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A Note on the kernel of the $\bar{\partial}$-Neumann operator on strongly pseudo-convex domains

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Abstract

In this paper, we discuss the relations between a special Heisenberg coordinate system and a normalized Levi metric on strongly pseudo-convex domains in $\mathbb{C}^n$ and see how they are related to the $\bar{\partial}$-Neumann operator.

1.Introduction

The study of boundary regularity for solution of $\bar{\partial}$-Neumann problem

(1.1) $\quad \square u=(\bar{\partial}\,\bar{\partial}^*+\bar{\partial}^*\bar{\partial})u=f$

(1.2) $\quad u\in \text{domain } (\bar{\partial}^*), \quad \bar{\partial} u\in \text{domain } (\bar{\partial}^*)$

on pseudo-convex domains Ω in $\mathbb{C}^{n+1}$ has been an interesting theme in the theory of several complex variables for many years. The existence and regularity properties of the $\bar{\partial}$-Neumann operator N (the parametrix for this problem) on strongly pseudo-convex domains were well understood by the results of Kohn [6], Greiner-Stein [8] (for "Levi metric" case), Beals-Greiner-Stanton [2], and Chang [4] (for "non-Levi metric" cases).

On the other hand, it is very interesting to give a more explicit constructions of the operator N, which expresses the solution u in terms of f. The result of the construction of N were essentially achieved by two different methods. The first method (Phong [15] and Phong-Stein [16],[17]) involves techniques in partial differential equations; the second method (Lieb and Range [13],[14]) uses integral formulas.

Phong's result [15] is based on the point of view of the Dirichlet problem for the complex Laplacian on the "model" case $D=\{(z_1,z_2,\ldots,z_{n+1})\in \mathbb{C}^{n+1}: \Im z_{n+1}>\sum_{1\le k\le n}|z_k|^2\}$ which was equipped with a Levi metric. He discovered that the kernel N_1 is a mixed type homogeneity kernel (mixed the Euclidean and the Heisenberg homogeneity). But unlike the parametrix of the boundary

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complex Laplacian $\Box_b$, Phong's result cannot be transferred to Ω directly by standard Heisenberg coordinates (we will explain this in section 2). After few years of effort, Phong and Stein [16],[17] achieved this goal by using **special Heisenberg coordinates.**

The method of Lieb and Range [12],[13],[14] is more fundamental, they used a generalized Bochner-Martinelli-Koppelman integral formula whose boundary term can be approximated with a Henkin type kernel. (Here the boundary geometry comes into play). In the papers [13] and [14], they found an integral operators T_1 and $(T_0)^*$ which are defined explicitly in terms of the geodesic distance function for the given Hermitian metric (the Euclidean homogeneity part) and the Levi polynomial of a plurisubharmonic defining function for Ω (the Heisenberg homogeneity part). T_1 and $(T_0)^*$ give the principal kernel to represent $\overline{\partial}^*N_1f$ and $\overline{\partial}N_0f$ in terms of f. Under the assumption that $\overline{\Omega}$ is equipped with a "**normalized Levi metric**" (we will explain this in section 2), Lieb and Range solved the system of partial differential equations:

$$(1.3) \qquad \overline{\partial}K_1=\ker(T_1), \qquad \overline{\partial}^*K_1=\ker[(T_0)^*]$$

to get the kernel K_1 for the $\overline{\partial}$-Neumann operator N_1.

It seems these two methods based on totally different philosophies. However, on the "model" **D** a straightforward computation shows that Lieb-Range's kernel is equal to Phong's kernel plus an acceptable error term. How about the general domain Ω? As we mentioned at the beginning, Phong-Stein need to use a special Heisenberg coordinates to transfer the kernel N_1 to Ω. But Lieb-Range just use a standard Heisenberg coordinates to simplify the computations in their works (but they need to assume $\overline{\Omega}$ is equipped with a normalized Levi metric to construct the kernel K for N_1). In this paper, we see how the special Heisenberg coordinates and a normalized Levi metric are related. The author would like to reiterate his thanks to his teacher and advisor E.M.Stein, not only for his many valuable suggestions but for his inspiring example in research and teaching. The author would also like to thank the referee for giving him some nice suggestions.

