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A NOTE ON COTYPE OF SMOOTH SPACES

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Abstract. For sufficiently smooth Banach spaces weak cotype 2 implies Hilbert space.

It is well known that differentiability of the norm on a given Banach space X has a significant impact on the structure of X. Speaking about higher order differentiability of norms, it was shown in [2] that a Banach space X is isomorphic to a Hilbert space if on both X and X^* there exist real valued functions φ with bounded nonempty support and such that φ' are locally Lipschitzian. It also is known that a Banach space X which admits a real valued twice continuously differentiable function φ with bounded nonempty support is isomorphic to a Hilbert space provided any infinite dimensional subspace of X contains an infinite dimensional subspace which is isomorphic to a Hilbert space [9]. There are many spaces which do not admit real valued twice differentiable functions with bounded support and yet have norms with differential which is Lipschitzian on the sphere [12], [4], [9]. In fact the result in this note puts the space of Tsirelson's kind [19], constructed by Johnson and Figiel and described in [5], Ex. 5.3 to this family of spaces. Indeed, this space is of type 2 with unconditional basis and superreflexive (cf. [1], [5], [8], [15]) and as such it admits an equivalent norm $\|\cdot\|$ whose modulus of smoothness is of power type 2 [3]. Thus the differential $\|\cdot\|'$ of the norm $\|\cdot\|$ is Lipschitzian on the sphere [2].

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In this note we shall work in real Banach spaces which will be assumed to be infinite-dimensional unless otherwise stated. From standard notions let us recall (cf. e.g. [8]) that a Banach space X is said to be of cotype q (with a constant $C_q > 0$) if for any n-tuple $x_1, x_2, \dots, x_n$ of points of X we have

$$2^n \sum \|x_i\|^q \leq C_q \sum_{\varepsilon_i = \pm 1} \|\sum \varepsilon_i x_i\|^q.$$

From not yet standard notions let us mention that, following Milman and Pisier [14], a Banach space X is said to be of weak cotype 2 if there exist constants $\delta \in (0,1)$ and $C > 0$ such that every finite-dimensional subspace E of X contains a subspace $F \subset E$ with $\dim F \geq \delta \dim E$ and such that the Banach-Mazur distance $d(F, \ell_2^{\dim F}) \leq C$. This property was shown in [14] to be equivalent to the condition of boundedness of so called volume ratio in the sense given in [17] (cf. [18]). Furthermore, following Pisier [15], a Banach space X is said to be asymptotically Hilbertian if a constant $C > 0$ exists such that for every positive integer n, there is a finite-codimensional subspace E_n of X such that for every n-dimensional subspace F_n of E_n we have that $d(F_n, \ell_2^n) \leq C$. An example of asymptotically Hilbertian space which is not isomorphic to a Hilbert space is the Tsirelson's kind space mentioned above [15]. Finally, a Banach space X is said to be a homogeneous space if all of its infinite dimensional subspaces are isomorphic to X. All the notions of differentiability will be understood in their continuous Frechet sense in this note.

We will say that a Banach space X is C^2-smooth respectively LUC^2-smooth respectively $LH^{2+\alpha}$-smooth ($\alpha \in (0,1]$) if X admits a real valued function φ with bounded nonempty support and such that the second order differential φ'' of φ is continuous, respectively locally uniformly continuous, respectively locally α-Hölder on X.

The main purpose of this note was to show that for sufficiently smooth spaces the notions of cotype 2 and weak cotype 2 coincide.

THEOREM 1. *Let X be a separable Banach space. Then each one of the two following conditions implies that X is isomorphic to a Hilbert space.*

(i) $2 + \sup\{\alpha > 0;\ X \text{ is } LH^{2+\alpha} - \text{smooth}\} > \inf\{q; X \text{ is of cotype } q\}$

(ii) *X is LUC^2-smooth and asymptotically Hilbertian.*

PROOF: An adaptation of the proof of Theorem 1 in [10]. We will use the following result of Figiel, which was used by Meshkov in [13] and the proof of which is based on the celebrated result of Kwapien in [7]. The result of Figiel reads:

If there are continuous symmetric bilinear forms $Q_1, Q_2, \dots, Q_n$ on a Banach space X such that

$$\sup_j |Q_j(x,x)| \geq \|x\|^2 \quad \text{for every} \quad x \in X, \tag{1}$$

then X is isomorphic to a Hilbert space.

To begin with the proof of the part (ii), let us first note that since an asymptotically Hilbertian space is reflexive [15] and thus does not contain a subspace isomorphic to c_0, the method of the proof of Proposition 2.1 in [2] gives that there is on X a real valued function φ such that φ'' is uniformly continuous on X and such that $\varphi(0) = 1$ and $\varphi(x) = 0$ for every $x \in X, |x\| \geq 1$.

