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Schwartz Spaces and Compact Holomorphic Mappings

MIKAEL LINDSTRÖM

Department of Mathematics, Åbo Akademi, Fänriksgatan 3, SF-20500 Åbo, Finland

Let E be a locally convex space. We investigate under which conditions on E it is true that every holomorphic mapping from E into c_0 is compact. We show that Schwartzity of E is a sufficient condition and also a necessary condition if E is quasi-normable.

INTRODUCTION

In this paper we prove that the following statements are equivalent for a quasi-normable locally convex space E: (a) E is Schwartz (b) Every equicontinuous, weak*-null sequence in the dual of E is also $L_\beta E$-null convergent. (c) Every continuous linear mapping from E into c_0 is compact (d) Every holomorphic mapping from E into c_0 is compact. The method of proof rests heavily on results concerning characterizations of Schwartz spaces carried out in [5, 10, 12]. Our result gives some insight into a question posed by Aron and Schottenloher in [1] whether there are any non-trivial Banach spaces E and F for which every holomorphic mapping from E into F is compact. If E is a quasi-normable locally convex space (or a Banach space) and $F = c_0$, then E is Schwartz (or finite-dimensional) if and only if every holomorphic mapping from E into F is compact. In [7] Freniche proved that if a completely regular space X contains an infinite compact subset, E is a separable Fréchet space and the space $C_{co}(X, E)$ of continuous functions from X into E endowed with the compact-open topology is Grothendieck, then E is Montel. He asks if this result is true without the separability assumption on E. From our result we obtain that E is Schwartz if X is not pseudofinite, E is a quasi-normable locally convex space and $C_{co}(X, E)$ is a Grothendieck space. This can be considered as a positive answer to the question of Freniche.

We recall some notations and definitions. All vector spaces in this paper are complex. Let E be a locally convex space (short lcs) and F a Banach space. By $H(E,F)$ we denote the vector space of all holomorphic mappings on E with values in F. Instead of $H(E,\mathbf{C})$ we write $H(E)$. When endowed with continuous convergence [2] or with the associated equable convergence [8] (local uniform convergence) it will be denoted by $H_c(E)$ and $H_e(E)$ respectively (cf. [5]). For a lcs E, $\mathcal{U}_E$ will denote the system of all closed balanced, convex zero-neighbourhoods U in E and E_U the associated normed spaces. We denote by $P_c(^mE,F)$ and $P_e(^mE,F)$ ($m \in \mathbf{N}$) the vector space of all continuous m-homogeneous polynomials from E into F endowed with continuous convergence and local uniform convergence respectively. In [4] we have proved that $P_c(^mE)$ can be represented in the form $\text{ind}_{U\in\mathcal{U}_E} P_s(^mE) \mid U^\circ$ and $P_e(^mE)$ in the form $\text{ind}_{U\in\mathcal{U}_E} P_\beta(^mE_U)$, where ind is the inductive limit in the category of convergence vector spaces and U° denotes the subset $\{p \in P(^mE) : |p(U)| \leq 1\}$ of $P(^mE)$. By $P_s(^mE)$ and $P_\beta(^mE)$ we denote $P(^mE)$ endowed with the topology of simple convergence and bounded convergence respectively. For $m = 1$ we get the well-known representations $L_cE = \text{ind}_{U\in\mathcal{U}_E}\sigma(LE,E) \mid U^\circ$ and $L_eE = \text{ind}_{U\in\mathcal{U}_E}(LE)_{U^\circ}$. A lcs E is called *Schwartz* if for every $U \in \mathcal{U}_E$ there is a $V \in \mathcal{U}_E$, $V \subseteq U$, such that U° is compact in the Banach space $(LE)_{V^\circ}$ and *quasi-normable*, if for any $U \in \mathcal{U}_E$ there exists a $V \in \mathcal{U}_E$, $V \subseteq U$, such that on U° the topology induced by $L_\beta E$ coincides with the norm topology induced by $(LE)_{V^\circ}$. The space E is Schwartz if and only if it is quasi-normable and bounded subsets of E are precompact. A lcs E is said to be *Montel*, if it is barrelled and every bounded subsets of E are relatively compact. If every null-sequence in $\sigma(LE,E)$ is equicontinuous, then E is called c_0*-barrelled.*