2.The special Heisenberg coordinates and a normalized Levi metric

Let us review some properties of strongly pseudo-convex domains. There are **many** equivalent definitions of strong pseudo-convexity, see Kerzman [9] and Krantz [11]. We will use the one which leads most quickly to the Heisenberg group (see [7],[20],[21] about the basic properties of the Heisenberg group):

(2.1) DEFINITION (Folland-Stein [7]):

A bounded smooth domain $\Omega \in \mathbf{C}^{n+1}$ is strongly pseudo-convex if and only if every $\xi \in \partial\Omega$ there are local holomorphic coordinates in which $\partial\Omega$ is osculated by the **Heisenberg group** $\mathbf{H}^n$ in the following sense ([7],[8],[18]):

There exists a map $\Theta_\xi : \partial\Omega \to \mathbf{H}^n$ which carries $\xi \in \partial\Omega$ to the origin and satisfies:

(i) Θ_ξ is a local diffeomorphism.

(ii) Θ_ξ is analytic (i.e., annihiliated by the $\overline{\partial_b}$-operator on $\partial\Omega$) to the third order at ξ, when $\mathbf{H}^n$ is viewed as sitting inside $\mathbf{C}^{n+1}$.

We may inspect this phenomena by Fefferman's result in [5], (p.17-p.18).

Moreover, we may make Θ depend smoothly on ξ. We write $\Theta(\xi,\zeta)$ for $\Theta_\xi(\zeta)$. Thus $\Theta : \partial\Omega \times \partial\Omega \to \mathbf{H}^n$. Furthermore,

(iii) for ζ close to ξ in $\partial\Omega$, $\Theta(\xi,\zeta) \approx [\Theta(\zeta,\xi)]^{-1}$ in $\mathbf{H}^n$.

For $\varepsilon > 0$ small, the hypersurface $\partial\Omega_\varepsilon = \{\zeta \in \Omega;\ \rho(\zeta) = \varepsilon\}$ is strongly pseudo-convex. For each $\xi \in \partial\Omega_\varepsilon$ we can thus find by the Levi procedure (see [8]) holomorphic coordinates $(w', w_{n+1}) \in \mathbf{C}^{n+1}$ for a neighborhood D_ξ of ξ in which the geodesic distance ρ becomes

$$\rho(\zeta) = \varepsilon + \Im w_{n+1} - |w'|^2 + O(|w_{n+1}|\cdot|w'| + \rho|w'|^2 + |w'|^3).$$

We set $\quad (w', s; \rho) \equiv (w', \Re w_{n+1}; \rho(\zeta)) \equiv (\Theta(\xi,\zeta); \rho(\zeta)) \in \mathbf{H}^n \times \mathbf{R}^+$

which will be referred to as the **standard Heisenberg coordinates** for ζ near ξ. From the above discussion, we can see the geometries of **D** and $\partial\mathbf{D} = \{\Im z_{n+1} = \sum_{j=1}^{n} |z_j|^2\} \equiv \mathbf{H}^n$ are "close" to those of Ω and $\partial\Omega$ respectively. This gives us a good reason to believe that we can investigate some problems on the "model" **D** or $\partial\mathbf{D}$ and then transfer to Ω or $\partial\Omega$. This phenomena had really occurred in [2],[3],[4],[7],[8],[18]. The property (iii) allows us to control the size of the error term when we transfer a parametrix $\Phi(z,w)$ of the boundary Laplacian $\square_b$ on $\partial\Omega$ to a parametrix $\Phi(\Theta(\xi,\zeta))$ of $\square_b$ on $\partial\mathbf{D}$.

