares (by assumption $r \geq 16$). Using that each not negative integer is a sum of four squares one easily deduces that F_r''' (so F_r'') represents all even not negative integers.

(b) By part (a) $\min_{x_r = 0,\ G_r(x) = A} F_r(x) = \mu_{r-1,A}$, where $A^2/(r-1) \leq \mu_{r-1,A}$. Then by symmetry it follows that $\{x : F_r(x) < A^2/(r-1),\ G_r(x) = A\} \subset \{x_j > 0,\ j=1,\ldots,r\}$.

Q.E.D. for lemma.

Proof of theorem. Let $a_1,\ldots,a_k$ be distinct big positive integers such that $(a_i, a_j - 1) = (a_i, a_j + 1) = 1$, $\forall\, i, j$, and $a_i \equiv 1 \pmod 4$, $\forall\, i$. Let

$$M = a_1^{s_1} \dots a_k^{s_k} \ [\ (a_1 - 1) \ \dots \ (a_k - 1)\ r - 3\ ]^2 ,$$

where for $j = 1,\dots,k$ and $\forall\, i \neq j$ the positive integer s_j is divisible by the numbers $2\phi(a_i - 1)$, $\phi(a_i^2 - 1)$ (ϕ is the Euler function). For each i we can write

$$M = a_i^r\ [\ (a_1)^{s_1/2} \dots (a_{i-1})^{s_{i-1}/2}\ (a_i)^{(s_i-r)/2}\ (a_{i+1})^{s_{i+1}/2}$$
$$\dots (a_k)^{s_k/2}\ (\ (a_1 - 1)\ \dots\ (a_k - 1) r\ -\ 3\ )\ ]^2.$$

Call b_i the number in square brackets; by construction $b_i \equiv -3 \pmod{(a_i - 1)}$, so there are solutions for

$$(a_i - 1)\ \Sigma_{j=1}\ m_{ij} - 3 = b_i \qquad (**)$$

with respect to the m_{ij}'s.

Set now

$$N = a_1^{s_1} \dots a_k^{s_k}\ [\ (a_1^2 - 1)\ \dots\ (a_k^2 - 1)\ r - 3\ ].$$

For each i one has

$$N = a_i^r\ [\ (a_1)^{s_1} \dots (a_{i-1})^{s_{i-1}}\ (a_i)^{(s_i-r)}\ (a_{i+1})^{s_{i+1}}$$
$$\dots (a_k)^{s_k}\ \ (\ (a_1^2 - 1)\ \dots\ (a_k^2 - 1) r\ -\ 3\ )\ ]\ ,$$

and if c_i is the integer in square brackets we have $c_i \equiv -3 \pmod{(a_i^2 - 1)}$. The second equation in (*) becomes

$$\Sigma_{j=1} m_{ij}^2 = (c_i + 3) / (a_i^2 - 1). \qquad (***)$$

By the lemma equation (***) is solvable under the conditions (**) and $m_{ij} > 0$ for all j if

$(b_i + 3)^2/r(a_i-1)^2 + \alpha_r \leq (c_i+ 3)/(a_i^2-1) < (1/(r-1))(b_i + 3)^2/(a_i-1)^2$, or

$$1 + r\,\alpha_r(a_i-1)^2/(b_i + 3)^2 \leq ((a_i - 1)/(a_i + 1))\, r(c_i +3)/(b_i+3)^2 < r/(r - 1) \qquad (\ddagger),$$

and $(c_i+ 3)/(a_i^2-1) \equiv (b_i + 3)/(a_i-1) \pmod 2$. It is easy to see that both $(c_i+3)/(a_i^2-1)$ and $(b_i + 3)/(a_i-1)$ become even (if they are not even already) by multiplying every exponent s_j by 2 (1) (we use here $r \equiv 0 \pmod 4$)).

When the a_i's are big enough $(\ddagger)$ is approximately the same as

$$1 + \alpha_r/ [r(a_1-1)^2 \ldots \widehat{(a_i-1)}^2 \ldots (a_k-1)^2 (a_1)^{s_1} \ldots (a_{i-1})^{s_{i-1}} (a_i)^{s_i-r} (a_{i+1})^{s_{i+1}} \ldots (a_k)^{s_k}] \leq (a_1+1) \ldots \widehat{(a_i+1)} \ldots (a_k+1) / (a_1-1) \ldots \widehat{(a_i-1)} \ldots (a_k-1) < r/(r-1)$$

which is verified for big a_i's. So (*) is solvable.

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(1). *Note*: we could assume from the beginning that $4\phi(a_i-1)$, $4\phi(a_i^2 -1)$ divide s_j.

It remains to see that $(3M + N)/24$ is odd. But $3M+N =$
$a_1^{s_1} \dots a_k^{s_k} \{ 3[(a_1-1)\dots(a_k-1)r -3]^2 + [(a_1^2-1)\dots(a_k^2-1)r$
$-3]\} =$
$a_1^{s_1} \dots a_k^{s_k} \{ 24 + r (a_1-1)\dots(a_k-1) [(a_1+1)\dots(a_k+1) +$
$r(a_1-1)\dots(a_k-1) - 6] \}$.

Since $a_i \equiv 1 \pmod 4$ the thesis follows. Q.E.D.

Note that we obtained odd values for M, so the intersection forms of the associated surfaces are odd.

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