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In this section we prove the following

THEOREM. *Let E be a quasi-normable lcs. The following statements are equivalent:*

(a) E is Schwartz

(b) Every L_cE-null sequence in LE is also L_eE-null

(c) Every continuous linear mapping $g : E \to c_0$ is compact

(d) Every holomorphic mapping $f : E \to c_0$ is compact.

Since every normed Schwartz space is finite-dimensional, the equivalence between (a) and (b) is formally a generalization of the Josefson-Nissenzweig theorem [11] to lcs (cf. [12]). In [1] Aron and Schottenloher ask if there are non finite-dimensional Banach spaces E and F such that every holomorphic mapping from E into F is compact. By a result of Pełczynski they obtained in a Note Added in Proof that if X is a dispersed compact Hausdorff space and F is a Banach space which does not contain c_0, then every holomorphic mapping from $C(X)$ into F is compact. Let $F = c_0$ and let E be a quasi-normable lcs. If E is Schwartz, then every holomorphic mapping from E into F is compact by the equivalence between (a) and (d). From this equivalence follows also that there always exists a non-compact holomorphic mapping from E into F, when E is an infinite-dimensional Banach space.

Let E be a lcs and F a Banach space. As in [1] we call a mapping $f \in H(E,F)$ *compact* if there for each $x \in E$ exists a neighbourhood V_x of x such that $f(V_x)$ is relatively compact in F.

FACT 1. *If E is a quasi-normable lcs, then (a) $\Leftrightarrow$ (b).*

Proof. According to [10] E is Schwartz if and only if $L_cE = L_eE$. By Corollary 3 in [12] E is Schwartz, if every L_cE-null sequence is also $L_\beta E$-null. Now the statement follows from the definition of quasi-normability.

A sequence $(p_n)_N$ in $P(^mE)$ converges to zero in $P_c(^mE)$ if and only if there exists a $U \in \mathcal{U}_E$ such that $p_n \in U^\circ$ and p_n converges to zero in $P_s(^mE)$. With this in mind, we shall now prove the following

FACT 2. *Let E be a lcs. If every $P_c(^mE)$-null sequence in $P(^mE)$ is also $P_e(^mE)$-null, then every continuous m-homogeneous polynomial $p : E \to c_0$ is compact.*

Proof. Take $p \in P(^mE, c_0)$ and let $(u_n)_\mathbf{N}$ be the standard basis in l_1. Then u_n converges to zero in $\sigma(l_1, c_0)$ and $p(x) = (u_n(p(x)))_\mathbf{N}$ for each $x \in E$. It is clear that $p_n := u_n \circ p$ is a $P_s(^mE)$-null sequence and that there exists a $U \in \mathcal{U}_E$ such that

$p_n \in U^\circ$. Hence p_n converges to zero in $P_c(^mE)$ and by the assumption in $P_e(^mE)$. Recall that for an m-homogeneous polynomial p from a lcs E into a Banach space F, $p^* : LF \to P(^mE)$ is defined by $p^*(l)x = l \circ p(x)$ for $l \in LF$ and $x \in E$, and that p^* is linear. Now it can be shown that the linear mapping $p^{**} : L_eP_e(^mE) \to l_\infty$ is given by $p^{**}(s) = \big(s(p_n)\big)_{\mathbf{N}}$ for $s \in LP_e(^mE)$. Since $P_e(^mE)$ is a polar bornological vector space [4], it follows that $L_eP_e(^mE)$ is a lcs [3] and that the canonical mapping $\hat{}: P_e(^mE) \to L_eL_eP_e(^mE)$ is an embedding [3]. Thus $\hat{p}_n$ converges to zero in $L_eL_eP_e(^mE)$ and

$$\|p^{**}(s)\| = \sup_{\mathbf{N}} |\hat{p}_n(s)| \quad \text{for every } s \in LP_e(^mE).$$

Now $p^{**} : L_eP_e(^mE) \to l_\infty$ is compact by Theorem 17.1.4 in [9]. Hence the restriction to $E \subseteq L_eP_e(^mE) \to l_\infty$ is compact, but this is just the mapping $p : E \to c_0$.