As in the proof of Lemma 2 in [10] we note that uniform continuity of φ'' gives: For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $z, h \in X$ with $\|h\| \leq \delta$ we have

$$|\varphi(z+h) - \varphi(z) - \varphi'(z)(h) - \frac{1}{2}\varphi''(z)(h,h)| \leq \frac{\varepsilon}{2}\|h\|^2. \tag{2}$$

Assume that X is not isomorphic to a Hilbert space. Fix $\varepsilon \in (0, (2C)^{-2})$, where C is the constant from the asymptotic Hilbertian property of X. Choose $\delta \in (0, \frac{1}{2})$ by (2) for the ε fixed and choose an integer $n > C^2\delta^{-2}$.

Let E_n be the finite codimensional subspace as in the definition of asymptotically Hilbertian spaces, chosen for C and n. Choose vectors $v_0, v_1, v_2, \dots, v_n$ by induction as follows:

(i) $v_0 = 0$

(ii) If $v_0, v_1, \dots, v_{k-1}$ have been chosen, put

$$S_{k-1} = \Big\{ \sum_{j=0}^{k-1} \varepsilon_j v_j;\ \varepsilon_j = \pm 1 \Big\} \quad \text{and}$$

$$H_{k-1} = \big\{ h \in E_n; \varphi'(z)(h) = 0 \quad \text{and} \quad |\varphi''(z)(h,h)| \le \frac{\varepsilon}{2}\|h\|^2 \quad \text{for each} \quad z \in S_{k-1} \big\}.$$

Since $E_n \cap \big(\bigcap_{z \in S_{k-1}} (\varphi'(z))^{-1}(0) \big)$ is a finite codimensional subspace of E_n which is, by our assumption, not isomorphic to a Hilbert space, it follows from (1) that $H_{k-1} \neq \phi$. Use homogeneity of H_{k-1} to pick a vector $v_k \in H_{k-1}$ such that $\|v_k\| = \delta$.

According to the definition of H_{k-1} and (2), we have that for every $z \in S_{k-1}$,

$$|\varphi(z \pm v_k) - \varphi(z)| \le \varepsilon \|v_k\|^2 = \varepsilon\delta^2. \tag{3}$$

Having the vectors $v_0, v_1, \dots, v_n$ constructed, put

$$S_n = \Big\{ \sum_{j=0}^{n} \varepsilon_j v_j; \varepsilon_j = \pm 1 \Big\} \quad \text{and} \quad V_n = sp\{v_0, v_1, \dots, v_n\}.$$

We claim that $S_n \not\subset B_1(E_n)$, where $B_1(E_n)$ denotes the closed unit ball in E_n. Indeed, since $d\big(V_n, \ell_2^{\dim V_n}\big) \le C$, it follows from the parallelogram equality in Hilbert spaces and from the choice of n, that

$$C^2 < n\delta^2 = \sum_{j=1}^{n} \|v_j\|^2 \le \frac{1}{2^n} C^2 \sum_{\varepsilon_j = \pm 1} \|\sum_{j=1}^{n} \varepsilon_j v_j\|^2 \le C^2 \sup_{\varepsilon_j = \pm 1} \|\sum_{j=1}^{n} \varepsilon_j v_j\|^2.$$

Thus $S_n \not\subset B_1(E_n)$. Let $n_0 \le n$ be defined as

$$n_0 = \min\{ j \le n : S_j \not\subset B_1(E_n)\}.$$

Then for some choice of ε_j's we have $\| \sum_{j=1}^{n_0} \varepsilon_j v_j \| > 1$. Assume, without loss of generality that $\| \sum_{j=1}^{n} v_j \| > 1$. By minimality of n_0, we have that $\sup_{\varepsilon_j = \pm 1} \| \sum_{j=1}^{n_0 - 1} \varepsilon_j v_j \| \le 1$ and since moreover $\|v_j\| \le 1$ for each $1 \le j \le n$, it follows that $\sup_{\varepsilon_j = \pm 1} \| \sum_{j=1}^{n_0} \varepsilon_j v_j \| \le 2$.

Thus, since $\varphi(0) = 1$ and $\varphi = 0$ outside $B_1(X)$ and since $d(V_n, \ell_2^{\dim V_n}) \leq C$, using (3) and the choice of ε, we can estimate:

$$1 = |\varphi\Big(\sum_{j=0}^{n_0} v_j\Big) - \varphi(0)| \leq \sum_{k=1}^{n_0} |\varphi\Big(\sum_{j=0}^{k} v_j\Big) - \varphi\Big(\sum_{j=0}^{k-1} v_j\Big)| \leq$$

$$\leq \varepsilon \sum_{k=1}^{n_0} \|v_k\|^2 \leq \frac{1}{2^{n_0}} \varepsilon C^2 \sum_{\varepsilon_k = \pm 1} \|\sum_{k=1}^{n_0} \varepsilon_k v_k\|^2 \leq \varepsilon\, C^2 \sup_{\varepsilon_k = \pm 1} \|\sum_{k=1}^{n_0} \varphi_k v_k\|^2 \leq$$

$$\leq 4\varepsilon C^2 < 1.$$

This contradiction finishes the proof of the part (ii).

