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ON THE EXTREMALITY OF MEASURE EXTENSIONS

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Let $(M,\mathfrak{A},p)$ be a probability space and $\mathfrak{A}_1$ be a σ-algebra on M with $\mathfrak{A}_1 \supset \mathfrak{A}$. We consider the set $\mathcal{F} = \mathcal{F}(p|\mathfrak{A}, \mathfrak{A}_1)$ of all (σ-additive) measure extensions $p_1|\mathfrak{A}_1$ of $p|\mathfrak{A}$ and the set ex $\mathcal{F}$ of all extremal elements of $\mathcal{F}$. There are well-known criteria for a $p_1 \in \mathcal{F}$ to be extremal (see [3], [4], [6], [7] et al.). In rather special cases, there exists an integral representation of $\mathcal{F}$ by ex $\mathcal{F}$ ([4], [5], [8], [9], [10] et al.). In this paper, we explicitly present ex $\mathcal{F}$, if $\mathfrak{A}_1$ is generated from $\mathfrak{A}$ by adjunction of countably many disjoint sets (Thm. 1 b)). For an arbitrary target σ-algebra $\mathfrak{A}_1$, we characterize the extremality of a $p_1 \in \mathcal{F}$ by the maximality of the null set ideal $\mathfrak{N}(p_1)$ (Thm. 2, as an additional criterion beside the well-known ones). If $\mathfrak{A}_1$ is generated from $\mathfrak{A}$ by a finite partition of M, $\mathcal{F}$ is the σ-convex hull of ex $\mathcal{F}$ (Thm. 3). The analogon of Thm. 3 for a countably infinite partition is not true, as an example shows. The fundamental lemma for the proofs of these theorems is a theorem of 1961 ([1], Satz 2 A) on the representation of $\mathcal{F}$ in case of $\mathfrak{A}_1$ being generated from $\mathfrak{A}$ by a countable partition.

Some results of this paper have been demonstrated in diploma theses which have been made under guidance of the two authors, especially Thm. 2 by Hans Gail and Thm. 3 with the following Notice by Ruth Bierlein.

1. PRELIMINARIES

According to the criteria of extremality mentioned above, a measure extension $p_1 \in \mathcal{F}$ is extremal (i.e. $p', p'' \in \mathcal{F}$ with $p_1 = \frac{1}{2}(p' + p'')$ implies $p' = p''$) if and only if

for any $K \in \mathfrak{A}_1$ there is a set $A \in \mathfrak{A}$ such that $p_1(K \Delta A) = 0$ [1)]

and consequently, if and only if

p_1 is minimal in $\mathcal{F}$ with respect to (Radon-Nikodym) dominance "$\ll$". [2)]

The latter condition is equivalent to condition

(1) There does not exist a $p' \in \mathcal{F}$ satisfying $p' \ll p_1$ and $p' \neq p_1$.

From (1) we get

(2) There does not exist a $p' \in \mathcal{F}$ satisfying $\mathfrak{N}(p') \supset \mathfrak{N}(p_1)$ and $\mathfrak{N}(p') \neq \mathfrak{N}(p_1)$.

Thus, as a trivial result, we have

(3) If $p_1 \in \mathrm{ex}\,\mathcal{F}$ then $\mathfrak{N}(p_1)$ is maximal on $\mathcal{F}$.

The other direction is not so trivial: For any $p_1 \in \mathcal{F} - \mathrm{ex}\,\mathcal{F}$ there exist different extensions p' and p'' of $\mathcal{F}$ satisfying $p_1 = \frac{1}{2}(p' + p'')$. Then, this p' is different from p_1 and keeps $\mathfrak{N}(p') \supset \mathfrak{N}(p_1)$ and therefore $p' \ll p_1$ (according to the negation of (1)), but, in generally, we have $\mathfrak{N}(p_1) = \mathfrak{N}(p')$. A $p_2 \in \mathcal{F}$ where $\mathfrak{N}(p_1)$ is a proper subset of $\mathfrak{N}(p_2)$, is much more scarce than a $p' \in \mathcal{F}$ which is dominated by p_1, as we will show next.

