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PARABOLIC Q-MINIMA AND MINIMAL SOLUTIONS TO VARIATIONAL FLOW

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We introduce the notion of parabolic Q-minima. Similarly to the elliptic case, where Q-minima were defined and studied by M. Giaquinta & E. Giusti, the purpose is to provide a unifying approach to various regularity results for parabolic problems. In addition, as a parabolic counterpart to the notion of elliptic minima of variational integrals, we analyse so-called minimal solutions to variational flows.

Introduction.

In the sequel we shall introduce the notion of parabolic Q-minima. As in the elliptic case, where Q-minima were defined and studied by M. Giaquinta & E. Giusti [9], the purpose is to provide a unifying approach to some of regularity results for parabolic systems and equations.

Referring for the detailed description of our results to the sections below, let us give here a brief summary. Starting with the definition and some examples of parabolic Q-minima in the first two sections, we prove in section 3 various basic estimates for general, i.e. vector-valued, parabolic Q-minima, among them an L^p-estimate of their spatial gradients. For scalar parabolic Q-minima we obtain Hölder regularity as a consequence of their membership in certain De Giorgi classes (section 4).

There is a special class of parabolic Q-minima in which each Q-minimum weakly solves an associated variational flow system, in the same way as a minimum of an elliptic variational problem solves its Euler-Lagrange system. Moreover, the regularity properties of these Q-minima fit well with those known for minima of regular elliptic variational integrals. We therefore call them minimal solutions (of their corresponding variational flow system). In fact, we shall obtain (in section 6) partial Hölder regularity for these minimal solutions without having to impose the well known smallness conditions that are needed in the regularity study of weak solutions to general second order quasilinear parabolic systems. (Cp. [11], and for the elliptic situation the review articles [8] and [12].) Encouraged by this analogy, one might also hope to get an existence theory for minimal solutions of equal generality as for elliptic minimizers. It turns out, however, that existence of minimal solutions requires in some sense stronger convexity conditions for the defining variational integral than in the elliptic case. We examine this point in section 5.

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1. Definition.

Let us start with some notationa. For $n \geq 2$, $\Omega \subset \mathbb{R}^n$ a domain denote by $D \equiv \Omega \times (0,T)$ a space-time region with generic point $z = (x,t)$. $W_2^1(\Omega)$ ($\overset{\circ}{W}{}_2^1(\Omega)$) stands for the usual Sobolev space of functions or vectorfunctions u, depending on the context, which are squareintegrable over Ω together with their generalized spatial gradient ∇u (and which vanish on the boundary $\partial\Omega$). We also need the spaces $W_2^{1,0}(D) \equiv L^2(0,T;W_2^1(\Omega))$, $\overset{\circ}{W}{}_2^{1,0}(D) \equiv L^2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$, and finally $W_2^1(D) \equiv \{u \in W_2^{1,0}(D) \mid u' \in L^2(D)\}$, where $u' \equiv \frac{\partial}{\partial t}u$.

Consider now a Carathéodory function $F = F(z,u,p)$: $D \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfying the growth condition

$$(1.1) \quad \lambda|p|^2 - b|u|^\gamma - g(z) \leq F(z,u,p) \leq \mu|p|^2 + b|u|^\gamma + g(z),$$

where N is a positive integer, g a nonnegative integrable function, and b,λ,μ,γ are numbers.

For $Q \geq 1$ we then define a function $u : D \to \mathbb{R}^n$, $u \in W_{2,loc}^{1,0}(D) \cap L_{loc}^\gamma(D)$, to be a *parabolic* Q-*minimum*, if for every $\phi \in C_0^\infty(D)$ we have the inequality

$$-\int_K u\phi' dz + E(u,K) \le Q \cdot E(u-\phi,K), \tag{1.2}$$

where $K \equiv \operatorname{spt}\phi$ and $E(w,K) \equiv \int_K F(z,w,\nabla w)dz$. To extend the range of applicability of our concept we shall also call u a *B-restricted* parabolic Q-minimum if (1.2) holds merely for all ϕ of a certain subset $\mathcal{B} \subset C_0^\infty(D)$.

For $Q = 1$ in (1.2) and $F = F(x,u,p)$ (i.e. time independent) (1.2) takes the same status for the variational flow (or gradient flow) system

$$(1.3)\qquad \begin{aligned} &(u^i)' - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} F_{p_\alpha^i}(x,u,\nabla u) + F_{u^i}(x,u,\nabla u) = 0 \\ &i = 1,\dots,N, \end{aligned}$$

as does the minimum problem $\int_\Omega F(x,u,\nabla u)dx = \min$ for the associated Euler-Lagrange system. In fact, to see that (1.2) (with $Q = 1$) implies the weak form of (1.3), replace ϕ by $h\phi$ in (1.2), $h \in \mathbb{R}\setminus\{0\}$, divide by h and let it tend to zero. Of course, for this to make sense, one has to assume that F is C^1 in u and p together with an appropriate growth behaviour of the derivatives, e.g. $|F_u| + |F_p|^2 \le C(1 + |p|^2 + |u|^{\gamma-1})$. Also, in the $\mathcal{B}$-restricted case, one has to impose the condition that for every $\phi \in C_0^\infty(D)$ $\mathcal{B}$ contains $h\,\phi$ if $|h| < h_0$, $h_0 \in \mathbb{R}^+$ possibly depending on ϕ. For reasons mentioned in the introduction we call these Q-minima *minimal solutions* and, respectively, $\mathcal{B}$-restricted minimal solutions.

2. Examples.

2.1

Under suitable structural hypothesis any weak solution of the quasilinear second order parabolic system

$$(2.1)\qquad \begin{aligned} &(u^i)' - \frac{\partial}{\partial x^\alpha} A_i^\alpha(z,u,\nabla u) + B_i(z,u,\nabla u) = 0 \\ &i = 1,\dots,N \end{aligned}$$

is a parabolic or $\mathcal{B}$-restricted parabolic Q-minimum, once F, $\mathcal{B}$ and $Q \geq 1$ are chosen properly.

In fact, let us assume for the leading term that

$$
\begin{aligned}
& A_i^\alpha(z,u,p)p_\alpha^i \geq |p|^2 - b|u|^\gamma - g(z) \ , \\
(2.2) \qquad & |A(z,u,p)| \leq \mu|p| + b|u|^{\gamma/2} + f(z) \ , \\
& g \in L^1(D), \ f \in L^2(D), \ 1 \leq \gamma < 2 \ ,
\end{aligned}
$$

while for the lower order term B we distinguish between the case of *controllable growth*,

$$
(2.3) \qquad \begin{aligned}
& |B(z,u,p| \leq a|p|^{2(\gamma-1)/\gamma} + b|u|^{\gamma-1} + c(z) \ , \\
& c \in L^{\gamma/(\gamma-1)}(D)
\end{aligned}
$$

and that of *natural growth*,

$$
(2.4) \qquad \begin{aligned}
& |B(z,u,p)| \leq a|p|^2 + c(z) \\
& \text{if } |u| \leq M \ , \text{ i.e. } a > 0 \text{ and } c \in L^1(D) \text{ may} \\
& \text{depend on } M \ .
\end{aligned}
$$

Then we have

PROPOSITION 2.1. Let $u \in W^{1,0}_{2,loc}(D)$ be a weak solution of (2.1) subject to conditions (2.2). Then

(a) in case (2.3) u is a parabolic Q-minimum for the functional $E(v,K) \equiv \lambda\int_K(|\nabla v|^2 + b|v|^\gamma + h)dz$, where $\lambda \in (0,1)$ and $h = 1 + g + f^2 + c^{\gamma/(\gamma-1)}$;

(b) in case (2.4) we have: If u is bounded, $|u| \leq M$, and if

$$(2.5) \qquad 2aM < 1$$

then u is a $\mathcal{B}$-restricted parabolic Q-minimum for the functional $E(v,K) \equiv \lambda\int_K(|\nabla v|^2 + h)dz$, where $\lambda \in (0,1)$, $h = g + f^2 + c$, and $\mathcal{B} = \{\phi \in C_0^\infty(D) \mid \|u-\phi\|_{\infty,D} \leq M\}$. In this case no restriction for the exponent γ in (2.3) is needed. On the other hand, Q depends on M .

Proof. We follow the lines of the elliptic proof in [9].

(a) $u \in W^{1,0}_{2,loc}(D)$ being a weak solution to (2.1), we have for $\phi \in C_0^\infty(D)$, $K = \operatorname{spt}\phi$,

$$
\begin{aligned}
&-\int_K u\,\phi' dz + \int_K A(z,u,\nabla u)\cdot\nabla u\,dz = \\
(2.6)\qquad &\int_K A(z,u,\nabla u)\cdot\nabla(u-\phi)dz + \int_K B(z,u,\nabla u)\cdot(u-\phi)dz \\
&-\int_K B(z,u,\nabla u)\cdot u\,dz .
\end{aligned}
$$

Using (2.2) and (2.3) as well as the Hölder and Young inequalities we easily derive from this,

$$
\begin{aligned}
&-\int_K u\phi' dz + (1-\varepsilon_1)\int_K(|\nabla u|^2 + b|u|^\gamma + h)dz \\
&\le C_1(\varepsilon_1)\int_K(|\nabla(u-\phi)|^2 + b|u-\phi|^\gamma + h)dz \\
&+ C_2\int_K |u|^\gamma\,dz \\
&= (*)\quad (\varepsilon_1 > 0\ \text{small}).
\end{aligned}
$$

In case that $b\int_K|u|^\gamma dz \le \int_K|\nabla(u-\phi)|^2dz$ we are obviously done. So let

$$(2.7)\qquad b\int_K|u|^\gamma dz \ge \int_K|\nabla(u-\phi)|^2 dz .$$

From the inequality $|u|^\gamma \le \varepsilon_2|\phi|^2 + C(\gamma,\varepsilon_2)\cdot(1+|u-\phi|^\gamma)$ we can further estimate

$$
\begin{aligned}
(*) &\le \tilde{C}_1(\varepsilon_1,\varepsilon_2)\int_K(|\nabla(u-\phi)|^2 + b|u-\phi|^\gamma + h)dz \\
&\quad + C_1\cdot\varepsilon_2\int_K|\phi|^2 dz
\end{aligned}
$$

and applying Poincaré's inequality together with (2.7),

$$
\begin{aligned}
&\int_K|\phi|^2dz \le C\int_K|\nabla\phi|^2dz \le C\int_K(|\nabla u|^2 + |\nabla(u-\phi)|^2)dz \\
&\le C\int_K(|\nabla u|^2 + b|u|^\gamma)dz ,
\end{aligned}
$$

we get the claim on taking ε_2 small enough. Note that in the elliptic case one can work with Sobolev's instead of Poincaré's inequality, which then allows a variation of the exponent γ in the range $[1,\frac{2n}{n-2})$.