(i) We point out only a few things needed to adjust the proof of the part (ii). Assume that for some $\alpha > 0$ and $q < 2 + \alpha, X$ is LH^α-smooth and of cotype q and that X is not isomorphic to a Hilbert space.

Similarly as in (ii) first note that since X has cotype $q \in (2, 2 + \alpha)$, X does not contain a subspace isomorphic to c_0 and again the proof of Proposition 2.1 in [2] gives that according to $LH^{2+\alpha}$-smoothness of X, there is a real valued function φ on X whose second differential is α-Hölder on X with a constant $K_\alpha > 0$ and such that $\varphi(0) = 1$ and $\varphi = 0$ outside $B_1(X)$. Choose $\varepsilon \in (0, 2^{-\alpha} K_\alpha)$ so that $\varepsilon^{\frac{q-2-\alpha}{\alpha}} > 2^q C_q K_\alpha^{\frac{q-2}{\alpha}}$ and then choose an integer n such that $n > C_q\big(\frac{\varepsilon}{K_\alpha}\big)^{q/\alpha}$ and fix these ε and n. Note that as in [10], p. 2515, for $z, h \in X$ a simple integration by parts gives

$$\varphi(z+h) - \varphi(z) - \varphi'(z)(h) - \frac{1}{2}\varphi''(z)(h,h) = \int_0^1 \big(\varphi''(z+th)(h,h) - \varphi''(z)(h,h)\big)(1-t)dt$$

and therefore using the α-Hölder property of φ'' it follows that we can choose $\delta = \big(\frac{\varepsilon}{K_\alpha}\big)^{1/\alpha}$ in (2).

Construct vectors $v_0, v_1 \ldots, v_n$, $\|v_j\| = \delta = \big(\frac{\varepsilon}{K_\alpha}\big)^{1/\alpha}$ for $j = 1, 2, \ldots, n$ as in the part (ii).

Using the choice of ε, n and δ and the q-cotype property of X we estimate:

$$C_q < n\left(\frac{\varepsilon}{K_\alpha}\right)^{q/\alpha} = \sum_{j=1}^{n} \|v_j\|^q \le \frac{1}{2^n} C_q \sum_{\varepsilon_j=\pm 1} \|\sum_{j=1}^{n} \varepsilon_j v_j\|^q \le$$

$$\le C_q \cdot \sup_{\varepsilon_j=\pm 1} \|\sum_{j=1}^{n} \varepsilon_j v_j\|^q.$$

Therefore $S_n \not\subset B_1(X)$.

Find $n_0 \le n$ and proceed as above. The final estimation reads:

$$1 = |\varphi\Big(\sum_{j=0}^{n_0} v_j\Big) - \varphi(0)| \le \sum_{k=1}^{n_0} |\varphi\Big(\sum_{j=0}^{k} v_j\Big) - \varphi\Big(\sum_{j=0}^{k-1} v_j\Big)| \le$$

$$\le \varepsilon \sum_{k=1}^{n_0} \|v_k\|^2 = \sum_{k=1}^{n_0} \varepsilon \cdot \left(\frac{\varepsilon}{K_\alpha}\right)^{2/\alpha} = n_0 (K_\alpha)^{-2/\alpha} \cdot \varepsilon^{1+2/\alpha} < n_0 (2^q C_q)^{-1} \left(\frac{\varepsilon}{K_\alpha}\right)^{q/\alpha} =$$

$$= (2^q C_q)^{-1} \sum_{k=1}^{n_0} \|v_k\|^q \le 2^{-q} 2^{-n_0} \sum_{\varepsilon_k=\pm 1} \|\sum_{k=1}^{n_0} \varepsilon_k v_k\|^q \le$$

$$\le 2^{-q} \sup_{\varepsilon_k=\pm 1} \|\sum_{k=1}^{n_0} \varepsilon_k v_k\|^q \le 1,$$

a contradiction which finishes the proof of Theorem 1.

COROLLARY 2. *Let X be a separable Banach space which is $LH^{2+\alpha}$-smooth for some $\alpha > 0$. Then each one of the two following conditions implies that X is isomorphic to a Hilbert space.*

(i) X is of weak cotype 2.

(ii) X is a homogeneous space.

PROOF: According to Theorem 1 (i), it is sufficient to point out that it is well known that each one of the conditions (i) and (ii) implies that $q(X) = \inf\{q; X \text{ is of cotype } q\} = 2$. Indeed, the result of Krivine and Maurey and Pisier (cf. [6], [11]) states that $\ell_{q(X)}$ is finitely representable in X. This combined with Example 3.1 in [5] gives that if $q(x) > 2$, then X is not of weak cotype 2. Furthermore if $q(X) > 2$, then according to the result of Szankowski [16], X contains a subspace without the approximation property. This can of course not be the case in (ii) since X does have an infinite dimensional subspace with Schauder basis.

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