manuscripta math. 61, 33 - 42 (1988)

manuscripta mathematica

ON THE DISTRIBUTION OF SEQUENCES CONNECTED WITH DIGIT-REPRESENTATION

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Let $N = e(s)b(s) + e(s-1)b(s-1) +\ldots+ e(1)b(1) + e(0)$ be the digit representation of the integer N to base b or to α-scale, that is with respect to the best approximation denominators of an irrational number α. Let $f: \mathbb{N}_0 \to \mathbb{Z}$ with $f(0) = 0$ be an arbitrary function and $r(0), r(1), \ldots$ be an arbitrary sequence of integers and $F(N) := f(e(s))r(s) +\ldots+ f(e(1))r(1) + f(e(0))r(0)$. Conditions for the uniform distribution modulo one of the sequence $\{F(N)x\}_{N\in\mathbb{N}}$, $x \in \mathbb{R}$ are given.

Introduction and statement of results

Let $1 = b(0) < b(1) < b(2) < \ldots$ be a sequence of integers, then for every positive integer N there is an s with $b(s) \leq N < b(s+1)$ and a unique $e(s) \in \mathbb{N}$ with $N = e(s)b(s) + N_{s-1}$ and with $0 \leq N_{s-1} < b(s)$.
By continuing in that way we get a representation of N, unique in the above sense, of the form:

$$(1)\qquad \begin{aligned} &N = e(s)b(s) + e(s-1)b(s-1) +\ldots+ e(0)b(0) \quad \text{with} \\ &0 \leq e(i) < \frac{b(i+1)}{b(i)} \quad \text{for all } i\ . \end{aligned}$$

Let now

$f: \mathbb{N}_0 \to \mathbb{Z}$ be an arbitrary integer-valued function with $f(0) = 0$

and let

$r(0), r(1), r(2), \ldots$ be an arbitrary sequence of integers.

Then we define the base-and-digit-changing function
$F : \mathbb{N}_0 \to \mathbb{Z}$ by $F(0) := 0$ and
$F(N) := f(e(s))r(s) + \ldots + f(e(0))r(0)$
if N has representation (1) .

In this paper we will be interested in the distribution modulo one of the sequence $\{F(N)x\}_{N\geq 0}$ where x is a real number, and where $\{y\} := y - [y]$ denotes the fractional part of y. Investigations in this direction have been done for example in [1], [2], [3], [4], [6], [7], [10], [11] and [12].

For the initial base $(b(i))_{i\geq 0}$ two cases are of outstanding interest:

i) $b(i) := b^i$ for all $i \geq 0$, and b is a fixed integer larger than one.

ii) Let α be an irrational number with continued fraction expansion $\alpha := [a(0);a(1),a(2),\ldots]$ and with best approximation denominators $1 = q(0) \leq q(1) < q(2) < \ldots$ If we take $b(i) := q(i)$ (or $b(i) := q(i+1)$ in case that $q(1) = 1$) for all $i \geq 0$, then we have the representation of N to α-scale .

In most of the previous papers the case $r(i) \equiv 1$ (that is the case of generalized sum-of-digit sequences) is considered. In this paper we will investigate the general case for the representations a) and b) and we will show ($\|y\|$ denotes the distance of y to the nearest integer) :

<u>THEOREM 1:</u> <u>If</u> N <u>is represented to base</u> b, <u>then</u>: $\{F(N)x\}_{N\geq 0}$ <u>is uniformly distributed modulo one if and only if there is an</u> $e \in \{1,2,\ldots,b-1\}$ <u>with</u> $f(e) \neq 0$ <u>and</u>

$$\sum_{n=1}^{\infty} \|r(n)hx\|^2 = \infty \quad \text{for all } h \in \mathbb{N} . \tag{2}$$

From this theorem moreover we will give some examples and applications.