(b) Again we start from (2.6), this time assuming that

(2.8) $\quad \|u-\phi\|_{\infty,D} \le M$.

Then (2.2) and (2.4) lead to the estimate

$$-\int_K u\phi' dz + (1-2aM)\int_K |\nabla u|^2 dz$$
$$\le b\int_K |u|^\gamma dz + \int_K g\ dz + \mu\int_K |\nabla u||\nabla(u-\phi)|dz$$
$$+ b\int_K |u|^{\gamma/2}|\nabla(u-\phi)|dz + \int_K f|\nabla(u-\phi)|dz$$
$$+ 2M\int_K c\ dz \ ,$$

which, further estimated and rearranged, yields

$$-\int_K u\phi' dz + \frac{1-2aM}{2}\int_K (|\nabla u|^2 + h)\ dz$$
$$\le C(M)\int_K (|\nabla(u-\phi)|^2 + h)\ dz \ ,$$

thereby proving the assertion.
At this instant we remark that, contrary to the elliptic proof, we cannot go further and show that u is actually an unrestricted Q-minimum (i.e. we cannot get free from condition (2.8) on ϕ). This is essentially due to the indefiniteness of the term $\int_K u\phi' dz$.

QED.

2.2

The obvious trivial example of a minimal solution is of course provided by any weak solution of the heat equation, taking $F = \frac{1}{2}|p|^2$. More generally, if F is a function *convex* and C^1 in the pair $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, than from the inequality

$$-\int_D u\phi' dz + E(u,D) - E(u-\phi,D) \le$$
$$-\int_D u\phi' dz + \int_D (F_p[u]\cdot\nabla\phi + F_u[u]\cdot\phi)dz,\ \phi \in C_0^\infty(D) \ ,$$

which follows from the convexity assumption, we see that any weak solution to (1.3) satisfies at the same time (1.2) with $Q = 1$, i.e. is a minimal solution.

To give a non-trivial example, consider the parabolic equation

$$
(2.9)\qquad \begin{aligned} &u' - \Delta u - u^3 = 0 \quad \text{on } \mathbb{R}^4 \times (0,T) \\ &u|_{t=0} = u_0 \ . \end{aligned}
$$

The corresponding F is $\frac{1}{2}p^2 - \frac{1}{4}u^4$, $(u,p) \in \mathbb{R} \times \mathbb{R}^4$, and is nowhere convex, if $u \neq 0$. Nevertheless , as we shall see in section 5, for initial values u_0 lying in a certain convex subset C of $\overset{\circ}{W}{}^1_2(\mathbb{R}^4)$, the weak solutions to (2.9) are $\mathcal{B}$-restricted minimal solutions, with $\mathcal{B} = \{\phi \in C_0^\infty(\mathbb{R}^4 \times (0,T)) \,|\, u(\cdot,t) - \phi(\cdot,t) \in C\}$.

2.3

Further examples of parabolic Q-minima can be constructed by taking the parabolic analogues of those mentioned in [9], e.g. parabolic variational inequalities with obstacles.

3. Estimates for Q-minima.

We continue with some frequently used notations. For $z_0 = (x_0,t_0) \in D$ and $r > 0$ set

$$
B_r(x_0) = \{x \in \mathbb{R}^n \,|\, |x-x_0| < r\} \ , \ \Lambda_r(t_0) \equiv (t_0 - r^2, t_0)
$$

and denote by

$$
\begin{aligned} &Q_r(z_0) \equiv B_r(x_0) \times \Lambda_r(t_0) , \\ &\Sigma_r(z_0) \equiv B_r(x_0) \times \{t_0 - r^2\} \cup \partial B_r(x_0) \times \Lambda_r(t_0) \end{aligned}
$$

the standard parabolic cylinder with its parabolic boundary. We shall omit the reference points and write Q_r, B_r etc. when no confusion will arise. Also set $⨍_{Q_r} f dz \equiv |Q_r|^{-1} \int_{Q_r} f dz$ for the mean of f over Q_r . For $h > 0$ and $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$,

$$\chi^h_{t_1,t_2}(t) \equiv \begin{cases} 0 & , t \le t_1 - h \text{ or } t \ge t_2 + h \\ 1 & , t_1 \le t \le t_2 \\ 1 + \frac{t-t_1}{h} & , t_1 - h \le t \le t_1 \\ 1 - \frac{t-t_2}{h} & , t_2 \le t \le t_2 + h \end{cases}$$

denotes a piecewise linear approximation of the characteristic function $\chi_{[t_1,t_2]}$.

By ψ_ε we shall mean the mollification of a function ψ w.r.t. the time variable by an even C^∞ kernel supported in the interval $|t| < \varepsilon$. In particular, if ψ is defined on $(0,T)$ and $t_1,t_2 \in (0,T)$, then $\operatorname{spt}(\psi_\varepsilon \chi^h_{t_1,t_2})_\varepsilon \subset\subset (0,T)$ for ε,h sufficiently small (as is always assumed).

A cut-off function for the pair (Q_r,Q_R) , $0 < r < R$, is a Lipschitz function $\tau(x,t)$, being equal to 1 on Q_r , equal to zero outside Q_R , and satisfying $|\tau'| + |\nabla\tau|^2 \le C/(R-r)^2$. The same applies to cut-off functions for (B_r,B_R) depending on x only.

Finally, unless otherwise stated, the letter C (occasionally labelled $C_0,\ldots$) will denote constants depending on given quantities (such as $\lambda,K,K_0,n,\ldots$) only, and its value is allowed to change within an estimate.

In the following we assume for simplicity of exposition that F grows like

$$(3.1) \quad \frac{\lambda}{2}p^2 - K_0 \le F(z,u,p) \le Kp^2 + K_0, \ \lambda,K > 0, \ K_0 \ge 0 ,$$

which can be weakened using the well-known embedding theorems (cp. [13], p. 74 - 78). Then in the defining relation (1.2) we are allowed to insert testfunctions $\phi \in \overset{\circ}{W}{}^1_2(D)$, whenever the Q-minimum $u \in W^{1,0}_2(D)$.

3.1

The first statement concerns an initial (low) amount of time regularity implied by the property of the function u being a Q-minimum. In particular, when inserting functions like $\phi = (\tau^2 u_\varepsilon \chi^h_{t_1,t_2})_\varepsilon$ into (1.2), this proposition allows us to pass to the limits $\varepsilon, h \to 0$ in the term $\int_D u\phi' dz$. But it will also be useful for the partial regularity proofs in section 6.

PROPOSITION 3.1. Let F satisfy (3.1) and $u \in W_2^{1,0}(D)$ be a Q-minimum. Then $u \in C([0,T]; L^2(\Omega))$.

Proof. From (1.2) we have

$$-\int_K u\phi' dz \le Q \cdot E(u-\phi,K) - E(u,K)$$

and also, replacing ϕ by $-\phi$,

$$\int_K u\phi' dz \le Q \cdot E(u+\phi,K) - E(u,K) .$$

Hence by (3.1),

$$\left|\int_K u\phi' dz\right| \le C\int_K |\nabla u|^2 dz + C\int_K |\nabla\phi|^2 dz ,$$

i.e. for the distribution u' defined as $\langle u',\phi\rangle \equiv -\int_D u\phi' dz$,

$$\sup\{|\langle u',\phi\rangle| \mid \phi \in C_0^\infty(D),\ \int_D |\nabla\phi|^2 = 1\} \le C .$$

Extend by continuity to get $u' \in L^2(0,T;W_2^{-1}(\Omega))$, $W_2^{-1}(\Omega)$ being the dual of $\overset{\circ}{W}{}_2^1(\Omega)$. Therefore,

$$u \in \{v \mid v \in L^2(0,T;W_2^1(\Omega)),\ v' \in L^2(0,T;W_2^{-1}(\Omega))\}$$
$$\subset C([0,T]; L^2(\Omega)) ,$$

by Theorems 3.1 and 2.4 of [14].

QED.

As an immediate consequence we have

COROLLARY 3.2.
Minimal solutions are unique.

Proof. By this we mean that, if $u, \bar u \in W_2^{1,0}(D)$ are minimal solutions (with same F of course) such that $u - \bar u \in \overset{\circ}{W}{}_2^{1,0}(D)$ and $u - \bar u|_{t=0} = 0$, then $u = \bar u$ a.e. .

In fact, place $\phi \equiv ((u - \bar{u})_\varepsilon \chi^h_{t_1,t_2})_\varepsilon$ into (1.2) , $t_1,t_2 \in (0,T)$. Then, since $Q = 1$,

$$\int_{spt\phi} F[u]\,dz \le \int_{spt\phi} F[u-\phi]dz + \int_{spt\phi} u\phi' dz$$

$$\le \int_{spt\phi} (F[u-\phi] - F[\bar{u}])dz + \int_{spt\phi} F[\bar{u}]dz + \int_{spt\phi} u\phi' dz$$

$$\le \int_{spt\phi} (F[u-\phi] - F[\bar{u}])dz + \int_{spt\phi} F[\bar{u}+\phi]dz$$

$$+ \int_{spt\phi} (u-\bar{u})\phi' dz ,$$

since also $\bar{u}$ satisfies (1.2). Here we used the shorthand notation $F[w] \equiv F(x,w,\nabla w)$. As $\varepsilon \to 0$ and $h \to 0$ (more precisely we always take some sequences $\varepsilon_i \to 0$, $h_i \to 0$) the first term on the r.h.s. vanishes, while the second cancels with the left hand term. Therefore

$$-\int_D (u-\bar{u})_\varepsilon\ ((u-\bar{u})_\varepsilon\ \chi^h_{t_1,t_2})' dz \le o(1) \quad \text{when} \quad \varepsilon,h \to 0 .$$

Since by Prop. 3.1 both $u,\bar{u} \in C([0,T];\ L^2(\Omega))$ we may pass to the limits on the left also, to get for $0 < t_1 < t_2 < T$

$$\int_\Omega (u(t_2) - \bar{u}(t_2))^2 dx \le \int_\Omega (u(t_1) - \bar{u}(t_1))^2 dx .$$

Letting $t_1 \to 0$ we get the uniqueness assertion.

QED.

