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CHEBYSHEV CENTERS, ε-CHEBYSHEV CENTERS AND THE HAUSDORFF METRIC

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In this paper, we affirmatively answer an open question raised by P.Szeptycki and Vlech in [9] and give a new characterization of p-uniformly convex Banach space. The Lipschitz stability of the set of ε-Chebyshev centers $G_\varepsilon(A)$ under the perturbations of A and G is also proved.

I. Introduction

For a normed linear space X, X* denotes its dual, $U(X)=\{x\in X; \|x\|\leq 1\}$ the closed unit ball and $S(X)=\{x\in X; \|x\|=1\}$ the unit sphere. Let A,G be two nonempty subsets of X, define the following:

$$d(y,A)=\inf_{x\in A}\|x-y\|, \quad r(y,A)=\sup_{x\in A}\|x-y\|,$$

$$h(A,G)=\max\left\{\sup_{x\in A} d(x,G),\ \sup_{y\in G} d(y,A)\right\},$$

the Hausdorff distance of A,G.

$r_A(G)=\inf_{y\in G} r(y,A)$, the Chebyshev radius of A with respect to G.

$G_\varepsilon(A)=\{g\in G;\ r(g,A)\leq r_A(G)+\varepsilon\}$, $\varepsilon\geq 0$,

the ε-Chebyshev centers of A in G.

$G(A)=G_0(A)$, the Chebyshev centers of A in G.

It is easy to see that $G_\varepsilon(A)\neq\emptyset$ for any $\varepsilon>0$ and $G(A)=\bigcap_{\varepsilon>0} G_\varepsilon(A)$. We denote the elements in G(A) (if not empty) by g_A.

Recall that the modulus of convexity of a Banach space X is

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2}\ ;\|x-y\|=\varepsilon,\ x,y\in S(X)\right\}$$

and X is called uniformly convex (p-uniformly convex) if $\delta_X(\varepsilon)>0$ ($\delta_X(\varepsilon)\geq c\varepsilon^p$ for some constant $c>0$) for every $\varepsilon>0$.

In [9], Szeptycki and Vlech proved the following

<u>THEOREM A:</u> If X is a Hilbert space, then for any two compact subsets A,B of X the following inequality holds

$$\|x_A-x_B\|^2\leq \left[r_A(X)+r_B(X)+h(A,B)\right]h(A,B) \qquad (1)$$

They also raised

<u>PROBLEM 1:</u> Does (1) remain valid without the

compact assumption?

PROBLEM 2: Does there exist a sharp estimate of $\|x_A-x_B\|$ similar to (1) in the case when X is a uniformly convex Banach space?

Recently, the problem 1 was solved in the affirmative in [8].

One purpose of this paper is to solve the problem 2. We show that $\|x_A-x_B\|$ admits an estimation similar to (1) if and only if X is a p-uniformly convex Banach space. Moreover, a sharp estimate of $\|g_A-f_B\|$ for any subsets A,B of X and closed convex subsets G,F of X when X is uniformly convex is obtained.

The another part of this paper is mainly devoted to the study of the way in which perturbations of ε and the sets A,G affect the $G_\varepsilon(A)$, and we show that $G_\varepsilon(A)$ is Lipschitzian stable under the perturbations of ε and A,G. In fact, the estimation of $h(G_\varepsilon(A),F_\eta(B))$ is given in general normed spaces in the case G,F are convex, which generalizes one of the results in [2].

II The Solution to Problem 2

We begin with some lemmas.

LEMMA 1. If G is a closed convex subset of X, $A \subset X$ and $g_A \in G(A)$, then for any $g \in G$, $\varepsilon > 0$, there exist $a_\varepsilon \in A$ and $x^* \in U(X^*)$ such that

$$x^*(a_\varepsilon - g_A) \geq r_A(G) - \varepsilon \text{ , and } x^*(g_A - g) \geq -\varepsilon .$$

PROOF: It follows immediately from the Freilich-Mclaughin-Amir result [3].

LEMMA 2. [5, p190, Prop. 1 & p194.] If X is uniformly convex, then for all $x, y \in X$ and $1 < p < +\infty$,

$$\left\|\tfrac{1}{2}(x+y)\right\|^p \leq \tfrac{1}{2}\left(1 - \delta_p\left(\frac{\|x - y\|}{\max(\|x\|, \|y\|)}\right)\right)(\|x\|^p + \|y\|^p)$$

where $\delta_p(\varepsilon) \geq c_p \delta(\varepsilon)$ for some constant c_p only depending on p.