<u>THEOREM 2:</u> *If* N *is represented to* α*-scale, then:*

$$\text{(3)}\qquad \text{If} \quad \sum_{\substack{n=1 \\ a<a(n+1) \text{ or} \\ a=a(n)=a(n+1)}}^{\infty} \frac{\|r(n)hx\|^2}{a(n+1)} = \infty \quad \text{for all } h \in \mathbb{N},$$

where $a := \min\{e \mid f(e) \neq 0\}$,
then $\{F(N)x\}_{N\geq 0}$ *is uniformly distributed modulo one.*

In this result the denominators a(n+1) in the summands of (3) can be neglected if α has bounded partial quotients. But otherwise these denominators seem to be disturbing since, if α is such that $\Sigma\, 1/a(n)$ is converging, then the condition (3) doesn't hold for any x . The result of Theorem 2 can be improved for special choices of the digit-changing function f , but it cannot be improved for general f as can be seen by the following:

<u>THEOREM 3:</u> *If* N *is represented to* α*-scale, then:*

a) *If* $f(e) = e$ *for all* $e \geq 0$ *then:*

$$\text{If} \quad \sum_{\substack{n=1 \\ a<a(n+1) \text{ or} \\ a=a(n)=a(n+1)}}^{\infty} \|r(n)hx\|^2 = \infty \quad \text{for all } h \in \mathbb{N},$$

then $\{F(N)x\}_{N\geq 0}$ *is uniformly distributed modulo one.*

b) *If* $f(1) = 1$ *and* $f(e) = 0$ *for* $e \neq 1$, *then for every choice of the new base* $(r(i))_{i\geq 0}$ *and for all* α *with* $\sum_{n=1}^{\infty} \frac{1}{a(n)} < \infty$ *the sequence* $\{F(N)x\}_{N\geq 0}$ *is for no* x *uniformly distributed modulo one.*

Proofs of the results

LEMMA 1: *There are positive constants* c, c' *such that for all integers* $q \geq 2$ *, all reals* $x(1),\dots,x(q-1)$ in $(-\frac{1}{2}, \frac{1}{2}]$ *and with* $x := \max_i |x(i)|$ *we have:*

$$q(1-cx^2) \leq |1 + \exp(x(1)) + \dots + \exp(x(q-1))| \leq q-c'x^2 \quad .$$

(*Here and in the following* $\exp(y) := e^{2\pi iy}$.)

We omit the easy proof .

Proof of Theorem 1: If $f(e) = 0$ for all $e \in \{0,\dots,b-1\}$ then $F(N) = 0$ for all N .
Let $f(e) \neq 0$ for an $e \in \{1,\dots,b-1\}$ and $\sum_{n=1}^{\infty} \|r(n)hx\|^2 = \infty$ for all $h \in \mathbb{N}$.
If N has representation (1) then with $L(s) = 0$ and $L(i) = e(s)b^s + \dots + e(i+1)b^{i+1}$ for $i = 0,1,\dots,s-1$, and because of $F(k + l.b^i) = F(k) + F(l.b^i)$ for all k with $0 \leq k < b^i$ $(k,l \in \mathbb{N}_0)$ we get:

$$S(N) := \sum_{n=0}^{N-1} \exp(F(N)hx) = \tag{4}$$
$$\sum_{j=0}^{s} \sum_{e=0}^{e(j+1)-1} \exp(F(L(j) + e.b^j)hx).S(b^j) \quad .$$