Let $p_1, p_2 \in \mathcal{F}$ and $p_1 \neq p_2$. Then we define the bounds

$$\lambda_1 = \lambda_1(p_1,p_2) := \inf\{\lambda \in \mathbb{R} : \lambda p_2(K) + (1-\lambda)p_1(K) \geq 0 \text{ for all } K \in \mathfrak{A}_1\}$$

$$\lambda_2 = \lambda_2(p_1,p_2) := \sup\{\lambda \in \mathbb{R} : \lambda p_2(K) + (1-\lambda)p_1(K) \geq 0 \text{ for all } K \in \mathfrak{A}_1\}$$

and the following "straight" subsets of $\mathcal{F}$

$$L(p_1,p_2) := \{\lambda p_2 + (1-\lambda)p_1|\mathfrak{A}_1 : \lambda_1 \leq \lambda \leq \lambda_2\}$$

$$L_o(p_1,p_2) := \{\lambda p_2 + (1-\lambda)p_1|\mathfrak{A}_1 : \lambda_1 < \lambda < \lambda_2\}.$$

If $\lambda_1(p_1,p_2)$ is negative (i.e. $p_1 \in L_o(p_1,p_2)$), then $\mathfrak{N}(p_1) = \mathfrak{N}(p')$ for all $p' \in L_o(p_1,p_2)$. An additonal null set may occur at best at a marginal element of $L(p_1,p_2)$. But, also this does not need, as one can see studying an *example* of the following kind: $\mathfrak{A}_1$ is generated by $\mathfrak{A} = \{\emptyset, M\}$ and the countably many

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1) See [3], [4], [6], [7] et al.

2) See [6], Korollar 1.5

disjoint sets $\{K_1, K_2, \dots, K_1', K_2', \dots\}$, $\Sigma K_n + \Sigma K_n' = M$; p_1 and p_2 are probabilities with the properties

(i) $p_1(K_n) < p_2(K_n)$ and $p_1(K_n') > p_2(K_n')$ for all $n \in \mathbb{N}$,

(ii) the zero points x_n of the functions $f_n(x) := xp_2(K_n) + (1-x)p_1(K_n)$ satisfy $x_n \nearrow \lambda_1$ and $x_n < \lambda_1$,

(iii) the points y_n defined analogously for K_n' instead of K_n satisfy $y_n \searrow \lambda_2$ and $y_n > \lambda_2$.

Then $\mathfrak{N}(p') = \{\emptyset\}$ for all $p' \in L(p_1, p_2)$.

Now, we return to the general case using the following definition of a "inner kernel" $\mathcal{F}^i$ of $\mathcal{F}$.

Definition: $p' \in \mathcal{F}^i$ if and only if for any $p_1 \in \mathcal{F}$ there exists a positive number $\varepsilon = \varepsilon(p', p_1)$ such that

$$\lambda p_1 + (1-\lambda)p' \in \mathcal{F} \text{ holds for all } \lambda \in [-\varepsilon, 1].$$

Then we have

LEMMA 1. a) $\mathfrak{N}(p')$ equals for all $p' \in \mathcal{F}^i$.

b) If $\mathcal{F}^i \neq \emptyset$, then $\mathfrak{N}(p') = \cap \{\mathfrak{N}(p_1) : p_1 \in \mathcal{F}\}$ for all $p' \in \mathcal{F}^i$ and, consequently, $p_1 \ll p'$ for all $p_1 \in \mathcal{F}$, $p' \in \mathcal{F}^i$.

Proof: Suppose $p' \in \mathcal{F}^i$, $p_1 \in \mathcal{F}$ and $p_1(A) > 0 = p'(A)$. Then, for $\varepsilon := \varepsilon(p', p_1)$ and $p_\varepsilon := -\varepsilon p_1 + (1+\varepsilon)p'$, we obtain $p_\varepsilon(A) < 0$ as a contradiction to $p_\varepsilon \in \mathcal{F}$. From this we get b) and hence a). ∎

2. ON REPRESENTATION OF ex $\mathcal{F}$

At first we consider the case of a target σ-algebra $\mathfrak{A}_1 = {}^B(\mathfrak{A} \cup \mathfrak{Z})$ where $\mathfrak{Z} = \{K_\nu : \nu \in \mathbb{N}\}$ is a partition of M. For this case we define D to be the set of all sequences $(d_\nu)_\nu$ of $\mathfrak{A}$-measurable mappings

$$d_\nu : M \to [0,1]$$

satisfying

$$\chi_{{}_*K_\nu} \le d_\nu \le \chi_{{}^*K_\nu} \quad p\text{-a.e. and } \sum_\nu d_\nu = 1 \text{ on } M,$$

where ${}_*K_\nu$ and ${}^*K_\nu$ are some p-measurable kernels and hulls of K_ν, respectively. For any $d = (d_\nu)_\nu \in D$ we define $p^{(d)}|\mathfrak{A}_1$ by

$$p^{(d)}(\sum_\nu A_\nu \cdot K_\nu) = \sum_\nu \int_{A_\nu} d_\nu \, dp \quad (\text{for } A_\nu \in \mathfrak{A} \text{ for all } \nu).$$