Now we are ready to prove

THEOREM 3. If X is uniformly convex, A,B are subsets of X, and G,F are closed convex subsets of X, then $\|g_A - f_B\|$ satisfies

$$\delta\left((\alpha+h)^{-1}\|g_A - f_B\|\right) \leq c_p' \frac{(\alpha+h+h_1)^p - \beta^p}{(\alpha+h)^p}$$

where $\alpha = \max(r_A, r_B)$, $\beta = \min(r_A, r_B)$, $r_A = r_A(G)$, $r_B = r_B(F)$, $h = h(A,B)$, $h_1 = h(F,G)$ and c_p' is a constant only depending on p, $1 < p < +\infty$.

PROOF: Without loss of generality, we assume $\alpha = r_B$, $\beta = r_A$. Since $f_B \in F$, by definition of $h(F,G)$, for any $\varepsilon > 0$, there exists $g_\varepsilon \in G$ such that

$$\|f_B - g_\varepsilon\| < h_1 + \varepsilon.$$

By lemma 1, there exist $a_\varepsilon \in A$ and $x_\varepsilon^* \in U(X^*)$ such that $x_\varepsilon^*(a_\varepsilon - g_A) \geq r_A - \varepsilon$ and $x_\varepsilon^*(g_A - g_\varepsilon) \geq -\varepsilon$.

Let $x = a_\varepsilon - f_B$, $y = a_\varepsilon - g_A$.

Obviously, $\|y\| \leq r_A$ (2)

For any n, choose $b_n \in B$ such that $\|a_\varepsilon - b_n\| \leq h + \frac{1}{n}$

then $\|x\| = \|a_\varepsilon - f_B\| \leq \|a_\varepsilon - b_n\| + \|b_n - f_B\| \leq h + \frac{1}{n} + r_B$

so $\|x\| \leq h + r_B$ (3)

By lemma 2,

$$2\left\|\frac{x+y}{2}\right\|^p \leq (\|x\|^p + \|y\|^p)\left(1 - \delta_p\left(\frac{\|g_A - f_B\|}{\max(\|x\|, \|y\|)}\right)\right)$$

$$\leq ((h + r_B)^p + r_A^p)\left(1 - c_p\delta\left(\frac{\|g_A - f_B\|}{\max(\|x\|, \|y\|)}\right)\right)$$

$$\leq 2(r_B + h)^p\left(1 - c_p\delta\left(\frac{\|g_A - f_B\|}{\alpha + h}\right)\right) \qquad (4)$$

On the other hand,

$$\left\|\frac{1}{2}(x+y)\right\| = \left\|a_\varepsilon - g_A + \frac{1}{2}(g_A - g_\varepsilon) + \frac{1}{2}(g_\varepsilon - f_B)\right\|$$

$$\geq \max\left(0,\ x_\varepsilon^*\left(a_\varepsilon - g_A + \frac{1}{2}(g_A - g_\varepsilon) + \frac{1}{2}(g_\varepsilon - f_B)\right)\right)$$

$$\geqslant \max(0, (r_A - \varepsilon - \frac{\varepsilon}{2} - \frac{1}{2}(h_1 + \varepsilon))) \qquad (5)$$

Combine (4) and (5), we have

$$1 - c_p \delta\left(\frac{\|g_A - f_B\|}{\alpha + h}\right) \geqslant (r_B + h)^{-p}(\max(0, r_A - \frac{1}{2}h_1 - 2\varepsilon))^p$$

Let $\varepsilon \longrightarrow 0$, then

$$1 - c_p \delta\left(\frac{\|g_A - f_B\|}{\alpha + h}\right) \geqslant (r_B + h)^{-p}(\max(0, r_A - \frac{1}{2}h_1))^p$$

Therefore,

$$\delta\left(\frac{\|g_A - f_B\|}{+h}\right) \leqslant c_p'(1 - (r_B + h)^{-p}(\max(0, r_A - \frac{1}{2}h_1))^p) \qquad (6)$$

We claim that

$$(1 - (r_B + h)^{-p}(\max(0, r_A - \frac{1}{2}h_1))^p) \leqslant \frac{(r_B + h + h_1)^p - r_A^p}{(r_B + h)^p} \qquad (7)$$