Further:

$$S(b^j) = \sum_{e=0}^{b-1} \exp(f(e)r(j-1)hx).S(b^{j-1}) \quad ,$$

and continuing in that way, because of Lemma 1, since $f(0) = 0$ and because of

$$S(b) = \sum_{e=0}^{b-1} \exp(f(e)r(0)hx) \quad , \qquad \text{we get:}$$

$$|S(b^j)/b^j| = \prod_{k=0}^{j-1} \frac{1}{b}.\left|\sum_{e=0}^{b-1} \exp(f(e)r(k)hx\right| \leq$$

$$\leq \prod_{k=0}^{j-1} \left(1 - \frac{c'}{b}.\max_{0<e<b} \|f(e)r(k)hx\|^2\right) \leq \tag{5}$$

$$\leq \prod_{k=0}^{j-1} \left(1 - \frac{c'}{b} \cdot \|f(l)r(k)hx\|^2\right) \qquad ,$$

where $l \in \{1,\dots,b-1\}$ is arbitrary such that $f(l) \neq 0$.
Since for $h' := f(l)h$ we have $\sum_{k=1}^{\infty} \|r(k)h'x\|^2 = \infty$,
the expression $|S(b^j)/b^j|$ tends to zero for j to infinity. Further from (4) :

$$|S(N)/N| \leq \sum_{j=0}^{s} b^{j+1-s} \cdot |S(b^j)/b^j|$$

and therefore $|S(N)/N|$ tends to zero for N to infinity and by Weyl's criterion (see [5] or [8]), $\{F(N)x\}_{N\geq 0}$ is uniformly distributed modulo one.

If x is rational then the sequence trivially is not uniformly distributed.
Assume now x irrational and $\sum_{n=1}^{\infty} \|r(n)h(0)x\|^2 = d < \infty$.
The polynomial $\sum_{e=0}^{b-1} y^{f(e)} =: p(y)$ has only finitely many zeros. Further $p(1) = b$, and because x is irrational, there is a $l(0)$ such that $p(\exp(lh(0)x)) \neq 0$ for all integers l with $|l| \geq l(0)$.
Therefore $\sum_{e=0}^{b-1} \exp(f(e)r(j)l(0)h(0)x) \neq 0$ for all j .
Because of $\|r(j)l(0)h(0)x\| \leq l(0) \cdot \|r(j)h(0)x\|$, because of (5) by Lemma 1 and with $h := h(0)l(0)$ we get:

$$|S(b^s)/b^s| = \prod_{k=0}^{s-1} \frac{1}{b} \cdot \left|\sum_{e=0}^{b-1} \exp(f(e)r(k)hx)\right| > c'' > 0$$

for all $s \in \mathbb{N}$ and by Weyl's criterion $\{F(N)x\}_{N\geq 0}$ is not uniformly distributed.

We give some examples:

<u>EXAMPLE 1:</u> If $(r(i))_{i\geq 0}$ is bounded (especially if $r(i) \equiv 1$) then (2) holds for every x irrational .

<u>EXAMPLE 2:</u> If $r(i) = g^i$ for an integer $g \geq 2$ and all $i \geq 0$, then (2) holds for every x irrational .

EXAMPLE 3: If $|r(i+1)/r(i)| \leq K$ for an absolute constant K and all i large enough, then (2) holds for all without countable many x, since under the above condition $\lim_{i\to\infty} ||r(i)x|| = 0$ can hold only for countable many x.

EXAMPLE 4: If α is irrational and $1 = q(0) \leq q(1) < \ldots$ are the best approximation denominators of α, then let $r(i) := q(i)$ for all i. If α has bounded partial quotients, then (2) holds if and only if x is not of the form $a\alpha + b$ with $a,b \in \mathbb{Q}$. This in general is not true for all α.

This follows from Theorem 1 in [9] which says that if α has bounded partial quotients, then $\lim_{n\to\infty} ||q(n)x|| = 0$ if and only if $x = a\alpha + b$ with $a,b \in \mathbb{Q}$, and from Theorem 2 in the same paper from which easily follows that there are α and x not of the above form for which (2) doesn't hold.