Let us consider $\tau = dt + 2\sum_{k=1}^{n}(x_k dy_k - y_k dx_k)$ as a 1-form on $\partial\mathbf{D}$. Note that $<\tau, Z_k> = <\tau, \overline{Z_k}> = 0$, for all $1 \le k \le n$. Here $Z_k = \frac{\partial}{\partial z_k} + \imath \bar{z}_k \frac{\partial}{\partial t}$ is the Heisenberg vector field. To compute the Levi form:

$$<Z_j, Z_k>_L = \frac{1}{2} <d\tau, [Z_j, \overline{Z_k}]> = \frac{1}{2} <\tau, -2\imath\delta_{jk}T> = \delta_{jk}.$$

Thus ∂D is strongly pseudo-convex. Suppose $\partial\Omega$ is smooth and strongly pseudo-convex, then from the definition (2.1) and the above computation, we can see that the strong pseudo-convexity is equivalent to the existence of a positive definite quadratic form on the complex tangent space which induced by the Levi form. We can define a Hermitian metric on the complex tangent space by this positive definite quadratic form. Hence we have the following definition:

(2.2) FIRST DEFINITION FOR A LEVI METRIC:

Suppose $\partial\Omega$ is strongly pseudo-convex and $\{Z_1, Z_2, \ldots, Z_n\}$ is a basis for the subbundle $T^{1,0}$ of the complex tangent bundle and $\omega_1, \omega_2, \ldots, \omega_n$ are the dual forms for Z_j's. A metric given by

$$ds^2 = \kappa(z) \sum_{i,j=1}^{n} <Z_i, Z_j>_L \omega_i \otimes \bar{\omega}_j$$

with a positive smooth function κ on $\partial\Omega$ is called a **Levi metric on** the complex tangent space $T^{(1,0)}(\partial\Omega) \oplus T^{(0,1)}(\partial\Omega)$. A **Levi metric on** Ω is a Hermitian metric which is a Levi metric when we restrict the metric to $T^{(1,0)}(\partial\Omega) \oplus T^{(0,1)}(\partial\Omega)$.

This definition is first given by Folland-Stein when they study the $\Box_b$ on $\mathbf{H}^n$. It is easy to see the metric defined in (2.2) allows one direction freedom (i.e., the normal direction). In [7], they just need to assume the metric is $ds^2 = \kappa(z) \sum_{j=1}^{n} \omega_j \otimes \bar{\omega}_j$, where ω_j is the dual form of Z_j. We also can look at the definition for a Levi metric in another way. According to a theorem of J.J.Kohn [6], the defining function γ for $\partial\Omega$ can be chosen strictly plurisubharmonic, i.e, $\sum_{1 \le i,j \le n+1} \partial^2\gamma(p)/\partial z_i \partial \bar{z}_j t_i \bar{t}_j > 0$ $\forall$ $t \in \mathbf{C}^{n+1} \setminus \{0\}$, $\forall p \in$ A neighborhood W of $\partial\Omega$. Using this theorem, we can have the following definition:

(2.3) SECOND DEFINITION FOR A LEVI METRIC:

Let γ be a strictly plurisubharmonic defining function for $\partial\Omega$. A metric given in local coordinates by

$$ds^2 = \kappa(z)\sum_{i,j=1}^{n+1}\frac{\partial^2\gamma(z)}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j$$

with a positive smooth function κ on a neighborhood W of $\partial\Omega$ is called a Levi metric on W with respect to γ. A Levi metric on Ω is a Hermitian metric which is a Levi metric in some neighborhood of $\partial\Omega$.