FACT 3. *If E is a lcs, then (b) $\Leftrightarrow$ (c).*

Proof. (b) $\Rightarrow$ (c) follows from Fact 2 for $m = 1$.
(c) $\Rightarrow$ (b): This implication is well-known. Simply consider the continuous linear mapping $h : E \to c_0$, $x \mapsto \big(l_n(x)\big)_{\mathbf{N}}$, where l_n is a L_cE-null sequence in LE.

Let us now consider statement (d). Let E be a lcs and F a Banach space. Every $f \in H(E,F)$ has a Taylor series expansion $f(x) = \sum_{m=0}^{\infty}(1/m!)\hat{d}^m f(0)(x)$ at zero valid for all $x \in E$, where $\hat{d}^m f(0)/m! \in P(^mE, F)$ for each $m \in \mathbf{N}$. Using this Taylor series expansion of holomorphic mappings we have proved in [5] that E is Schwartz if and only if $H_c(E) = H_e(E)$.

FACT 4. *If E is a lcs, then (a) $\Rightarrow$ (d).*

Proof. Take an arbitrary holomorphic mapping $f : E \to c_0$. In the case E is normed Proposition 3.4 in [1] tells us that f is compact, if $\hat{d}^m f(0)/m! : E \to c_0$ is compact for each $m \in \mathbf{N}$. The proof for a lcs E is similar so we omit it. Since $H_c(E) = H_e(E)$, it follows that $P_c(^mE) = P_e(^mE)$ for each $m \in \mathbf{N}$. Thus Fact 2 yields that $f : E \to c_0$ is compact.

Proof of the theorem. By Facts 1 and 3 (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). Since (d) $\Rightarrow$ (c) is trivial, Fact 4 completes the proof.

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In this section we shall apply our theorem on two results concerning Grothendieck spaces.

We say that a lcs E is a *Grothendieck space* when every L_cE-null sequence in LE is also $\sigma(LE, LL_\beta E)$-null. This definition of a Grothendieck space was introduced by Freniche in [7]. There he also noticed that Grothendieck spaces and lcs E in which the $\sigma(LE, E)$- and $\sigma(LE, LL_\beta E)$-sequential convergences coincide do not coincide in general. For c_0-barrelled lcs they of course coincide.

Let E and F be lcs, and suppose that F contains a subspace topologically isomorphic to the normed subspace H_0 of c_0, whose elements have only finitely many non-zero coordinates. Then we have, by Theorem 2.1 in [6], that if the injective tensor product $E \otimes_\epsilon F$ is a Grothendieck space, then the $\sigma(LE, E)$- and $L_\beta E$-sequential convergences coincide in the equicontinuous subsets of LE. Thus we may formulate:

COROLLARY 1. *Let E and F be lcs. Suppose that F contains a subspace topologically isomorphic to H_0 and that $E \otimes_\epsilon F$ is Grothendieck. If E is quasi-normable, then every holomorphic mapping from E into c_0 is compact.*

It is not difficult to prove that E is Grothendieck, if every continuous linear mapping from E into c_0 is compact.

In [7] Freniche proved that the $\sigma(LE, E)$-and $L_\beta E$-sequential convergences coincide in the equicontinuous subsets of LE, if $C_{co}(X, E)$ is a Grothendieck space and X contains an infinite compact subset. Using this result our theorem yields:

COROLLARY 2. *If E is a quasi-normable, X is not pseudofinite and $C_{co}(X, E)$ is a Grothendieck space, then E is Schwartz.*

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