Using these notations we have

THEOREM 1. Let $\mathfrak{Z}$ be a countable partition of M and $\mathfrak{A}_1 = {}^B(\mathfrak{A} \cup \mathfrak{Z})$. Then the following representations are true:

a) $\mathcal{F} = \{p^{(d)} | \mathfrak{A}_1 : \ d \in D\}$.

b) $\mathrm{ex}\,\mathcal{F} = \{p^{(d)} | \mathfrak{A}_1 : d \in D \text{ with } p(d_\nu \in \{0,1\}) = 1\}$.

Proof: Part a) is due to [1], Satz 2A.

The first direction of part b) may be proved indirectly:

Let $p_1 = p^{(d)}$, $d \in D$ with $p(0 < d_{\nu_0} < 1) > 0$. Then there exists an integer $\nu_1 \neq \nu_0$ and a positive ε such that also the set

$$A := \{x \in M: \ \varepsilon < d_{\nu_0}(x) < 1-\varepsilon, \quad \varepsilon < d_{\nu_1}(x) < 1-\varepsilon\}$$

has a positive p-measure. Now, we define two elements d' and d" of D by

$$d'_\nu(x) := \begin{cases} d_\nu(x) - \varepsilon\chi_A(x) & \text{for } \nu = \nu_0 \\ d_\nu(x) + \varepsilon\chi_A(x) & \text{for } \nu = \nu_1 \\ d_\nu(x) & \text{otherwise} \end{cases} \qquad d''_\nu(x) := \begin{cases} d_\nu(x) + \varepsilon\chi_A(x) & \text{for } \nu = \nu_0 \\ d_\nu(x) - \varepsilon\chi_A(x) & \text{for } \nu = \nu_1 \\ d_\nu(x) & \text{otherwise.} \end{cases}$$

The measure extensions $p^\lambda := p^{(d^\lambda)}$ satisfy

$$p_1 = \frac{1}{2}(p' + p'') \text{ with } p''(AK_{\nu_0}) - p'(AK_{\nu_0}) = 2\varepsilon\, p(A) > 0.$$

Therefore p_1 belongs to $\mathcal{F} - \mathrm{ex}\,\mathcal{F}$.

To prove the other direction of part b), let $p_1 = p^{(d)}$, $d \in D$ with $p(0 < d_\nu < 1) = 0$ for all ν. Now, for $p^\lambda = p(d^\lambda)$, $d^\lambda \in D$ with $p_1 = \frac{1}{2}(p' + p'')$ we have p-a.e.

$$d_\nu = \frac{1}{2}(d'_\nu + d''_\nu) \text{ and, thus,}$$

$$d'_\nu = d''_\nu = \begin{cases} 0 & \text{where } d_\nu = 0 \\ 1 & \text{where } d_\nu = 1, \end{cases}$$

and for that reason $p' = p''$.

Therefore p_1 is an element of $\mathrm{ex}\,\mathcal{F}$. ∎

For the *general* case of any target σ-algebra $\mathfrak{A}_1$, the maximality of the null set ideal is equivalent to the extremality of a measure extension:

THEOREM 2. $\mathrm{ex}\,\mathcal{F} = \{p_1 \in \mathcal{F} : \mathfrak{N}(p_1) \text{ is maximal on } \mathcal{F}\}$.

Proof: On account of our consideration above in section 1 item (3), it is sufficient to prove that for any $p_1 \in \mathcal{F} - \mathrm{ex}\,\mathcal{F}$, there exists a $p_0 \in \mathcal{F}$ such that $\mathfrak{N}(p_1)$ is a proper subset of $\mathfrak{N}(p_0)$.

According to the definition of ex $\mathcal{F}$, for $p_1 \in \mathcal{F}$ - ex $\mathcal{F}$ there exist two elements p' and p" of $\mathcal{F}$ and a set $L \in \mathfrak{A}_1$ such that

(1) $\quad p_1 = \frac{1}{2}(p' + p")$ and $p'(L) < p"(L)$.