Remark:

From the second definition, it is easy to see that the metric controls all directions,i.e., the "tangential" and "normal" directions. For example, when Lieb-Range studied the $\overline{\partial}$- Neumann problem on the other "model" - the unit ball $B^{n+1} \subset C^{n+1}$; they assumed a Levi metric defined on $\overline{B^{n+1}}$:

$$ds^2 = \sum_{i,j=1}^{n+1}\frac{\partial^2(|z_1|^2+|z_2|^2+\cdots+|z_{n+1}|^2-1)}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j = \sum_{j=1}^{n+1} dz_j \otimes d\overline{z}_j,$$

which just the Euclidean metric on $\overline{B^{n+1}}$. Suppose Ω is strongly pseudo-convex, it is easy to see the Levi metric as in the Definition (2.3) is also a Levi metric as in the Definition (2.2). (We just need to restrict ds^2 to the complex tangent subspace $\sum_{j=1}^{n+1}\frac{\partial\gamma(z)}{\partial z_j} t_j=0$, $\forall\, t \in (C^{n+1}\setminus\{0\})$. On the other hand, once we have a defining function γ for $\partial\Omega$, we always can find another strictly plurisubharmonic defining function γ' (e.g., $\gamma'=(e^{A\gamma}-1)/A$ for some constant A) such that the metric is defined by

$$ds^2 = \sum_{i,j=1}^{n+1} g_{i\overline{j}} dz_i \otimes d\overline{z}_j = \sum_{i,j=1}^{n+1}\frac{\partial^2\gamma'}{\partial z_i \partial \overline{z}_j}(z) dz_i \otimes d\overline{z}_j$$

is a Levi metric on a neighborhood W of $\partial\Omega$.

As we mentioned in Section 1, Phong's result of the Neumann operator N cannot be transferred to Ω directly by standard Heisenberg coordinates. Few years later, Phong and Stein [16], [17] achieved this goal by using **special Heisenberg coordinates** which is any coordinate system $(z'^{\dagger}, t^{\dagger}; \rho^{\dagger}) \in H^n \times R^+$ for $\zeta \in D_\xi$ (where D_ξ is a small neighborhood of ξ), depending smoothly on ξ, which satisfies the conditions

$$|z'^{\dagger}-z'|=\overline{O}^2,\quad |t^{\dagger}-t|=O^3,\quad |\rho^{\dagger}-\rho|=O^3$$

where $(z',t;\rho)$ are standard Heisenberg coordinates. Evidently such a system will also satisfy all the properties which standard Heisenberg coordinates have. We write $(z'^{\dagger},t^{\dagger})=\Theta^{\dagger}(\xi,\zeta)$. Here $\overline{O}^k$ and O^k denote respectively C^∞ functions $f(\xi,\zeta)$ and $g(\xi,\zeta)$ satisfying $|f(\xi,\zeta)|\le(|z'|+|t|+\rho(\xi)+\rho(\zeta))^k$ and $|g(\xi,\zeta)|\le(|z'|+|t|^{1/2}+\rho(\xi)^{1/2}+\rho(\zeta)^{1/2})^k$. The existence theorem of special Heisenberg coordinates is the following:

THEOREM (Phong-Stein):

For each strongly pseudo-convex domain, there exists a special Heisenberg coordinate system $(\Theta^{\dagger}(\xi,\zeta);\rho^{\dagger}(\xi,\zeta))=(z'^{\dagger}(\xi,\zeta),t^{\dagger}(\xi,\zeta);\rho^{\dagger}(\xi,\zeta))$ which satisfies the additional conditions:

(2.4) $\overline{Z}_{n+1}(t^{\dagger}+i\rho^{\dagger})=O^3,\quad \overline{Z}_{n+1}(|z'^{\dagger}|^2)=O^3.$