In fact, (I) if $r_A - \frac{h_1}{2} \leqslant 0$, then

$$\frac{(r_B + h + h_1)^p - r_A^p}{(r_B + h)^p} \geqslant \frac{(r_B + h)^p + h_1^p - r_A^p}{(r_B + h)^p} \geqslant 1 + \frac{h_1^p - (\frac{h_1}{2})^p}{(r_B + h)^p} \geqslant 1 =$$

$$= (1 - (r_B + h)^{-p}(\max(0, r_A - \frac{1}{2}h_1))^p).$$

(II) if $r_A - \frac{h_1}{2} > 0$, then $r_B + h \geqslant r_A - \frac{1}{2}h_1$ (since $r_B = \alpha \geqslant \beta = r_A$), so

$$(r_B + h)^p - (r_A - \frac{1}{2}h_1)^p \leqslant (r_B + h + \frac{1}{2}h_1)^p - (r_A - \frac{1}{2}h_1 + \frac{1}{2}h_1)^p$$

$$\leqslant (r_B + h + h_1)^p - r_A^p$$

(here we use the inequality: $x^p - y^p \leqslant (x + \varepsilon)^p - (y + \varepsilon)^p$

for $x \geqslant y > 0$, $\varepsilon > 0$, $1 \leqslant p < +\infty$)

From (I), (II), the claim follows.

By (6) and (7)

$$\delta\left(\frac{\|g_A - f_B\|}{\alpha+h}\right) \leqslant c_p' \frac{(r_B+h+h_1)^p - r_A^p}{(r_B+h)^p}$$

$$= c_p' \frac{(\alpha+h+h_1)^p - \beta^p}{(\alpha+h)^p} \qquad \text{Q.E.D.}$$

<u>COROLLARY 4:</u> If X is uniformly convex, A, B, G and F as in theorem 3, then

$$\delta\left(\frac{\|g_A - f_B\|}{\alpha+h}\right) \leqslant c \frac{r_A+r_B+h+h_1}{(\alpha+h)^2}(h+h_1),$$ where c is a constant.

<u>PROOF:</u> Choose p=2 in theorem 3 and note that

$$\alpha - \beta \leqslant h+h_1. \qquad \text{Q.E.D.}$$

<u>REMARK:</u> The corollary 4 shows that if X is a uniformly convex Banach space, then the mapping $T: \mathcal{B}_1(X) \times \mathcal{B}_2(X) \longrightarrow X$ defined by $T(G,A)=g_A$ is uniformly continuous on bounded subsets of its domain, where $\mathcal{B}_1(X)$ denotes all the bounded closed convex subsets of X and $\mathcal{B}_2(X)$ all the bounded subsets.

If X is p-uniformly convex, the situation is more satisfactory. We have

<u>COROLLARY 5:</u> If X is p-uniformly convex, then there exists a constant c such that

$$\|g_A - f_B\|^p \leqslant c((\alpha + h + h_1)^p - \beta^p)$$

<u>PROOF:</u> Use the definition of "p-uniformly convex" and theorem 3. Q.E.D.

<u>COROLLARY 6:</u> With the same assumption on X as in corollary 5, the following estimation holds $\|g_A - g_B\|^p \leqslant c((\alpha + h)^p - \beta^p)$ with c a constant.

<u>REMARK 1:</u> In the case where X is 2-uniformly convex, we get the estimate of Theorem A up to a constant.

<u>REMARK 2:</u> As well-known, $L_p(\mu)$ is p-uniformly convex when $p \geqslant 2$, and is 2-uniformly convex when $1 < p < 2$ (see [7]). Also, every superreflexive Banach space can be renormed to be p-uniformly convex for some $p \geqslant 2$ (see [5 , p273]).

The following result shows that the converse of corollary 6 also holds.

<u>THEOREM 7:</u> If X is a Banach space and there exists a constant c such that for any $A, B \subset X$,

$\|x_A - x_B\|^p \leqslant c((r_B(X) + h(A,B))^p - r_A^p(X))$,

where $x_A \in X(A)$, $x_B \in X(B)$, then X is a p-uniformly convex space.

PROOF: For any $x,y \in S(X)$, $\|x-y\| = \varepsilon$, we will use the same planar sets used in [1]. Let

$A = co(x, -y, \frac{x+y}{\|x+y\|}, -\frac{x+y}{\|x+y\|})$ and

$B = co(x, -y, \frac{x+y}{2}, -\frac{x+y}{2})$, then $0 \in X(A)$ and

$\frac{x-y}{4} \in X(B)$, $h(A,B) = 1 - \frac{\|x+y\|}{2}$, $r_A(X)=1$ and $r_B(X) \leqslant 1$.