3.2

We proceed with some basic estimates. Varying an idea of M . Struwe [15] we shall use a competing function involving the weighted mean

$$u^{\tau,\varepsilon}_\sigma(t) \equiv \int_{B_\sigma} \tau^2(x,t)\,u_\varepsilon(x,t)\,dx / \int_{B_\sigma} \tau^2(x,t)\,dx$$

$$(\equiv 0 \quad \text{outside} \quad spt\,\tau = Q_\sigma),$$

where τ is a cut-off function for (Q_ρ,Q_σ), $0 < \rho < \sigma$, with the further property that

(3.2) $$\sup_{x\in B_\sigma} \tau(x,t) \le C \fint_{B_\sigma} \tau(x,t)\,dx , \quad t \in \Lambda_\sigma .$$

The corresponding weighted mean of the Q-minimum u is denoted by $u_\sigma^\tau(t)$. When compared with the simple mean $u_\rho(t) \equiv ⨍_{B_\rho} u(x,t)\,dx$, it turns out that

$$(3.3) \qquad \int_{B_\rho} |u-u_\rho(t)|^2 dx \le \int_{B_\sigma} |u-u_\sigma^\tau(t)|^2 dx$$

(since $c = u_\rho(t)$ minimizes for $t \in \Lambda_\sigma$ the integral $\int_{B_\sigma} |u(x,t) - c|^2 dx)$, and

$$(3.4) \qquad \int_{B_\sigma} |u-u_\sigma^\tau(t)|^2 dx \le C\int_{B_\sigma} |u-u_\sigma(t)|^2 dx$$

(which follows from (3.2)).

Also, since $\int_{B_\sigma} (u(t) - u_\sigma(t))\,dx = 0$, the Sobolev-Poincaré inequalities hold, i.e. with $2^+ \equiv \frac{2n}{n+2}$, if $n \ge 3$, and $2^+ \in (1,2)$ arbitrary, if $n = 2$,

$$(3.5) \qquad \sigma^{-2} ⨍_{B_\sigma} |u(t) - u_\sigma(t)|^2 dx \le C\left(⨍_{B_\sigma} |\nabla u(t)|^{2^+} dx\right)^{2/2^+}$$

and

$$(3.6) \qquad \sigma^{-2}\int_{B_\sigma} |u(t) - u_\sigma(t)|^2 dx \le C\int_{B_\sigma} |\nabla u(t)|^2 dx \ .$$

To get a preliminary estimate, with the notation above insert into (1.2) the function

$$\phi(x,t) \equiv (\tau^2(x,t)(u_\varepsilon(x,t) - u_\sigma^{\tau,\varepsilon}(t))\chi^h_{0,t_1})_\varepsilon, \ t_1 \in \Lambda_\rho \ .$$

($Q_\sigma \subset D$, therefore $\mathrm{spt}\phi \subset D$ for ε, h small.)

As $\varepsilon, h \to 0$, we observe that

$$\int_{\mathrm{spt}\phi} (F[u]-Q\cdot F[u-\phi])\,dz \to$$

$$\int_{Q_\sigma \cap \{t \le t_1\}} \left(F[u]-Q\cdot F[u-\tau^2(u-u_\sigma^\tau(t))\chi_{[0,t_1]}]\right) dz$$

$$= \int_{Q_\sigma \cap \{t\le t_1\}} F[u]\,dz - Q \ \cdot$$

$$\int_{(Q_\sigma \setminus Q_\rho)\cap\{t\le t_1\}} F[u-\tau^2(u-u_\sigma^\tau(t))]\,dz - Q\cdot\int_{Q_\rho\cap\{t\le t_1\}} F[u_\sigma^\tau(t)]\,dz,$$

while

$$-\int_D u\phi' dz = -\int_D (u_\varepsilon - u_\sigma^{\tau,\varepsilon})(\tau^2(u_\varepsilon - u_\sigma^{\tau,\varepsilon})\chi^h)' dz$$
$$-\int_D u_\sigma^{\tau,\varepsilon}(\tau^2(u_\varepsilon - u_\sigma^{\tau,\varepsilon})\chi^h)' dz$$
$$= -\frac{1}{2}\int_D (u_\varepsilon - u_\sigma^{\tau,\varepsilon})^2(\tau^2\chi^h)' dz + \int_D (u_\sigma^{\tau,\varepsilon})'\tau^2(u_\varepsilon - u_\sigma^{\tau,\varepsilon})\chi^h dz$$
$$\equiv (*)$$

It is easily checked that by Fubini's theorem the last term vanishes (this is precisely the reason for introducing the weighted mean), hence as $\varepsilon, h \to 0$

$$(*) \to \frac{1}{2}\int_{B_\sigma} \tau^2(x,t_1)(u(x,t_1) - u_\sigma^\tau(t_1))^2 dx$$
$$- \int_{Q_\sigma \cap \{t \le t_1\}} |u-u_\sigma^\tau|^2 \tau\tau' dz \; .$$

Collecting the facts we arrive at the inequality

$$\frac{1}{2}\int_{Q_\sigma} \tau^2(t_1)(u(t_1) - u_\sigma^\tau(t_1))^2 dx + \int_{Q_\sigma \cap \{t\le t_1\}} F[u] dz \le$$
$$Q \cdot \int_{(Q_\sigma \setminus Q_\rho)\cap\{t\le t_1\}} F[u-\tau^2(u-u_\sigma^\tau)] dz +$$
$$\int_{Q_\sigma \cap\{t\le t_1\}} |u-u_\sigma^\tau|^2 \tau\tau' \, dz + Q \cdot \int_{Q_\rho \cap\{t\le t_1\}} F[u_\sigma^\tau] dz \; .$$

Using that τ is a cut-off function and passing to $\sup_{t_1\in\Lambda_\rho}$ we finally get on account of (3.1): for every $0 < \rho < \sigma$ with $Q_\sigma \subset D$,

$$\sup_{t\in\Lambda_\rho} \int_{B_\rho} |u(t)-u_\sigma^\tau(t)|^2 dx + \int_{Q_\rho} |\nabla u|^2 dz$$
$$\le C_0 \int_{Q_\sigma\setminus Q_\rho} |\nabla u|^2 dz + \frac{C}{(\sigma-\rho)^2}\int_{Q_\sigma} |u-u_\sigma^\tau|^2 dz + C\,K_0\sigma^{n+2} \tag{3.7}$$
$$\le C_0 \int_{Q_\sigma\setminus Q_\rho} |\nabla u|^2 dz + \frac{C}{(\sigma-\rho)^2}\int_{Q_\sigma} |u-u_\sigma(t)|^2 dz + C\,K_0\sigma^{n+2} ,$$

where in the last step we used (3.4).
(3.7) is the starting inequality for the following two obwervations.

<u>PROPOSITION 3.2.</u> (<u>Caccioppoli inequality</u>). <u>A</u> Q-<u>minimum</u> $u \in W_2^{1,0}(D)$ <u>satisfies for every</u> $0 < \rho < R$ <u>such that</u> $Q_\rho = Q_\rho(z_0) \subset Q_R = Q_R(z_0) \subset D$,

$$(3.8) \qquad \int_{Q_\rho} |\nabla u|^2 dz \le \frac{C}{(R-\rho)^2} \sup_{\sigma\in[\rho,R]} \int_{Q_\sigma} |u-u_\sigma(t)|^2 dz + CK_0 R^{n+2} .$$

(*Note that for fixed* z_0 $\sigma \to \int_{Q_\sigma} |u-u_\sigma(t)|^2 dz$ *is continuous*).

PROPOSITION 3.3. (*First Poincaré type inequality*). *For a Q-minimum* $u \in W_2^{1,0}(D)$ *and every* $Q_R \subset D$

$$(3.9) \qquad \sup_{t\in\Lambda_{R/2}} \int_{B_{R/2}} |u(t)-u_R^\tau(t)|^2 dx \le C \left\{\int_{Q_R} |\nabla u|^2 dz + K_0 R^{n+2}\right\} ,$$

where τ *is a cut-off function for* $(Q_{R/2}, Q_R)$.

Proofs. Prop. 3.2: By hole-filling in (3.7),

$$\int_{Q_\rho} |\nabla u|^2 dz \le \frac{C_0}{C_0+1} \int_{Q_\sigma} |\nabla u|^2 dz + \frac{C}{(\sigma-\rho)^2} \int_{Q_\sigma} |u-u_\sigma(t)|^2 dz + CK_0\sigma^{n+2}$$

Now for fixed ρ and R, $0 < \rho < R$, $Q_R \subset D$, iterate this inequality taking the sequence

$$\rho_0 \equiv \rho \ , \ \rho_{i+1} - \rho_i = (1-\lambda)\lambda^i (R-\rho) , \ i = 0,1,\ldots$$

With $\lambda > 0$ small enough this implies (3.8). (Compare [7], Lemma 3.1, p. 161).

Proposition 3.3: Choose $\rho = R/2$, $\sigma = R$ in (3.7) and estimate the second term of its r.h.s. using (3.6):

$$\frac{C}{R^2} \int_{Q_R} |u-u_R(t)|^2 dz \le C \int_{Q_R} |\nabla u|^2 \, dz .$$

This gives (3.9) .

QED.

As a corollary we obtain what is called a "reverse Hölder inequality with increasing domains", first introduced as a local refinement of Gehrings's lemma [6] in [10] to conclude L^p-estimates.

<u>COROLLARY 3.4.</u> (<u>Giaquinta-Modica estimate</u>). <u>For every</u> Q_r <u>with</u> $Q_{4r} \subset D$,

$$(3.10) \qquad ⨍_{Q_r} |\nabla u|^2 dz \le C \left(⨍_{Q_{4r}} |\nabla u|^{2^+} dz \right)^{2/2^+} + \theta ⨍_{Q_{4r}} |\nabla u|^2 dz + C K_0 ,$$

2^+ <u>as defined in</u> (3.5), <u>and</u> $\theta \in (0,1)$.