Applying Thm. 1a for $\mathfrak{A}_L := {}^B(\mathfrak{A} \cup \{L\})$ as target σ-algebra, we can use the representations

$$p_1 = p^{(d)} \quad \text{and } p^\lambda = p^{(d^\lambda)} \text{ on } \mathfrak{A}_L,$$

where $d^\cdot = (d_L^\cdot, 1 - d_L^\cdot)$ for $d^\cdot \in \{d, d', d"\}$. From (1) we get

$$d_L = \frac{1}{2}(d_L' + d_L") \quad p\text{-a.e.}$$

and, using the notation $B_o := \{x \in M: 0 < d_L(x) < 1\}$,

(2) $\quad p(B_o) \geq p(d_L' > d_L") > 0.$

We define $p^{(d^o)} | \mathfrak{A}_L$ by

$$d_L^o(x) := \begin{cases} 0 & \text{for all } x \text{ with } d_L(x) < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then we have

(3) $\quad \mathfrak{N}\{p_1 | \mathfrak{A}_L) \subset \mathfrak{N}(p^{(d^o)} | \mathfrak{A}_L);$

for $p_1(A_1 L + A_2 \bar{L}) = 0$ implies

$$d_L = \begin{cases} 0 \text{ on } A_1 \\ 1 \text{ on } A_2 \end{cases} \quad p\text{-a.e.}$$

and hence

$$d_L^o = d_L \text{ on } A_1 \cup A_2 \quad p\text{-a.e..}$$

Furthermore, $B_o L \in \mathfrak{N}(p^{(d^o)} | \mathfrak{A}_L) - \mathfrak{N}(p_1 | \mathfrak{A}_L)$ is true because

(4) $\quad p^{(d^o)}(B_o L) = \int_{B_o} d_L^o \, dp = 0$

and, as a consequence of (2),

(5) $\quad p_1(B_o L) = \int_{B_o} d_L \, dp > 0.$

Because of (3) there exists a $\mathfrak{A}_L$-measurable Radon-Nikodym derivative $f|M$ satisfying

(6) $\quad p^{(d^o)}(K) = \int_K f \, d p_1 \quad$ for all $K \in \mathfrak{A}_L$.

Using this integral representation, we extend $p^{(d^o)} | \mathfrak{A}_L$ to a measure p_o on $\mathfrak{A}_1$:

$p_0(K) := \int_K f\, dp_1$ for all $K \in \mathfrak{A}_1$.

This $p_0|\mathfrak{A}_1$ is an element of $\mathcal{F}$ because of $p_0|\mathfrak{A} = p^{(d^0)}|\mathfrak{A} = p|\mathfrak{A}$, and $\mathfrak{N}(p_1|\mathfrak{A}_1)$ is a proper subset of $\mathfrak{N}(p_0|\mathfrak{A}_1)$ because of (4), (5) and (6). ■

3. ON REPRESENTATION OF $\mathcal{F}$ BY ex $\mathcal{F}$.

It is well known that ex$\mathcal{F}$ can be empty also in case of non empty $\mathcal{F}$ (see e.g. [2]). In special cases of non empty ex $\mathcal{F}$ one can represent $\mathcal{F}$ by integrals defined on suitable σ-algebras of subsets of ex $\mathcal{F}$ (see [4], [5], [8], [9], [10] et al.). Even if $\mathfrak{A}_1 = {}^B(\mathfrak{A} + \{K\})$, $\mathcal{F}$ is not the convex hull of ex $\mathcal{F}$ in general, as one can see easily. Here we will demonstrate that $\mathcal{F}$ equals the σ-convex hull of ex $\mathcal{F}$, if $\mathfrak{A}_1$ is generated by $\mathfrak{A}$ and finite many additional sets.

Definition: If $\mathcal{G}$ is a set of measures, we define the σ-convex hull of $\mathcal{G}$ by

$$\sigma\mathrm{co}\, \mathcal{G} := \{ \sum_{i\in I} \lambda_i p_i : p_i \in \mathcal{G},\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1,\ I \text{ countable}\}.$$

Now, the following statement is true:

THEOREM 3. If $\mathfrak{A}_1 = {}^B(\mathfrak{A} \cup \{K_1,\dots K_n\})$ with $\sum_{v=1}^{n} K_v = M$, then $\mathcal{F} = \sigma\mathrm{co\,ex}\, \mathcal{F}$.