By the assumption, $\|\frac{x-y}{4}\|^p \leqslant c((r_B + 1 - \|\frac{x+y}{2}\|)^p - 1)$

hence $1 - \|\frac{x+y}{2}\| \geqslant (\frac{1}{c}(\frac{\varepsilon}{4})^p + 1)^{1/p} - 1$

i.e $\delta_X(\varepsilon) \geqslant (1 + \frac{1}{c}(\frac{\varepsilon}{4})^p)^{1/p} - 1 \geqslant 1 + \frac{1}{c}\frac{1}{p}(\frac{\varepsilon}{4})^p - 1 = c'\varepsilon^p$

where $c' = \frac{1}{c}\frac{1}{p}(\frac{1}{4})^p$, thus X is p-uniformly convex. Q.E.D.

REMARK 1: From the proof, it is enough to assume that the condition of theorem 7 holds for any two-dimensional subspace of X with a common constant c.

REMARK 2: Corollary 6 and theorem 7 give the solution to Problem 2.

To conclude this section, we give a result concerning the metric projection which is only a particular case of corollary 5.

COROLLARY 8: If X is p-uniformly convex, then for any $x,y \in X$ and closed convex subsets G,F,

$$\|P_G x - P_F y\|^p \leqslant c((\alpha + \|x-y\| + h(G,F))^p - \beta^p)$$

holds with some constant c, where P_G (P_F) is the metric projection on G (F), and

$\alpha = \max\{d(y,F),\ d(x,G)\}$, $\beta = \min\{d(y,F),\ d(x,G)\}$.

III. The Stability of ε-Chebyshev Centers

In this section, we shall prove

THEOREM 9: If X is a normed linear space, A,B are subsets of X and G,F are convex subsets of X, then $h(G_\varepsilon(A),F_\eta(B)) \leqslant h_1 + (2h+2h_1+|\varepsilon-\eta|)k(\varepsilon,\eta)$, where $h=h(A,B)$, $h_1=h(G,F)$ and

$$k(\varepsilon,\eta) = \frac{2\max\{r_A(G),r_B(F)\}+2h+2h_1+\min\{\varepsilon,\eta\}}{2h+2h_1+\min\{\varepsilon,\eta\}}$$

For the proof of this theorem, we need

LEMMA 10: If A is a subset of a normed linear space X and G is a convex subset of X, then for any $\varepsilon > 0$, $\eta > 0$, $h(G_{\varepsilon+\eta}(A),G_\varepsilon(A)) \leqslant f(\varepsilon,\eta)\eta$

where $\quad f(\varepsilon,\eta) = \dfrac{2r_A(G)+\varepsilon+\eta}{\varepsilon+\eta}$

PROOF: Let $z \in G_{\varepsilon+\eta}(A)$. Take, for sufficiently small $\delta > 0$, $x \in G_\delta(A)$ and let $\lambda = \dfrac{\eta}{\varepsilon+\eta-\delta}$,

$y=\lambda x+(1-\lambda)z$. Then, for any $a\in A$,

$\|a-y\|\leqslant\lambda(r_A(G)+\delta)+(1-\lambda)(r_A(G)+\varepsilon+\eta)=r_A(G)+\varepsilon$,

i.e $y\in G_\varepsilon(A)$, while $\|z-y\|=\lambda\|z-x\|$

$\leqslant\frac{\eta}{\varepsilon+\eta-\delta}(2r_A(G)+\delta+\varepsilon+\eta)$.

Since δ is arbitrary,

$d(z,G_\varepsilon(A))\leqslant\lim_{\delta\to 0}\frac{\eta}{\varepsilon+\eta-\delta}(2r_A(G)+\delta+\varepsilon+\eta)=\frac{\eta}{\varepsilon+\eta}(2r_A(G)+\varepsilon+\eta)$

Therefore $h(G_{\varepsilon+\eta}(A),G_\varepsilon(A))\leqslant f(\varepsilon,\eta)\eta$

where $f(\varepsilon,\eta)=\dfrac{2r_A(G)+\varepsilon+\eta}{\varepsilon+\eta}$. Q.E.D.