<u>Proof.</u> Take $\rho = R/4$, $\sigma = R/2$ in (3.8), and let $\sigma_0 \in [R/4, R/2]$ be such that

$$\int_{Q_{\sigma_0}} |u - u_{\sigma_0}(t)|^2 dz = \sup_{\sigma \in [R/4, R/2]} \int_{Q_\sigma} |u - u_\sigma(t)|^2 dz .$$

Then by (3.3), for τ a cut-off function for $(Q_{\sigma_0}, Q_{2\sigma_0})$ (satisfying (3.2)),

$$\int_{Q_{\sigma_0}} |u - u_{\sigma_0}(t)|^2 dz \le \int_{Q_{\sigma_0}} |u - u^\tau_{2\sigma_0}(t)|^2 dz .$$

Hence (3.8) gives

$$⨍_{Q_{R/4}} |\nabla u|^2 dz \le \frac{C}{R^2} ⨍_{Q_{\sigma_0}} |u - u^\tau_{2\sigma_0}(t)|^2 dz + C K_0$$

$$\le \frac{C}{R^2} ⨍_{\Lambda_{\sigma_0}} \left(⨍_{B_{\sigma_0}} |u - u^\tau_{2\sigma_0}(t)|^2 dx \right)^{1-2^+/2} \left(⨍_{B_{2\sigma_0}} |u - u^\tau_{2\sigma_0}(t)|^2 dx \right)^{2^+/2} dt$$

$+ C K_0$, and by (3.4),

$$\le \frac{C}{R^2} \left(\sup_{t \in \Lambda_{\sigma_0}} ⨍_{B_{\sigma_0}} |u - u^\tau_{2\sigma_0}(t)|^2 dx \right)^{1-2^+/2} \cdot$$

$$\cdot ⨍_{\Lambda_{\sigma_0}} \left(⨍_{B_{2\sigma_0}} |u - u_{2\sigma_0}(t)|^2 dx \right)^{2^+/2} dt + C K_0$$

which, using (3.9) and (3.5), is

$$\le \frac{C}{R^2} \left(\sigma_0^2 ⨍_{Q_{2\sigma_0}} |\nabla u|^2 dz + K_0 \sigma_0^2 \right)^{1-2^+/2} \cdot$$

$$\cdot ⨍_{\Lambda_{\sigma_0}} \sigma_0^{2^+} ⨍_{B_{2\sigma_0}} |\nabla u|^{2^+} dx\, dt + C K_0 .$$

Hence, since $\sigma_0 \in [R/4, R/2]$,

$$⨍_{Q_{R/4}} |\nabla u|^2 dz \le C \left(⨍_{Q_R} |\nabla u|^2 dz + K_0 \right)^{1-2^+/2} \cdot$$

$$\cdot ⨍_{Q_R} |\nabla u|^{2^+} dz + C K_0 .$$

Setting $r = R/4$, (3.10) follows by Young's inequality.

QED

Next we state an L^p-estimate for the spatial gradient of a parabolic Q-minimum. Besides being of interest in its own as an inherent property of Q-minima, estimates of this kind have been used as a tool to prove partial Hölder regularity results in elliptic and parabolic theory (e.g. [7],[11]). We shall need it in section 6, too.

THEOREM 3.5. (L^p-*estimate*) *Let* F *satisfy* (3.1) *and* $u \in W_2^{1,0}(D)$ *be a Q-minimum. Then* $\nabla u \in L^p_{loc}(D)$ *for some* $p > 2$, *and*

$$(3.11) \qquad \left(⨍_{Q_r} |\nabla u|^p dz \right)^{1/p} \le C \left(⨍_{Q_{4r}} |\nabla u|^2 dz \right)^{1/2} + C K_0$$

holds for all $Q_{4r} \subset D$.

Proof. Follows from the reverse Hölder inequality (3.10) by direct application of the Giaquinta-Modica lemma [7], Prop. 11, p. 122. Although formulated for Euclidian cubes, the lemma clearly extends to parabolic ones, and therefore applies to the case at hand, as has already been remarked by Giaquinta & Struwe [11] in a similar situation.

QED

3.3

Proposition 3.2 up to Theorem 3.5 carry over the case of β-restricted Q-minima, once we assume that $u \in C([0,T];L^2(\Omega))$ (which, in applications, can be shown by other means) and that, in the limit $\varepsilon, h \to 0$, the test function introduced at the beginning of subsection 3.2 lies in the $W_2^{1,0}(D)$-closure of $\mathcal{B}$. This is true e.g. for the case mentioned in Prop. 2.1 (b), since, if $\|u\|_{\infty,D} = M$, then

$$\|u-\tau^2(u-u_R^\tau(t))\chi_{[0,t_1]}\|_{\infty,D}$$
$$\leq \max\{\|((1-\tau^2)u+\tau^2\ u_R^\tau(t))\chi_{[0,t_1]}\|_{\infty,D}, \|u(1-\chi_{[0,t_1]})\|_{\infty,D}\} \leq M\ .$$

3.4

We close this section with another Poincaré type estimate being one of the main ingredients of the partial regularity proof for minimal solutions in section 6. Let us point out, however, that this estimate already holds for general Q-minima. The idea of its proof is taken from [15], but is technically more involved caused by the need of a suitable modification in the choice of testing function. For $Q_r \subset D$ set $u_r \equiv ⨍_{Q_r} u\,dz$ (an N-vector with constant components).

PROPOSITION 3.6. (Second Poincaré type estimate). Let $u \in W_2^{1,0}(Q)$ be a Q-minimum with F satisfying (3.1). Then for all Q_r with $Q_{2r} \subset D$,

$$r^{-2}\int_{Q_r} |u-u_r|^2 dz \leq C \int_{Q_{2r}} |\nabla u|^2 dz + C\ K_0 r^{n+2}\ . \tag{3.12}$$

Proof. 1) For fixed $s \in \Lambda_{2r}\backslash\Lambda_r$ set $u_{2r}(s) \equiv ⨍_{B_{2r}} u(x,s)dx$ and let $\tau = \tau(x)$ be a cut-off function for (B_r,B_{2r}) , so that for some $C_0 > 0$

$$|\nabla\tau|^2 \leq C_0/r^2\ . \tag{3.13}$$

Then define $t_s \in \Lambda_{2r}^s \equiv \Lambda_{2r} \cap \{t\geq s\}$ by

$$\int_{B_{2r}} \tau^2|u(t_s)-u_{2r}(s)|^2 dx = \sup_{t\in\Lambda_{2r}^s} \int_{B_{2r}} \tau^2|u(t)-u_{2r}(s)|^2 dx\ .$$

(By Proposition 3.1 such t_s exists.)

Clearly, since $c = u_r$ minimizes the integral $\int_{Q_r} |u-c|^2 dz$,

$$\int_{Q_r}|u-u_r|^2dz \le \int_{Q_r}|u-u_{2r}(s)|^2dz \le \int_{\Lambda_r}\int_{B_{2r}}\tau^2|u-u_{2r}(s)|^2dz$$

(3.14)

$$\le r^2\int_{B_{2r}}\tau^2|u(t_s)-u_{2r}(s)|^2dx \ .$$

2) Insert into (1.2) the function

$$\phi(x,t) \equiv (\tau^2(x)(u_\varepsilon(x,t)-u_{2r}(s))\psi(t)\chi^h_{s,t_s})_\varepsilon \ ,$$

where the auxiliary function ψ is defined as

$$\psi(t) \equiv \frac{1}{8\gamma+\frac{\gamma}{r^2}(t-t_o)} \ , \ t\in\Lambda_{2r} = \Lambda_{2r}(t_o) \ ,$$

so that

(3.15) $\psi'(t) + \frac{\gamma}{r^2}\psi^2(t) = 0$ in Λ_{2r}

and moreover,

(3.16) $1/(8\gamma) \le \psi(t) \le 1/(4\gamma)$.

The constant $\gamma > 0$ will be chosen in a moment.

As before, let us first discuss the terms appearing in (1.2) seperately.

We check that

$$-\int_D u\phi' dz = \frac{1}{2}\frac{1}{h}\int_{t_s}^{t_s+h}\int_{B_{2r}}\tau^2(u_\varepsilon-u_{2r}(s))^2dx\psi(t)dt$$

$$-\frac{1}{2}\frac{1}{h}\int_{s-h}^{s}\int_{B_{2r}}\tau^2(u_\varepsilon-u_{2r}(s))^2dx\psi(t)dt$$

$$-\frac{1}{2}\int_D\tau^2(u_\varepsilon-u_{2r}(s))^2\psi'(t)\chi^h_{s,t_s}dz$$

$$\to \frac{1}{2}\int_{B_{2r}}\tau^2(u(t)-u_{2r}(s))^2\psi(t)dx\Big|_{t=s}^{t=t_s}$$

$$-\frac{1}{2}\int_s^{t_s}\int_{B_{2r}}\tau^2(u-u_{2r}(s))^2dx\psi'(t)\ dt \ ,$$

as $\varepsilon, h \to 0$.

Passing to the same limits in the other terms of (1.2) we get

$$\int_{\mathrm{spt}\phi}(F[u]-Q\cdot F[u-\phi])dz$$

$$\to \int_s^{t_s}\int_{B_{2r}}\left(F[u]-Q\cdot F[u-\tau^2(u-u_{2r}(s))\psi(t)]\right)dz$$

which, using (3.1), (3.13), and (3.16), is estimated

$$\geq \frac{\lambda}{2}\int_s^{t_s}\int_{B_{2r}}|\nabla u|^2dz-2KQ\int_{Q_{2r}}|\nabla u|^2dz-4KQ\int_{Q_{2r}}|\nabla u|^2\psi^2dz$$

$$-\ 4KQ\int_s^{t_s}\int_{B_{2r}}|\nabla\tau^2|^2\ (u-u_{2r}(s))^2dx\psi^2(t)dt-C\ K_0r^{n+2}$$

$$\geq -\ C\ (1+\frac{1}{\gamma^2})\int_{Q_{2r}}|\nabla u|^2dz-C\ K_0r^{n+2}$$

$$-\ \frac{16KQC_0}{r^2}\int_s^{t_s}\int_{B_{2r}}\tau^2(u-u_{2r}(s))^2dx\psi^2(t)dt\ .$$

Hence,

$$\frac{1}{2}\int_{B_{2r}}\tau^2(u(t_s)-u_{2r}(s))^2\psi(t_s)dx$$

$$\leq \frac{1}{2}\int_{B_{2r}}\tau^2(u(s)-u_{2r}(s))^2\psi(s)dx+C(1+\frac{1}{\gamma^2})\int_{Q_{2r}}|\nabla u|^2dz$$

$$+\ CK_0r^{n+2}+\frac{1}{2}\int_s^{t_s}\int_{B_{2r}}\tau^2(u-u_{2r}(s))^2dx\{\psi'(t)+\frac{32KQC_0}{r^2}\psi^2(t)\}dt$$

Thus, choosing $\gamma \equiv 32\ KQC_0$ in the definition of ψ , we see that by (3.15) the last term vanishes, and for the rest we conclude using (3.16) and (3.6),

$$\int_{B_{2r}}\tau^2(u(t_s)-u_{2r}(s))^2dx \leq C\ r^2\int_{B_{2r}}|\nabla u(x,s)|^2\ dx$$

$$+\ C\int_{Q_{2r}}|\nabla u|^2dz+C\ K_0r^{n+2}\ .$$

3) With the last inequality we further estimate (3.14) ,

$$\int_{Q_r} |u-u_r|^2 dz \le C\, r^4 \int_{B_{2r}} |\nabla u(x,s)|^2 dx$$
$$+ C\, r^2 \int_{Q_{2r}} |\nabla u|^2 dz + C\, K_0 r^{n+4} ,$$

and averaging this w.r.t $s \in \Lambda_{2r} \backslash \Lambda_r$ delivers the estimate (3.12).