Proof: Of course, any element of σ co ex $\mathcal{F}$ belongs to $\mathcal{F}$. For the other direction of the proof, we have according to Thm. 1 this task:

For any $d = (d_1,\dots d_n) \in D$ we have to construct sequences

(1) $(\lambda_i)_i$ of non negative numbers with $\Sigma\lambda_i = 1$

and

(2) $(B_v^i)_i$ of members of $\mathfrak{A}$ for $v = 1,\dots n$ satisfying $g^i \in D$ for all $i \in \mathbb{N}$,
where $g_v^i := \chi_{B_v^i}$ and $g^i := (g_1^i,\dots g_n^i)$,

such that

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \Sigma \lambda_i \begin{pmatrix} g_1^i \\ \vdots \\ g_n^i \end{pmatrix} \quad \text{on } M.$$

To fullfil this task we use an iterative exhaustion procedure:

Starting with $i = 1$ for any $x \in M$ there exists an integer v_x satisfying

(3) $d_{v_x}(x) \geq \frac{1}{n} \sum_{v=1}^{n} d_v(x) = \frac{1}{n} =: \lambda_1$.

(It is evident that, in case of finite many K_v, such a v_x does exist - in case of infinite many disjoint sets $K_1, K_2, \dots$ our procedure does not work at this point.) We define

$$B_v^1 := \{x \in M: d_v(x) \geq \lambda_1,\ d_\mu(x) < \lambda_1 \text{ for all } \mu < v\}$$

and obtain

(4) $\sum_v B_v^1 = M, \quad \sum_v g_v^1 = 1$ on M.

In step κ, we consider $d^\kappa := d - \sum_{i<\kappa} \lambda_i g^i$ instead of d. For any $x \in M$ there exists an integer v_x satisfying

(5) $d_{v_x}^\kappa(x) \geq \frac{1}{n} \sum_{v=1}^{n} d_v^\kappa(x) = \frac{1}{n}(1 - \sum_{i<\kappa} \lambda_i) = \frac{1}{n}(1 - \frac{1}{n})^{\kappa-1} =: \lambda_\kappa.$

Now we define

$$B_v^\kappa := \{x \in M: d_v^\kappa(x) \geq \lambda_\kappa,\ d_\mu^\kappa(x) < \lambda_\kappa \text{ for all } \mu < v\}.$$

For all $\kappa \in \mathbb{N}$ we have

(6) $\sum_v B_v^\kappa = M, \quad \sum_v g_v^\kappa = 1$ on M.

The numbers λ_i and sets B_v^i defined above constitute a solution of our task; for the following statements (7)...(9) are true:

(7) $\sum_i \lambda_i = 1.$

(8) $d = \sum_i \lambda_i g^i$ on M,

because

$$d_v - \sum_{i<\kappa} \lambda_i g_v^i = d_v^\kappa \leq \sum_v d_v^\kappa = (1 - \tfrac{1}{n})^{\kappa-1} \to 0 \text{ on M for all } v.$$

(9) $g^i \in D$,

because of (4) and (6) where we take notice of the fact that the relation

$$_*K_v \subset B_v^i \subset {}^*K_v \quad \text{for all } v$$

is true except p-null sets. ∎

<u>NOTICE.</u> The analogon of Thm. 3 for the case of infinite many disjoint sets K_1, $K_2, \dots$ does not hold. To prove this we study the following

Counterexample: $\mathfrak{A} = {}^B\{A_n: n \in \mathbb{N}\}$ with $\sum A_n = M$ and $p(A_n) > 0$ for all n

$\mathfrak{Z} = \{K_{n,v}: v \leq n,\ n \in \mathbb{N}\}$ with $\sum_{v=1}^{n} K_{n,v} = A_n$ for all n

and $K_{n,v} \neq \emptyset$ for all n and v

$$p_1 = p^{(d)}, \text{ where } d_{n,v} = \frac{1}{n} \chi_{A_n} \text{ for all } v \text{ and } n.$$

Now let us suppose, that $p^{(d)} = \sum_i \lambda_i p^{(g^i)}$ is a representation of p_1 in the sense of Thm. 3, i.e. in detail we assume

$$d_{n,v} = \sum_i \lambda_i \, g^i_{n,v} \text{ on } A_n \text{ for all } n, v$$

where

(1) $g^i_{n,v}$ is the indicator function of a suitably choosen set $B_{n,v} \in \mathfrak{A}$,

(2) $\sum_{v \leq n} g^i_{n,v} = 1$ on A_n for all n,

(3) $\lambda_i \geq 0, \quad \sum_i \lambda_i = 1.$

Then, for any n,v and i, we have

$$\lambda_i \, g^i_{n,v}(x) \leq d_{n,v}(x) = \frac{1}{n} \text{ for all } x \in A_n.$$

Because of (1) and (2), this implies

$$\lambda_i \leq \frac{1}{n} \text{ for all } n \text{ and } i$$

and therefore

$$\lambda_i = 0 \text{ for all } i$$

being inconsistent with (3). ∎

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