This theorem is crucial when we consider the differential operator $\Box$ and the boundary operator act on the "transferred kernel"! We can compare the equation $\Box$ on D and Ω, the crucial differences between these two situations are terms SZ_{n+1} and $\overline{Z}_{n+1}Z_{n+1}$. When we consider the model case, $S\equiv 0$ and the kernel K for N_1 is holomorphic in $t+\iota\rho$. If we just plug in the standard Heisenberg coordinates in Phong's kernel, the kernel $K(\Theta(\xi,\zeta);\rho(\xi)+\rho(\zeta))$ is not holomorphic in $t+\iota\rho$. The first two terms K_1 and K_2 of $K(\Theta(\xi,\zeta);\rho(\xi)+\rho(\zeta))$ are smoothing kernel of order -1 and -3/2 respectively. $\Box_b(K_1)$ is a kernel of order zero and $\Box_b(K_2)$ is a kernel of order -1/2, but $(SZ_{n+1}-\overline{Z}_{n+1}Z_{n+1})(K_2)$ will produce a kernel of order 1/2 which is really bad! In this situation, $(\Box_b-\overline{Z}_{n+1}Z_{n+1}+SZ_{n+1})(K(\Theta(\xi,\zeta);\rho(\xi)+\rho(\zeta)))$ will not satisfy the condition $\Box K=\delta_0$+at least type two operators. This is the reason why Phong and Stein need the extra property (2.4) to overcome the difficulties in their "transferred" process.

If we go back to Lieb and Range's paper, the only crucial assumption in their method is the **"normalized Levi metric"** ([14], p.153).

(2.5) DEFINITION FOR A NORMALIZED LEVI METRIC:

A normalized Levi metric is a Levi metric

$$ds^2=\kappa(z)\sum_{i,j=1}^{n+1}\frac{\partial^2\gamma(z)}{\partial z_i\partial\overline{z}_j}dz_i\otimes d\overline{z}_j$$

such that $\|\partial\gamma(z)\|_{ds^2}\equiv1$, for all z in a small neighborhood W of $\partial\Omega$ and $\kappa\equiv1$ on the subbundle

$T_z^{(1,0)}(\partial\Omega)\oplus T_z^{(0,1)}(\partial\Omega)$, $z\in\partial\Omega$.

Remark:

The property $\partial\gamma(z)$ has length 1 for z in a neighborhood U of $\partial\Omega$ is important! In Lieb and Range's papers (see [13], p.286-p.287), they always assume $\omega_{n+1}=\partial\gamma$ on U. According to the Definition (2.5), now the normalized Levi metric has the following form:

$$ds^2=\sum_{j=1}^{n}\omega_j\otimes\overline{\omega}_j+\omega_{n+1}\otimes\overline{\omega}_{n+1},$$

where $\{\omega_1,\dots,\omega_n\}$ are the dual forms of the "tangential" vector fields.

When we solve the differential system:

(2.6) $\quad\overline{\partial}K_1=\ker(T_1),\qquad\overline{\partial}^*K_1=\ker[(T_0)^*]$

we need to know the coefficients of T_1 and $(T_0)^*$ exactly. If $\kappa\neq1$ in the definition (2.5), the above remark gives us the geodesic function on D as follows:

$$R^2(z,w)=\sum_{j,k=1}^{n+1}\kappa(z)\frac{\partial^2\gamma(z)}{\partial z_j\partial\overline{z}_k}(z_j-w_j)(\overline{z}_k-\overline{w}_k)=\kappa(z)\Big\{\sum_{j=1}^{n}2|z_j-w_j|^2+(t-s)^2+(\rho-\mu)^2\Big\}.$$

and $\overline{\partial}_z\partial_zR^2(z,w)=\kappa(z)\overline{\partial}_z\partial_z\psi^2(z,w)$+error terms. When we consider the transition kernel $\mathcal{Q}_1(z,w)$ (see [22], p.279-p.286) on the boundary, the kernel will depend on different power of κ and we don't know how to solve the system! We also know Lieb and Range just used the standard Levi procedure to construct the kernel (so they just used the **standard Heisenberg coordinates**). It is very interesting to know why their method can be worked without the **special Heisenberg coordinates**. Is it true that the **"normalized Levi metric"** gives us some properties which can recover those particular properties of special Heisenberg coordinates?