<u>QED</u>

4. Hölder continuity of scalar Q-minima.

We now study the special case of *scalar* parabolic Q-minima, i.e. we take $N = 1$ in the following. The goal will be to prove their membership in certain De Giorgi classes (see below), from which, as is well-known, we can conclude that scalar Q-minima are in fact Hölder continuous. For the time-independent case Di Benedetto & Trudinger [4] have shown that nonnegative functions lying in a De Giorgi class satisfy a Harnack inequality. It seems likely that this also holds for the time-dependent situation.

Since in the sequel we also want to cover the case of parabolic equations (2.1) with natural growth (2.4), we shall slightly generalize (1.2) to:

$$\begin{aligned} -\int_K u\phi' dz + E(u,K) &\le QE(u-\phi,K) \\ &+ a \int_K (|\nabla u|^2 + c)|\phi|\, dz, \quad a \ge 0 , \end{aligned} \tag{4.1}$$

where we assume (1.1) with g and γ satisfying

$$g \in L^\sigma(D) \;,\; \gamma \cdot \sigma \le q \equiv \frac{2(n+2)}{2} \;,\; \sigma > \frac{q}{q-1} \tag{4.2}$$

and

$$c \in L^\sigma(D) \;,\; \text{if} \;\; a > 0 . \tag{4.3}$$

Setting for $z_0 = (x_0,t_0) \in D$ and $\rho,\, \tau > 0$ $Q_{\rho,\tau} \equiv Q_{\rho,\tau}(z_0) \equiv B_\rho(x_0) \times (t_0-\tau,t_0)$, we shall prove

THEOREM 4.1. *Assume* $N = 1$, (1.1), (4.2), (4.3) *and let* $u \in V_2^{1,0}(D) \equiv W_2^{1,0}(D) \cap C([0,T]; L^2(\Omega))$ *satisfy* (4.1) *for all* $\phi \in C_0^\infty(D)$. *If* $a > 0$ *in* (4.1) *assume further that* $u \in L^\infty(D)$, $|u| \leq M$.

Then there exist positive constants L *and* κ *depending only on the structure conditions and, in case of* L , *on the* $V_2^{1,0}(D)$*-norm of* u , *such that every* $Q_{\rho,\tau} \subset D$ *and* $\sigma_1, \sigma_2 \in (0,1)$ *the following inequalities hold*

$$\max_{t\in[t_0-\tau,t_0]} \int_{B_{(1-\sigma_1)\rho}} |(u-k)^\pm(t)|^2 dx \leq \int_{B_\rho} |(u-k)^\pm(t_0-\tau)|^2 dx \tag{4.4}$$
$$+ L \cdot \{(\sigma_1\rho)^{-2} \int_{Q_{\rho,\tau}} |(u-k)^\pm|^2 dz + \left|A^\pm_{k,\rho,\tau}\right|^{\frac{1+\kappa}{q}}\}$$

and

$$\max_{t\in[t_0-(1-\sigma_2)\tau,t_0]} \int_{B_{(1-\sigma_1)\rho}} |(u-k)^\pm(t)|^2 dx + \int_{Q_{(1-\sigma_1)\rho,(1-\sigma_2)\tau}} |\nabla(u-k)^\pm|^2 dz \tag{4.5}$$
$$\leq L \cdot \{[(\sigma_1\rho)^{-2} + (\sigma_2\tau)^{-1}] \int_{Q_{\rho,\tau}} |(u-k)^\pm|^2 dz + \left|A^\pm_{k,\rho,\tau}\right|^{\frac{1+\kappa}{q}}\},$$

where $k \in \mathbb{R}$ *if* $a = 0$ *in* (4.1), *while in case* $a > 0$, *setting* $M^\pm \equiv \sup_{Q_{\rho,\tau}} \pm u$, k *respectively ranges between*

$$M^+ - \frac{\lambda}{4a} \leq k \leq M^+ \quad \text{or} \quad - M^- \leq k \leq -(M^- - 4a) .$$

(*Above*, $(u-k)^\pm \equiv \max(\pm u \mp k, 0)$ *and* $A^\pm_{k,\rho,\tau} \equiv Q_{\rho,\tau} \cap \{(u-k)^\pm > 0\}$.)

Postponing the proof to the end of this section, we remark that in case $a = 0$ $u \in C([0,T]; L^2(\Omega))$ by Proposition 3.1; moreover, in the same way as in [4], one can deduce from (4.5) that u is locally bounded. Denote therefore $M' \equiv \sup_{D'}|u|$ for $D' \subset\subset D$.

We notice further that (4.4) and (4.5) are the defining inequalities for a function $u \in V_2^{1,0}(D)$ to be element of a De Giorgi class. More precisely, adopting the notation of [13], p. 110, we have in case $a = 0$,

$u \in B_2(D', M', L, q, \infty, \kappa)$ for all $D' \subset\subset D$,

and in case $a > 0$,

$u \in B_2(D, M, L, q, \frac{\lambda}{4a}, \kappa)$.

As a direct consequence of Theorem 4.1 we therefore conclude from [13], Thm. 7.1, p. 120;

THEOREM 4.2. Under the conditions of Theorem 4.1 there exists $\alpha \in (0,1)$ depending only on the structural assumptions, the $V_2^{1,0}(D)$-norm of u and, in case $a > 0$, on M , such that $u \in C^{\alpha}(D;\delta)$ where $\delta(z,z_0) \equiv \max\{|x-x_0|, |t-t_0|^{1/2}\}$ denotes the parabolic metric on D .

Proof of Theorem 4.1. By (4.1) and (1.1) we have for $\phi \in C_0^{\infty}(D)$, $K \equiv \operatorname{spt}\phi$,

$$
\begin{aligned}
& -\int_K u\phi' dz + \lambda\int_K |\nabla u|^2 dz \leq \mu\int_K |\nabla(u-\phi)|^2 dz \\
(4.6) \qquad & + (1+Q)\int_K (b|u-\phi|^{\gamma} + b|u|^{\gamma} + g)dz \\
& + a\int_K (|\nabla u|^2 + c)|\phi| dz \equiv I + II + III .
\end{aligned}
$$

Taking k as described in the theorem the claim follows by successively inserting into (4.6) the testfunctions

$$\phi = \pm\, \eta^2 (u-k)^{\pm}\, \chi_{[t_0-\tau, t_1]}$$

(more precisely, time regularizations of them, as done in the preceding section).

For the verification of (4.4) t_1 is arbitrarily fixed in $(t_0-\tau, t_0)$ while $\eta = \eta(x)$ is a cut-off function for the pair (B_r, B_R) where $0 < r < R \leq \rho$. To prove (4.5) we fix for $0 < s < S \leq \tau$ and $0 < r < R \leq \rho$ $t_1 \in [t_0-s, t_0]$ and let $\eta = \eta(x,t)$ be a cut-off function for the pair $(Q_{r,s}, Q_{R,S})$.

Since the procedure is the same in all cases let us concentrate on the proof of the "+"-part of (4.4), leaving the other cases to the reader.

Then, with the conventions made, insert $\phi = \eta^2(u-k)^+\chi_{[t_0-\tau,t_1]}$ into (4.6). We estimate the r.h.s. of (4.6), using (4.2) and (4.3).

$$I \le \mu Q \int_{Q_{R,\tau}\setminus Q_{r,t}} |\nabla(u-k)^+|^2 dz + \frac{C}{(R-r)^2}\int_{Q_{R,\tau}} |(u-k)^+|^2 dz .$$

$$II \le C\left(\int_D (|u|^{\gamma\sigma} + |g|^{\sigma})dz\right)^{1/\sigma} \left|A^+_{k,R,\tau}\right|^{\frac{1+\kappa}{q}} ,$$

where $\kappa \equiv q(\frac{\sigma-1}{\sigma} - \frac{1}{q})$ $(q = \frac{2(n+2)}{n}$ as above),

$$\le C(\|g\|_{L^{\sigma}(D)}, \|u\|_{V_2^{1,0}(D)})\ |A^+_{k,R,\tau}|^{\frac{1+\kappa}{q}} ,$$

by [13], Chap. II, § 3.

$$III \le a\,\|\phi\|_{L^{\infty}(K)} \int_{A^+_{k,R,\tau}\cap\{t\le t_1\}} (\eta^2|\nabla(u-k)^+|^2 + c)\,dz$$

$$\le \frac{\lambda}{4}\int_{Q_{R,\tau}} \eta^2|\nabla(u-k)^+|^2 dz + \frac{\lambda}{4}\left(\int_D c^{\sigma} dz\right)^{1/\sigma} |A^+_{k,R,\tau}|^{\frac{1+\kappa}{q}} .$$

Calculating the first term on the l.h.s. of (4.6) as usual we obtain therefore, setting $\Lambda_\tau \equiv [t_0-\tau, t_0]$

$$\frac{1}{2}\max_{t\in\Lambda_\tau} \int_{B_r} |(u-k)^+(t)|^2 dx + \frac{3}{4}\lambda\int_{Q_{r,\tau}} |\nabla(u-k)^+|^2 dz \le$$

$$c_1\left(\frac{3}{4}\lambda\int_{Q_{R,\tau}\setminus Q_{r,\tau}} |\nabla(u-k)^+|^2 dz\right) + \frac{1}{2}\int_{B_r} |(u-k)^+(t_0-\tau)|^2 dx$$

$$+ c_2\left\{\frac{1}{(R-r)^2}\int_{Q_{R,\tau}} |(u-k)^+|^2 dz + \left|A^+_{k,R,\tau}\right|^{\frac{1+\kappa}{q}}\right\} ,$$

with $c_1 = 4\eta Q/(3\lambda)$ and c_2 depending on $\|u\|_{V_2^{1,0}(D)}$.

Adding on both sides c_1 times the second term of the l.h.s., followed by division through $1 + c_1$ ("hole-filling"), yields the inequality

$$\varphi(r) \le \theta\varphi(R) + C\left\{\frac{1}{(R-r)^2} b(\rho) + c(\rho)\right\} + d(R) - a(r) ,$$

where we abbreviated $\varphi(r) \equiv \frac{3}{4}\lambda\int_{Q_{r,\tau}} |\nabla(u-k)^+|^2 dz$, $\theta = \frac{c_1}{1+c_1}$

$b(\rho) = \int_{Q_{\rho,\tau}} |(u-k)^+|^2 dz$, $c(\rho) = |A^+_{k,\rho,\tau}|^{\frac{1+\kappa}{q}}$, $d(R) =$

$\frac{1}{2(1+c)_1} \int_{B_R} |u-k)^+(t_0-\tau)\,^2 dx$, and

$$a(r) = \frac{1}{2(1+c)_1} \max_{t\in\Lambda_\tau} \int_{B_r} |(u-k)^+(t)|^2 dx .$$

As in [7], Lemma 3.1, p. 161, iterating (4.7) with the sequence of radii

$$r_0 = r,\ r_{i+1} - r_i = (1-\varepsilon)\varepsilon^i(R-r),\ \varepsilon \in (\sqrt{\theta},1) ,$$

we arrive at

$$\varphi(r) \le \theta^k\varphi(r_k) + C\,\{\frac{b(\rho)}{(R-r)^2(1-\varepsilon)^2} + c(\rho)\} \sum_{i=0}^{k+1} (\theta\varepsilon^{-2})^i$$
$$+ \frac{1}{1-\theta}\,(d(R) - a(r)) .$$

As $k \to \infty$ we conclude (4.4) by choosing $R = \rho$ and $r = (1-\sigma_1)\rho$. The same line of reasoning yields the "-"-part of (4.4) as well as (4.5).