<u>The proof of theorem 9:</u> Fix an arbitrary $x\in G_\varepsilon(A)$, choose $f_n\in F$ such that $\|f_n-x\|<h_1+\frac{1}{n}$.

For any $b\in B$, by the definition of h, there is $a_b\in A$ such that $\|a_b-b\|<h+\frac{1}{n}$.

Now, $\|f_n-b\|\leqslant\|f_n-x\|+\|x-a_b\|+\|a_b-b\|$

$$\leqslant h_1+\frac{1}{n}+r_A(G)+\varepsilon+h+\frac{1}{n}$$

Since $|r_A(G)-r_B(F)|\leqslant h+h_1$,

$$\|f_n-b\|\leqslant r_B(F)+2h+2h_1+\varepsilon+\frac{2}{n}$$

hence $f_n\in F_{2h+2h_1+\varepsilon+2/n}(B)$.

By lemma 10, $d(f_n,F_\eta(B))\leqslant(2h+2h_1+\varepsilon-\eta+\frac{2}{n})f$,

where $f=(2r_B(F)+2h+2h_1+\varepsilon+\frac{2}{n})/(2h+2h_1+\varepsilon+\frac{2}{n})$.

So $d(x,F_\eta(B)) \leqslant \|x-f_n\|+d(f_n,F_\eta(B))$
$\leqslant h_1+\frac{1}{n}+(2h+2h_1+\varepsilon-\eta+\frac{2}{n})f.$

Let $n \longrightarrow +\infty$, $d(x,F_\eta(B)) \leqslant h_1+(2h+2h_1+\varepsilon-\eta)\bar{f}$

where $\bar{f}=(2r_B(F)+2h+2h_1+\varepsilon)/(2h+2h_1+\varepsilon)$.

Since x is arbitray in $G_\varepsilon(A)$,

$$\sup_{x\in G_\varepsilon(A)} d(x,F_\eta(B)) \leqslant h_1+(2h+2h_1+\varepsilon-\eta)\bar{f}$$

A symmetric argument yields

$$\sup_{y\in F_\eta(B)} d(y,G_\varepsilon(A)) \leqslant h_1+(2h+2h_1+\eta-\varepsilon)\bar{f}',$$

where $\bar{f}'=(2r_A(G)+2h+2h_1+\eta)/(2h+2h_1+\eta)$.

Hence $h(G_\varepsilon(A),F_\eta(B)) \leqslant h_1+(2h+2h_1+|\varepsilon-\eta|)k(\varepsilon,\eta)$,

where $k(\varepsilon,\eta)=\dfrac{2\max\{r_A(G),r_B(F)\}+2h+2h_1+\min\{\varepsilon,\eta\}}{2h+2h_1+\min\{\varepsilon,\eta\}}$, Q.E.D

<u>COROLLARY 11</u>: If G,F are convex subsets of a normed linear space X, then

$$h(prj_\varepsilon(x,G),prj_\eta(y,F)) \leqslant h+(2h+2\|x-y\|+|\varepsilon-\eta|)g$$

where $g=\dfrac{2\max\{d(x,G),d(y,F)\}+2h+2\|x-y\|+\min\{\varepsilon,\eta\}}{2h+2\|x-y\|+\min\{\varepsilon,\eta\}}$

$h=h(G,F)$, $prj_\varepsilon(x,G)=\{g\in G;\ \|g-x\| \leqslant d(x,G)+\varepsilon\}$

and similarly for $prj_\eta(y,F)$.

<u>PROOF:</u> Let $A=\{x\}$ and $B=\{y\}$, so $h(A,B)=\|x-y\|$

and $prj_\varepsilon(x,G)=G_\varepsilon(A)$, $prj_\eta(y,F)=F_\eta(B)$.

An appeal to theorem 9 yields the result. Q.E.D.

COROLLARY 12: ([4], theorem 2.1) Let C,D be two convex subsets of a normed linear space X. Given any $\varepsilon > 0$ and $x_0 \in X$, the following estimation holds

$$h(prj_{\varepsilon}(x_0,C),\ prj_{\varepsilon}(x_0,D)) \leq \rho_{\varepsilon}(\|x_0\|)h(C,D)$$

with $\rho_{\varepsilon}(\|x_0\|)=3+\frac{4}{\varepsilon}(d(x_0,C)+d(x_0,D))$.

REMARK: Theorem 9 shows that the set $G_{\varepsilon}(A)$ is Lipschitzian stable under the perturbation of G, A respectively.

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