Once we choose one defining function γ, we always can construct another strictly plurisubharmonic defining function $\gamma=\frac{e^{A\gamma}-1}{A}$ such that

$$ds^2=\kappa(z)\sum_{i,j=1}^{n+1}\frac{\partial^2\gamma(z)}{\partial z_i\partial \bar{z}_j}dz_i\otimes d\bar{z}_j=\kappa(z)\sum_{i,j=1}^{n+1}g_{i\bar{j}}dz_i\otimes d\bar{z}_j$$

is a normalized Levi metric. It is easy to see that ds^2 is not only a Hermitian metric but also a Kähler metric on the subbundle $T_z^{(1,0)}(\partial\Omega)\oplus T_z^{(0,1)}(\partial\Omega)$. Now we use the Levi procedure to change the defining function γ to

$$\gamma'=\frac{e^{A(|z'|^2-\Im z_{n+1}+O^4)}-1}{A}$$

and change the Euclidean vector field $\{\partial/\partial z_j,\partial/\partial\bar{z}_j\}_{j=1,\dots,n+1}$ to the Heisenberg vector field $\{Z_j,\bar{Z}_j\}_{j=1,\dots,n+1}$, it is easy to get

$$\Delta=2\Box=2\left(\Box_b-Z_{n+1}\bar{Z}_{n+1}+SZ_{n+1}+\varepsilon(Z,\bar{Z})+\text{zero order terms}\right)$$

with $|S|\approx O^2$. On the other hand, suppose we consider the other Kähler metric on thre subbundle $T_z^{(1,0)}(\partial\Omega)\oplus T_z^{(0,1)}(\partial\Omega)$:

$$ds'^2=f(z)\sum_{i,j=1}^{n+1}\frac{\partial^2\gamma(z)}{\partial z_i\partial\bar{z}_j}dz_i\otimes d\bar{z}_j=\sum_{i,j=1}^{n+1}g_{i\bar{j}}dz_i\otimes d\bar{z}_j$$

(in this situation, $f(z)$ has to satisfy some PDE system). Will this metric ds'^2 contradict the argument we have made? Now, if we study the metric defined by (2.3) more carefully, the assumption of $\{\omega_{n+1}=\partial\gamma,\omega_n,\dots,\omega_1\}$ (Once again, we use Lieb-Range's assumption; see [13], p.286-p.287) is an orthonormal basis for (0,1)-forms will give us a lot of restrictions. Suppose γ is a strictly plurisubharmonic defining function and

$$\omega_{n+1}=\partial\gamma=\sum_{m=1}^{n+1}\frac{\partial\gamma}{\partial z_m}dz_m.$$

Consider $Z_{n+1}=\|\partial\gamma\|^{-2}\cdot\sum_{m=1}^{n+1}\frac{\partial\gamma}{\partial\bar{z}_m}\frac{\partial}{\partial z_m}$. Then $Z_{n+1}(\omega_{n+1})=1$. Suppose the "tangential" vector fields are $Z_j=\sum_{k=1}^{n+1}a_k^j\frac{\partial}{\partial z_k}$, $j=1,2,\dots,n$. Then we can prove the following theorem:

(2.7) THEOREM:

Suppose the metric defined on $\overline{\Omega}$ is given by Definition (2.3) and

$$<Z_j, Z_{n+1}>_{ds^2}=0, \quad \forall j=1,2,...n; \quad \text{and} <Z_{n+1}, Z_{n+1}>_{ds^2}=1,$$

Then γ is uniquely definied on the boundary $\partial\Omega$.

Proof:

Suppose γ' is another such function, then there exist a positive function h such that $\gamma'=h\gamma$.