<u>QED</u>

5. Existence of minimal and ß-minimal solutions.

In this section we assume that the integral

(5.1) $E(u,\Omega) \equiv \int_\Omega F(x,u(x),\nabla u(x))\,dx$
is a bounded functional on $W^1_2(\Omega)$.

This is true for example if (1.1) holds with $\gamma = 2^* = \frac{2n}{n-2}$. Furthermore, let there be giben a closed, convex subset $\mathcal{C} \subset W^1_2(\Omega)$, such that

(5.2) E is bounded from below and weakly lower semicontinuous (w.l.s.c.) on $\mathcal{C}$.

Let us then seek for functions $u : D \to \mathbb{R}^N$, belonging to $L^2(0,T;C)$ and having a time derivative $u' \in L^2(0,T; W_2^{-1}(\Omega))$, such that

(5.3) $\quad \langle u'(t),\varphi\rangle \le E(u(t) - \varphi,\Omega) - E(u(t),\Omega)$

holds for a.e. $t \in [0,T]$ and every $\varphi \in \mathcal{B}_t \equiv \{\psi \in \overset{o}{W}{}_2^1(\Omega) \mid u(t) - \psi \in C\}$. The case $C = W_2^1(\Omega)$ leads to minimal solutions, while in general, setting

(5.4) $\quad \mathcal{B} \equiv \{\phi \in C_0^\infty(D) \mid \phi(.,t) \in \mathcal{B}_t\}$,

we obtain $\mathcal{B}$-restricted minimal solutions (simply called $\mathcal{B}$-minimal solutions in the following). As examples for C one may have in mind the intersection of $W_2^1(\Omega)$ with either of the balls $\{\|\nabla u\|_{2,\Omega} \le C\}, \{\|u\|_{2*,\Omega} \le C\}$, or $\{\|u\|_{\infty,\Omega} \le C\}$.

In addition we impose the initial-boundary condition

(5.5) $\quad u_{|\Omega\times\{0\} \cup \partial\Omega \times (0,T)} = u_0 = u_0(x)$

where we assume that $u_0 \in C$ depends only on the space variable, i.e. (5.3), (5.4) is viewed as determining a deformation $\{u(.,t)\}_{t\ge 0}$ of the initial state u_0 leaving its boundary value fixed.

We shall show in the following that existence of $\mathcal{B}$-minimal solutions (5.3), (5.5) for all $u_0 \in C$ is equivalent to the convexity of the functional E on C . In which way this latter property effects the structure of the integrand F seems to be a delicate problem. A partial answer in the cases $F = F(x,u)$ and $F = F(x,p)$ can be found in [5]. On the other hand, of course, the convexity of F in *both* u and p implies that of the functional E . The question therefore is whether such restrictive assumption on F can be substantially weakened. That this should indeed be possible is already indicated by the examples examined in 5.2 which show that at least if $C \neq W_2^1(\Omega)$ the convexity of the functional need not be related to a convexity assumption on the integrand.

5.1

To proceed with the details, define for given $u_o \in C$ the functional $e : \overset{\circ}{W}{}^1_2(\Omega) \to \mathbb{R} \cup \{\infty\}$ as

$$(5.6) \qquad e(v) \equiv e(v,u_o) \equiv \begin{cases} E(u_o + v, \Omega), & \text{if } u_o + v \in C \\ + \infty & \text{, else} \end{cases}$$

We then observe that (5.3), (5.5) can be equivalently translated into: Find $v \in \overset{\circ}{W}{}^{1,0}_2(D)$ such that

$$(5.7) \qquad \begin{aligned} &-v'(t) \in \partial e(v(t)) \quad \text{a.e. } t \in [0,T] \\ &v_{|t=0} = 0 , \end{aligned}$$

where $\partial e(w) = \{q \in W_2^{-1}(\Omega) \,|\, <q,\varphi> \le e(w+\varphi) - e(w)$ for every $\varphi \in \overset{\circ}{W}{}^1_2(\Omega)\}$ denotes the subdifferential of e at w.

Finally, let e^{**} be the bidual of e,

$$e^{**}(v) \equiv \sup_{q \in W_2^{-1}(\Omega)} \left\{ <q,v> - \sup_{w \in \overset{\circ}{W}{}^1_2(\Omega)} \Big(<q,w> - e(w) \Big) \right\} ,$$

which, on the other hand, is nothing else but the greatest <u>convex</u> l.s.c. lower bound of the functional e (cp. [5]). Note also that, by (5.2), e^{**} is bounded from below. With this we can state

<u>THEOREM 5.1. Let</u> C <u>be a closed convex subset of</u> $W_2^1(\Omega)$ <u>and</u> $\mathcal{B}$ <u>be defined by</u> (5.4). <u>Assume</u> (5.1), (5.2) <u>for the integral functional</u> $E(\cdot,\Omega)$.

<u>(a): For</u> $u_o \in C$, e <u>given by</u> (5.6) <u>let</u> $v \in W_2^1(D) \cap \overset{\circ}{W}{}^{1,0}_2(D)$ <u>be the solution to</u>

$$(5.8) \qquad \begin{aligned} &-v'(t) \in \partial e^{**}(v(t)) \quad \text{a.e. } t \in [0,T] \\ &v_{|t=0} = 0 . \end{aligned}$$

<u>Then a</u> $\mathcal{B}$<u>-minimal solution</u> (5.3), (5.5) <u>exists and is given by</u> $u(x,t) = u_o(x) + v(x,t)$ <u>if and only if</u>

$$(5.9) \qquad e(v(t)) = e^{**}(v(t))$$

holds for every $t \in [0,T]$.

(b): A $\mathcal{B}$*-minimal solution* (5.3), (5.5) *exists for every* $u_o \in C$ *if and only if the functional* $E(\cdot,\Omega)$ *is convex and l.s.c. on the set* C .

Proof: (a): Since e^{**} is convex l.s.c. and bounded from below, and since the initial value 0 lies in the essential domain of $e^{**}, D(e^{**}) = \{w | e^{**}(w) < \infty\}$, there exists always a unique solution v of (5.8), satisfying

$$
\text{(5.10)} \quad \begin{array}{l} v \in W_2^1(D) \cap \overset{\circ}{W}{}_2^{1,0}(D) \text{ and the map } t \to e^{**}(v(t)) \\ \text{is absolutely continuous; moreover for } 0 \le s \le t \le T : \\ \int_s^t \int_\Omega |v'|^2 dz + e^{**}(v(t)) \le e^{**}(v(s)) \end{array}
$$

(cp. e.g. Prop. 3.1 and Theorem 3.6 of Brézis' book [1]). Hence, if (5.9) holds, v satisfies also (5.7), i.e. $u = u_o + v$ is a $\mathcal{B}$-minimal solution.

Conversely, if $u \in L^2(0,T;C)$ satisfies (5.3), (5.5), then $v = u - u_o$ solves (5.7). But this implies, as a necessary condition for $\partial e(v(t))$ to be a non-empty set, that $e(v(t)) = e^{**}(v(t))$ for a.e. $t \in [0,T]$, thereby also that $\partial e(v(t)) = \partial e^{**}(v(t))$ (cp. [5], p. 21). Then $v \in \overset{\circ}{W}{}_2^{1,0}(D)$ solves (5.8). Moreover, $v(t) \in D(e^{**})$ for a.e. $t \in [0,T]$, which, in the same way as in Cor. 3.2, yields uniqueness of v . Thus we can redefine v on a measure zero set of $[0,T]$ to obtain (5.10) and from this the continuity of $e^{**}(v(t))$, thence (5.9) holds for every $t \in [0,T]$.

(b): First assume E convex l.s.c. on C . Then for $u_o \in C$ also e is convex l.s.c., i.e. $e = e^{**}$, so that existence of $u \in L^2(0,T;C) \cap W_2^1(D)$ satisfying (5.3), (5.5) follows from step 1 of (a).

To prove the converse, let $u_1 \in C$ and $w \in \overset{\circ}{W}{}_2^1(\Omega)$ be given such that $u_1 + w \in C$. Setting $u_o \equiv u_1 + w$ there exists by assumption $v \in \overset{\circ}{W}{}_2^{1,0}(D)$ (actually $\in W_2^1(D)$) , solution of (5.7). By (a) we therefore have in particular $e(0,u_o) = e^{**}(0,u_o)$. Since, by definition of e and u_o ,

$$e(w,u_1) = e(0,u_o) \ ,$$

and also

$$\begin{aligned} e^{**}(w,u_1) &= \sup_{q\in W_2^{-1}(\Omega)} \left\{ \langle q,w\rangle - \sup_{\varphi\in \overset{o}{W}{}_2^1(\Omega)} \Big(\langle q,\varphi\rangle - e(\varphi,u_1) \Big) \right\} \\ &= \sup_q \left\{ \langle q,w\rangle - \sup_\varphi \Big(\langle q,\varphi\rangle - e(\varphi + u_1 - u_o,u_o) \Big) \right\} \\ &= \sup_q \left\{ \langle q,w+u_1-u_o\rangle - \sup_\varphi \Big(\langle q,\varphi\rangle - e(\varphi,u_o) \Big) \right\} \\ &= e^{**}(0,u_o) \ , \end{aligned}$$

we therefore obtain $e(w,u_1) = e^{**}(w,u_1)$, i.e. for every $u_1 \in C$ $e(\cdot,u_1)$ is a convex l.s.c. functional on $\overset{o}{W}{}_2^1(\Omega)$. But then clearly, for $u_o, u_1 \in C$ and $\alpha \in [0,1]$,

$$\begin{aligned} &E(\alpha u_1 + (1-\alpha)u_o, \Omega) = e(\alpha(u_1 - u_o), u_o) \\ &= e(\alpha(u_1-u_o) + (1-\alpha)\cdot 0, u_o) \\ &\le \alpha \ e(u_1-u_o, u_o) + (1-\alpha) \ e(0,u_o) \\ &= \alpha \ E(u_1,\Omega) + (1-\alpha) \ E(u_o,\Omega) \ . \end{aligned}$$

QED

5.2

We want to apply part (b) of the last theorem to the following two examples, each having a nonconvex integrand $F(u,p)$ in the definition of the corresponding energy integral $E(u)$. At the same time they illustrate how to restrict the domain of definition of the energy functional in order to fulfill the convexity requirement in Theorem 5.1.