For the proof of Theorem 2 we need the following rather technical

LEMMA 2: Let $q(-1) = S(-1) = 0$, $q(0) = S(0) = 1$ and
$q(i+1) = a(i+1)q(i) + q(i-1)$,
$S(i+1) = b(i+1)S(i) + c(i+1)S(i-1)$ for $i > 0$
with $a(i) \in \mathbb{N}$, $b(i), c(i) \in \mathbb{C}$, $|b(i)| \leq a(i)$, $|c(i)| \leq 1$ for all i and moreover:
$b(i) = a(i)$ if $a(i) \geq a$ and $c(i) = 1$ if $a(i) > a$ where a is fixed, then:

If $$\sum_{i=1}^{\infty} \left(1 - \frac{|b(i)|}{a(i)}\right) + \sum_{\substack{i=2 \\ a(i-1)=a(i)=a}}^{\infty} ||\arg c(i)||^2 = \infty$$

then $$\lim_{i\to\infty} \frac{|S(i)|}{q(i)} = 0 .$$

Proof: If $\sum_{\substack{i=2 \\ a(i-1)=a(i)=a}}^{\infty} ||\arg c(i)||^2 = \infty$ then there is an l (without restriction of generality say $l = 0$) such that

$$\sum_{\substack{i=2 \\ i\equiv 1(4) \\ a(i-1)=a(i)=a}}^{\infty} \| \arg c(i) \|^2 = \infty .$$

Let $B(i) := \dfrac{2 + |b(i)|/a(i)}{3}$, $C(i) := \dfrac{2 + |a^2 + c(i)|/(a^2+1)}{3}$

then $\frac{2}{3} \leq B(i), C(i) \leq 1$ and

$$\sum_{i=1}^{\infty} \left(1 - \frac{|b(i)|}{a(i)}\right) + \sum_{\substack{i=2 \\ i\equiv 0(4) \\ a(i-1)=a(i)=a}}^{\infty} \| \arg c(i) \|^2 \quad \text{diverges if}$$

$$\text{and only if} \quad \sum_{i=1}^{\infty} \left(1 - \frac{4 + B(i)}{5}\right) + \sum_{\substack{i=2 \\ i\equiv 0(4) \\ a(i-1)=a(i)=a}}^{\infty} \left(1 - \frac{4 + C(i)}{5}\right)$$

diverges.

We define $t(-1) := t(0) := 1$, $t(1) := |b(1)|/a(1)$ and

$$t(i+1) := \begin{cases} \text{Case 1:} \quad C(i+1)t(i-1) & \text{if } \dfrac{t(i-2)}{t(i-1)} \leq 2 - C(i+1) \\[2mm] \text{Case 2:} \quad \dfrac{t(i-1) + t(i-2)}{2} & \text{otherwise} \\[2mm] \text{Case 3:} \quad B(i+1)t(i) & \text{if } \dfrac{t(i-1)}{t(i)} \leq 2 - B(i+1) \\[2mm] \text{Case 4:} \quad \dfrac{t(i) + t(i-1)}{2} & \text{otherwise} \end{cases} ,$$

where Case 1 and Case 2 are applied if $a(i) = a(i+1) = a$ and $i \equiv 3 \pmod 4$, and Case 3 and Case 4 are applied otherwise, and where $i \geq 1$.

By induction and some easy calculations it can be seen, that $|S(i)| \leq t(i)q(i)$ for all $i \geq 0$.

Further we have: If $t(i+1)$ was defined by

Case 1: Then $t(i)$ was defined by Case 3 or 4 .

a) If $t(i)$ was defined by Case 3 then $t(i) \leq t(i-1)$

b) If $t(i)$ was defined by Case 4 then $t(i-1) < t(i-2)$ and $t(i-1)$ was defined by Case 3 and $t(i) = \frac{1 + B(i-1)}{2} . t(i-2) \leq t(i-2)$.

So we have $\quad t(i+1) = C(i+1)t(i-1)$

and $\quad t(i) \leq t(i-1)$

or $\quad t(i+1) = C(i+1).B(i-1).t(i-2)$

and $\quad t(i),t(i-1) \le t(i-2)$.