According to the assumption, when we consider the restriction of γ to the boundary:

$$0=\sum_{k,m}^{n+1} a_k^j \frac{\partial\gamma'}{\partial z_k \partial z_m}\frac{\partial(h\gamma)}{\partial z_m}=\sum_{k,m}^{n+1} a_k^j \frac{\partial(h\gamma)}{\partial z_k \partial z_m}\{h\frac{\partial\gamma}{\partial z_m}+\gamma\frac{\partial h}{\partial z_m}\}$$

$$=\sum_{k,m}^{n+1} a_k^j \frac{\partial\gamma}{\partial z_m}\frac{\partial h}{\partial z_k}\frac{\partial\gamma}{\partial z_m}+\sum_{k,m}^{n+1} a_k^j \frac{\partial h}{\partial z_m}\frac{\partial\gamma}{\partial z_k}\frac{\partial\gamma}{\partial z_m}+\sum_{k,m}^{n+1} a_k^j \frac{\partial^2\gamma}{\partial z_k \partial z_m}\frac{\partial\gamma}{\partial z_m}$$

$$=\sum_{k=1}^{n+1} a_k^j \frac{\partial h}{\partial z_k}\cdot\left(\sum_{m=1}^{n+1}\left|\frac{\partial\gamma}{\partial z_m}\right|^2\right)+\left(\sum_{k=1}^{n+1} a_k^j \frac{\partial\gamma}{\partial z_k}\right)\cdot\left(\sum_{m=1}^{n+1}\frac{\partial h}{\partial z_m}\frac{\partial\gamma}{\partial z_m}\right).$$

This tells us $Z_k h=0$ and $\overline{Z_k}h=0$, $\forall$ k=1,2,...,n. Strong pseudo-convexity implies that $[Z_k,\overline{Z_j}]h=0$ and we know the function h is a constant on the boundary. If we also assume the metric is "normalized" i.e., $<Z_{n+1},Z_{n+1}>_{ds^2}=1$, then $h\equiv1$ and γ is uniquely determined on the boundary!

Remark:

In this theorem, we just use the local property of strong pseudo-convexity. Since the function h $\equiv1$, so the local solution can gives us the global solution. In fact, we can prove this theorem without the assumption of strong pseudo-convexity. But in this case we need to apply a theorem due to Bochner: $\overline{Z_j}h=0$ for all j=1,2,...,n, then h is a CR function and which is a boundary value of a holomorphic function. By Bochner's theorem [10], h can be extended holomorphically to a neighborhood of $\partial\Omega$. But we also know that h is a real-valued function, which implies h is a constant. Without the assumption of storng pseudoconvexity, we only can consider this problem globally!

Now the problem reduces to consider the existence of such function γ for a strongly pseudo-convex domain. But the existence of such a defining function was proved by Lieb-Range [12], [13], we will not go through all the details. The assumption of a normalized Levi metric not only allows us to solve the system (2.6), but also gives us the properties $|S| \approx O^2$ and

$$\left(\frac{\partial}{\partial \rho} + \iota \frac{\partial}{\partial t}\right)(K(\Theta(\xi,\zeta);\rho(\xi)+\rho(\zeta)) \approx O^3.$$

These are the crucial properties of a special Heisenberg coordinates! This explain why standard Heisenberg coordinates can be worked in this case. If we study Phong's theorem ([1], p.313 -p.322) more carefully, it is not so hard to generalize his result to the following metrics:

$$ds^2 = \alpha \sum_{j=1}^{n} \omega_j \otimes \overline{\omega}_j + \beta \omega_{n+1} \otimes \overline{\omega}_{n+1}, \quad \text{where } \alpha,\beta : \mathbf{C}^{n+1} \to \mathbf{R}^+; \quad \alpha,\beta \in C^\infty(\overline{\mathbf{D}}).$$

Hence we need more efforts to control the normal direction and put in an extra term

$\mathfrak{I}(\xi,\zeta)(\Sigma_{k,l\geq 0}(\Phi^{-k/2})^\dagger \cdot (\Psi^{-l/2})^\dagger)$ (see [17], p.106-p.112) to approximate the parametrix.

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