Example 1. Take $\Omega = \mathbb{R}^n$, $n \ge 3$, $N = 1$, and consider for $(u,p) \in \mathbb{R}\times\mathbb{R}^n$ the function

$$F(u,p) \equiv \frac{1}{2}\, p^2 - \frac{1}{2^*}\, |u|^{2^*} \ , \tag{5.11}$$

where $2^* = \dfrac{2n}{n-2}$ is the Sobolev exponent coming from the embedding $\overset{o}{W}{}_2^1(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n)$. Denote by S_n the quantity

$$(5.12)\qquad S_n = \inf_{u\in \overset{\circ}{W}{}_2^1(\mathbb{R}^n)} \frac{\|\nabla u\|_2^2}{\|u\|_{2*}^2}$$

$(\|.\|_q \equiv L^q(\mathbb{R}^n)\text{-norm.})$

We observe that, except for the set $\{(0,p)\,|\,p \in \mathbb{R}^n\}$ F is nowhere convex, and also that the functional

$$(5.13)\qquad E(u) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{2*}\int_{\mathbb{R}^n} |u|^{2*} dx ,$$

though bounded on $\overset{\circ}{W}{}_2^1(\mathbb{R}^n)$, is neither bounded from below nor w.l.s.c. .

However, set $C \equiv \{u \in \overset{\circ}{W}{}_2^1(\mathbb{R}^n)\,|\,\|u\|_{2*} \le C^* \equiv \frac{1}{2}\left(\frac{2_n}{2*-1}\right)^{\frac{1}{2*-2}}\}$
Then we have

<u>LEMMA 5.2.</u> <u>(a)</u> E <u>is convex and w.l.s.c. on</u> C .

<u>(b) With</u> $\alpha = 1 - \left(\frac{1}{2}\right)^{2*-2} \frac{2}{(2*-1)2*} > 0$,

$$E(w) \ge \frac{\alpha}{2}\|\nabla w\|_2^2 , \quad w \in C .$$

<u>Proof.</u> (a) For the convexity we check that the derivative $dE\colon \overset{\circ}{W}{}_2^1(\mathbb{R}^n) \to W_2^{-1}(\mathbb{R}^n)$,

$$dE(u) = -\Delta u - |u|^{2*-2}u ,$$

is a monotone operator when restricted to C . Indeed, since

$$\int_{\mathbb{R}^n}(|u|^{2*-2}u - |v|^{2*-2}v)(u-v)dx$$
$$= \int_{\mathbb{R}^n}\int_{v(x)}^{u(x)} \frac{d}{ds}|s|^{2*-2}\,ds(u-v)dx$$
$$= (2*-1)\int_{\mathbb{R}^n} |\theta u+(1-\theta)v|^{2*-2}(u-v)^2 dx , \quad (\theta=\theta(x) \in (0,1)) ,$$
$$\le (2*-1)(\|u\|_{2*} + \|v\|_{2*})^{2*-2}\|u-v\|_{2*}^2$$

we get with (5.12)

$$\langle dE(u)-dE(v),u-v\rangle \ge \|\nabla(u-v)\|_2^2 - (2*-1)\left(\|u\|_{2*} + \|v\|_{2*}\right)^{2*-2}$$

$$\cdot \; \|u-v\|_{2^*}^2 \geq \|\nabla(u-v)\|_2^2 \, \{1 - \frac{2^*-1}{S_n} (\|u\|_{2^*} + \|v\|_{2^*})^{2^*-2}\}$$

≥ 0 , for $u,v \in C$.

The weak lower semicontinuity on C follows in the standard way from the convexity and the Banach-Saks theorem. (b) is obtained by direct computation using (5.12).

QED

Through application of Theorem 5.1 we are therefore lead to the

COROLLARY 5.3. *To every* $u_o \in W_2^1(\mathbb{R}^n)$ *satisfying* $\|u_o\|_{2^*} \leq C^*$ *there is a unique* $u \in W_2^1(\mathbb{R}^n \times (0,T))$, *such that* $u_{|t=0} = u_o$, $\|u(t)\|_{2^*} \leq C^*$, *and*

$$\int_{\mathbb{R}^n} u'(t)\varphi dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t)|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u(t)|^{2^*} dx$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla(u(t)-\varphi)|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u(t)-\varphi|^{2^*} dx$$

hold for a.e. $t \in [0,T]$ *and every* $\varphi \in \overset{\circ}{W}{}_2^1(\mathbb{R}^n)$ *with* $\|u(t) - \varphi\|_{2^*} \leq C^*$.

Moreover, once we can show the strict inequality

(5.14) $\quad \|u(t)\|_{2^*} < C^* \quad$ a.e. $\quad t \in [0,T]$

then certainly for *every* $\varphi \in \overset{\circ}{W}{}_2^1(\mathbb{R}^n)$ we find $h_o > 0$ such that $\|u(t)-h\varphi\|_{2^*} \leq C^*$ is valid whenever $|h| < h_o$. Using the argument at the end of section 1 we therefore conclude from (5.14) that u weakly solves

$$u' - \Delta u - |u|^{2^*-2}u = 0 \quad \text{on } \mathbb{R}^n \times (0,T)$$

$$u_{|t=0} = u_o \; .$$

(5.14) holds e.g. if $E(u_o) < \frac{\alpha S_n}{2} C^{*2}$, since by Lemma 5.2 (b) and the energy inequality in (5.10),

$$\frac{\alpha}{2} S_n \|u(t)\|_{2^*}^2 \leq \frac{\alpha}{2}\|\nabla u(t)\|_{2^*}^2 \leq E(u(t)) \leq E(u_o) \; .$$

Example 2. (I wish to thank Prof. S. Hildebrandt for pointing my attention to this example.) Here we take $n = 2$, $N = 3$ and let Ω be the plane unit disk. For some real constant H we then define $F : \mathbb{R}^3 \times \mathbb{R}^{2\cdot 3} \to \mathbb{R}$ as

$$F(u,p) \equiv \frac{1}{2}\, p\cdot p + \frac{2}{3}\, H\, u\cdot(p_1 \wedge p_2) ,$$

$p_1 \wedge p_2$ denoting the usual product of the 3-vectors (p_i^1, p_i^2, p_i^3), $i = 1,2$. The corresponding energy

$$E_H(u) \equiv \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{2}{3}\, H \int_\Omega u\cdot(u_{x_1} \wedge u_{x_2})\,dx \tag{5.15}$$

plays the central role in the variational formulation of Plateau's problem for surfaces of constant mean curvature H .

For given $u_o \in W_2^1(\Omega) \cap L^\infty(\Omega)$, $E_H(u)$ is well-defined on the space $u_o + \overset{o}{W}{}_2^1(\Omega) \cap L^\infty(\Omega)$ but again is neither bounded from below nor w.l.s.c. there. However, making use of an inequality due to Heinz (see [18], Lemma 3.3), namely

$$\frac{1}{2}(E_H(u)+E_H(v)) \geq E_H\left(\frac{u+v}{2}\right) + \frac{1}{4}(1-2|H|R)\int_\Omega |\nabla(u-v)|^2 dx , \tag{5.16}$$

valid for all $u,v \in V_R \equiv \{w \in W_2^1(\Omega) \mid \|w\|_\infty \leq R\}$ such that $u - v \in \overset{o}{W}{}_2^1(\Omega)$, one can immediately deduce

LEMMA 5.4. Choose R such that $|H|R \leq \frac{1}{2}$ and let $u_o \in V_R$. Then E_H defined by (5.15) is convex and w.l.s.c. on the set

$$C_R(u_o) \equiv \{u \in V_R \mid u - u_o \in \overset{o}{W}{}_2^1(\Omega)\} .$$

Moreover, for $u \in C_R$,

$$E_H(u) \geq \frac{1}{6}\int_\Omega |\nabla u|^2 dx .$$

Proof. The convexity follows from (5.16) since the last summand is nonnegative, if $u,v \in C_R$. The lower semicontinuity proof can be found in [18], Lemma 3.4. Finally, the asserted lower bound is straightforwardly calculated, using the inequality $2|u_{x_1} \wedge u_{x_2}| \leq |\nabla u|^2$.

QED

Thus, applying again Theorem 5.1, wo obtain

COROLLARY 5.5. *If* $u_o \in W_2^1(\Omega)$ *satisfies* $|H|\|u_o\|_\infty \le \frac{1}{2}$ *then there is a unique* $u \in W_2^1(D)$ *such that*

(5.17) $\qquad u_{|\Omega \times \{0\} \cup \partial\Omega \times (0,T)} = u_o \;, \; |H|\|u(t)\|_\infty \le \frac{1}{2}$

and

(5.18) $\qquad \int_\Omega u'(t)\varphi \, dx + E_H(u(t)) \le E_H(u(t) - \varphi)$

hold for a.e. $t \in (0,T)$ *and for every* $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$ *with* $|H|\|u(t) - \varphi\|_\infty \le \frac{1}{2}$.

Assume in addition that $u_o \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$. By well-known local existence theorems there exists a $t_o > 0$ and a function $v : \bar{\Omega} \times [0,t_o] \to \mathbb{R}^3$, smooth in the interior and continuously attaining the initial-boundary value u_o , which satisfies the mean curvature flow system

(5.19) $\qquad v' - \Delta v + 2H \, v_{x_1} \wedge v_{x_2} = 0 \quad \text{on} \quad \Omega \times (0,t_o]$.

Since $|H|\|u_o\|_\infty \le \frac{1}{2}$, we may assume that $|H|\|v(t)\|_\infty \le 1$ for $t \in [0,t_o]$ (by taking t_o small enough). But then

$$\Delta\left(\frac{v^2}{2}\right) - \left(\frac{v^2}{2}\right)' = |\nabla v|^2 + 2H \, v \cdot (v_{x_1} \wedge v_{x_2})$$
$$\ge |\nabla v|^2(1 - |H||v|) \ge 0 \;,$$

so the maximum principle implies that actually $|H|\|v(t)\|_\infty \le \frac{1}{2}$, i.e. the solution to (5.19) stays in the convexity domain of the functional E_H . Therefore also v satisfies (5.18), and by uniqueness we have $v(t) \equiv u(t)$ for $t \in [0,t_o]$. Continuing this argument sucessively to the intervals $[t_o,2t_o]$, $[2t_o,3t_o]$ etc. we have thus proved:

THEOREM 5.6. *If, in addition to the assumption of Corollary 5.5,* $u_o \in C^\alpha(\bar{\Omega})$ *for some* $\alpha \in (0,1)$, *then besides* (5.17) *and* (5.18) u *also solves*

$$u' - \Delta u + 2H \, u_{x_1} \wedge u_{x_2} = 0 \quad \text{on} \quad D \;,$$

i.e. u is a minimal solution for the mean curvature flow.