Case 2: Then $t(i+1) = \dfrac{t(i-1)+t(i-2)}{2}$, $\dfrac{t(i-2)}{t(i-1)} > 2-C(i+1)$

and especially $t(i-2) > t(i-1)$, therefore $t(i-1)$ was defined by case 3 and $t(i)$ was defined by case 3 or 4 .

a) If $t(i)$ was defined by Case 3 then $t(i) \le t(i-1)$.

b) If $t(i)$ was defined by Case 4 then

$$t(i) = \frac{t(i-1)+t(i-2)}{2} = t(i+1) .$$

In any case $t(i+1) = \dfrac{B(i-1)+1}{2}.t(i-2)$ and

$\dfrac{1}{B(i-1)} = \dfrac{t(i-2)}{t(i-1)} > 2 - C(i+1)$ and therefore

$$t(i+1) \le \frac{4+C(i+1)}{5}.\frac{4+B(i-1)}{5}.t(i-2)$$

and $\quad t(i-1) \le t(i-2)$

and $\quad (t(i) \le t(i-1)$ or $t(i) \le t(i+1))$.

Case 3: Then $\quad t(i+1) \le B(i+1).t(i)$.

And finally by analogous considerations as above we get in

Case 4: Then

$$t(i+1) \le \frac{4+B(i+1)}{5}.\frac{4+B(i)}{5}.t(i-1)$$

and $\quad t(i) \le t(i-1)$

or

$$t(i+1) \le \frac{4+B(i+1)}{5}.\frac{4+C(i)}{5}.\frac{1+B(i-2)}{5}.t(i-3)$$

and $\quad t(i),t(i-1),t(i-2) \le t(i-3)$.

From all this it follows that $t(i)$ tends to zero for i to infinity and the result follows.

Proof of Theorem 2: If $N = e(s)q(s) + \ldots + e(0)q(0)$, then as in the proof of Theorem 1 we have:

$$\left| \sum_{n=0}^{N-1} \exp(F(n)hx) \right| \le \sum_{j=0}^{s} e(j+1).\left| \sum_{k=0}^{q(j)-1} \exp(F(k)hx) \right| .$$

Further:

$$S(j) := \sum_{k=0}^{q(j)-1} \exp(F(k)hx) =$$

$$= \left(\sum_{e=0}^{a(j)-1} \exp(f(e)r(j-1)hx)\right).S(j-1) +$$

$$+ \exp(f(a(j))r(j-1)hx).S(j-2)$$

and by Lemma 1 and Lemma 2 we get that $\lim_{j\to\infty} \frac{|S(j)|}{q(j)} = 0$

and from this we easily get that $\frac{1}{N}.\sum_{n=0}^{N-1} \exp(F(n)hx)$ tends to zero for N to infinity, and the theorem is proved.

<u>Proof of Theorem 3:</u>

a) In this case we have

$$\left| \sum_{e=0}^{a(j)-1} \exp(f(e)r(j-1)hx) \right| = \left| \frac{1-\exp(a(j)r(j-1)hx)}{1-\exp(r(j-1)hx)} \right| \leq$$

$$\leq a(j)(1 - \|r(j-1)hx\|^2) ,$$

that is an improvement of the right side of the inequality in Lemma 1. From this and from the proof of Theorem 2 the result follows.

b) Let $a(j) > 2$ for $j \geq j(0) - 1$ then for $j \geq j(0)$:

$$A(j+1) := \kappa(\{n | q(j) \leq n < q(j+1) , F(n) = 0\}) \geq$$

$$\geq (a(j+1) - 2).A(j) .$$

Therefore for $j \geq j(0)$:

$$\frac{A(j)}{q(j)} \geq \prod_{i=j(0)}^{j} (a(i) - 2) / \prod_{i=0}^{j} (a(i) + 1) \geq c''' > 0$$

because of $\sum_n 1/a(n) < \infty$, and therefore for no x the sequence $\{F(N)x\}_{N\geq 0}$ is uniformly distributed .

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(Received November 15, 1987)