6. Partial regularity of minimal solutions.

For simplicity of exposition, we shall restrict the partial regularity investigation to the *quadratic case*

$$(6.1) \qquad F(x,u,p) \equiv \frac{1}{2} A[u] \cdot p \cdot p \equiv \frac{1}{2} \sum_{i,j=1}^{n} A_{ij}^{\alpha\beta}(x,u) p_\alpha^i p_\beta^j$$

for which (3.1) is assumed to hold with $K_0 = 0$, and where the coefficients

$$(6.2) \qquad A_{ij}^{\alpha\beta} \text{ are uniformly continuous, and } A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} .$$

We recall the definition of q-dimensional Hausdorff measure $\mathcal{H}_\delta^p$ of a subset $Q' \subset D$ w.r.t. the parabolic metric $\delta(z,z_0) \equiv \max\{|x-x_0|, |t-t_0|^{1/2}\}$,

$$\mathcal{H}_\delta^q(Q') \equiv \lim_{\varepsilon\to 0+} \inf\{ \sum_i (d_\delta(U_i))^q | \bigcup_i U_i \supset Q', d_\delta(U_i) < \varepsilon\} ,$$

where the U_i's are open sets and $d_\delta(U_i)$ denote their diameters w.r.t. the metric δ. Then Hausdorff dimension of Q' is the infimum of the numbers q such that $\mathcal{H}_\delta^p(Q') \neq \infty$.

Finally, let us write $C^\alpha(Q';\delta)$ for the space of (vector) functions being Hölder continuous on Q' with exponent $\alpha \in (0,1)$ in the metric δ. Then we can prove for the case of quadratic F.

THEOREM 6.1. (Partial Hölder regularity). *For F of the form (6.1) with coefficients satisfying (6.2) let $u \in W_2^{1,0}(Q)$ be a minimal solution. Then $u \in C^\alpha(D\setminus S;\delta)$ for every $\alpha \in (0,1)$, where S is a closed set of Hausdorff dimension strictly less than n.*

THEOREM 6.2. (Partial Hölder regularity for ∇u). *In addition to the assumptions of Theorem 6.1 let the coefficients $A_{ij}^{\alpha\beta}$ be uniformly Hölder continuous on $\bar{\Omega} \times \mathbb{R}^N$ with exponent $\beta \in (0,1)$.*

Then, with some exponent $\gamma > 0$ possibly less than β, u possesses Hölder continuous x-derivatives outside the singular set S .

Let us compare these results for the quadratic case with their analogues for weak solutions to the corresponding variational flow system

$$(6.3) \quad (u^i)' - \frac{\partial}{\partial x^\alpha}\left(A^{\alpha\beta}_{ij}(x,u)\frac{\partial u^j}{\partial x^\beta}\right) + \frac{1}{2}A^{\alpha\beta}_{k\ell,u^i}(x,u)\frac{\partial u^k}{\partial x^\alpha}\frac{\partial u^\ell}{\partial x^\beta} = 0 .$$

First then, we have to assume that the coefficients $A^{\alpha\beta}_{ij}$ are C^1 in the variable u , together with a bound

$$\sup_{\substack{x \in \Omega \\ u \in \mathbb{R}^N}} \left| \frac{1}{2} A_u(x,u) \right| \leq a .$$

Then, in order to get the same conclusions as in the theorems above, one has to impose the a priori smallness condition

$$2\, a\, \|u\|_{L^\infty(D)} < \lambda ,$$

relating the growth factor a , the ellipticity constant λ from (3.1) and the assumed L^∞-bound of u . This follows from the paper of M . Giaquinta & M. Struwe [11]. Concerning their analogue of our Theorem 6.2 they even get $\gamma = \beta$, but apart from this faxt the situation for minimal solutions is definitely better, in that here we don't even have to assume that u be bounded, which fits well with the results for elliptic minimizers. On the other hand, as we have seen in section 5, the conditions for a minimal solution of (6.3) to exist are more restrictive than in the elliptic case.

We also point out that both Theorems carry over to B-restricted minimal solutions, once the results of section 3 are valid for them, cp. with the remark in 3.3. This holds in particular for the second example in 5.2. There, however, we can obtain full smoothness on account of the special (diagonal) structure of the corresponding variational flow system.

6.1. preliminaries.

As in the case of general quasilinear systems, the main idea of the partial regularity proof for a minimal solution u is to compare it locally in a neighborhood of a point $z_o \in D$ with the solution w of the linear system with frozen coefficients $A^o \equiv \left(A_{ij}^{\alpha\beta}(x_o,u_r)\right)$, where again $u_r \equiv ⨍_{Q_r} u\, dz$ and $Q_r(z_o) \subset D$,

$$\text{(6.4)} \qquad \begin{aligned} &-\int_{Q_r} w\phi' dz + \int_{Q_r} A^o \cdot \nabla w \cdot \nabla\phi dz = 0 \ , \ \forall\ \phi \in \overset{o}{W}{}_2^1(Q_r) \ , \\ &w\big|_{\Sigma_r} = u\big|_{\Sigma_r} \ , \end{aligned}$$

$\Sigma_r = \Sigma_r(z_o)$ being the parabolic boundary of Q_r. Let us immediately check that (6.4) has a solution $w \in W_2^{1,0}(Q_r) \cap C([t_o - r^2, t_o]; L^2(B_r))$. Indeed, first solve for v the proble

$$\text{(6.5)} \qquad \begin{aligned} &-\int_{Q_r} v\phi' dz + \int_{Q_r} A^o\cdot\nabla v\cdot\nabla\phi dz = -\int_{Q_r} u\phi' dz + \int_{Q_r} A^o\cdot\nabla u\cdot\nabla\phi dz \ , \\ &\forall\ \phi \in C_0^\infty(Q_r) \ , \text{ and} \\ &v\big|_{\Sigma_r} = 0 \ . \end{aligned}$$

From the proof of Proposition 3.1 we have that $\langle u',\phi\rangle \equiv \int_{Q_r} u\phi' dz$, $\phi \in C_0^\infty(Q_r)$, extends to $u' \in W_2^{-1,0}(Q_r) = L^2(\Lambda_r; W_2^{-1}(B_r))$. So r.h.s. of (6.5) lies in $W_2^{-1,0}(Q_r)$, which by linear theory (e.g. [16], Theorem 41.1) yields unique existence of $v \in W_2^{1,0}(Q_r) \cap C([t_o-r^2,t_o]; L^2(B_r))$ satisfying (6.5). Setting $w \equiv u - v$ we get solvability of (6.4) in the same class. Note that, through the intervention of Proposition 3.1 in the above argument, we have crucially used that u, the initial-boundary value for w in (6.4), is a Q-minimum. Replacing u by an arbitrary function $g \in W_2^{1,0}(Q_r) \cap C([t_o-r^2,t_o]; L^2(B_r))$ it will in generell *not* even be true that $w \in W_2^{1,0}(Q_r)$.

As the essential tools for the proofs of Theorems 6.1 and 6.2, we recall the Campanato estimates for w, [2],

$$(6.6)\qquad \int_{Q_\rho} |\nabla w|^2\, dz \le C\left(\frac{\rho}{r}\right)^{n+2} \int_{Q_r} |\nabla w|^2 dz \ ,$$

$$(6.7)\qquad \int_{Q_\rho} |\nabla w-(\nabla w)_\rho|^2 dz \le C\left(\frac{\rho}{r}\right)^{n+4} \int_{Q_r} |\nabla w-(\nabla w)_r|^2 dz \ ,$$

valid for every $\rho \in (0,r)$ $\quad ((\nabla w)_\rho \equiv ⨍_{Q_\rho} \nabla w\, dz)$.

Next we have

LEMMA 6.3. *There is a constant* C *such that for any* $Q_r \subset D$ and w satisfying (6.4)

$$(6.8)\qquad \int_{Q_r} |\nabla w|^2 dz \le C\int_{Q_r} |\nabla u|^2 dz \ .$$

Remark. Contrary to the situation in the elliptic case, (6.8) is not a simple estimate resulting from linear theory. (By the remark make above it is even false in general.) Again we have to use that u is a Q-minimum.

Proof. The difference $v \equiv u - w$ satisfies (6.5), of which the r.h.s. is estimated by use of (3.1) (for the quadratic case):

$$\int_{Q_r} A^0 \cdot \nabla u \cdot \nabla\phi dz - \int_{Q_r} u\phi' dz \le \int_{Q_r} A^0 \cdot \nabla u \cdot \nabla\phi dz$$
$$+ \frac{1}{2}\int_{Q_r} A[u-\phi]\cdot\nabla(u-\phi)\cdot\nabla(u-\phi)\, dz - \frac{1}{2}\int_{Q_r} A[u]\cdot\nabla u\cdot\nabla u dz \ ,$$

thus, rearranging,

$$-\int_{Q_r} v\phi' dz + \int_{Q_r} A^0 \cdot \nabla v \cdot \nabla\phi dz \le$$
$$(6.9)\qquad \int_{Q_r} A^0 \cdot \nabla u \cdot \nabla\phi dz - \int_{Q_r} A[u-\phi] \cdot \nabla u \cdot \nabla\phi dz$$
$$+ \frac{1}{2}\int_{Q_r} A[u-\phi]\cdot\nabla\phi\cdot\nabla\phi + \frac{1}{2}\int_{Q_r} (A[u-\phi]-A[u])\cdot\nabla u\cdot\nabla u dz \ .$$

Insert in particular $\phi \equiv (v_\varepsilon \chi^h_{t_0+\sigma-r^2,t-\sigma})_\varepsilon \cdot \eta$, where $\sigma > 0$ is chosen arbitrarily small (so that the time component of the spt ϕ lies in Λ_r) and the number $\eta > 0$ will be fixed in the course of the estimate. Then divide (6.9) by η .

As ε, h, $\sigma \to 0$, we infer that on account of (3.1)

$$\tfrac{1}{2}\int_{B_r} v^2(t_0)\,dx + \lambda\int_{Q_r} |\nabla v|^2 dz \le 2\,K \int_{Q_r} |\nabla u|\,|\nabla v| dz$$

$$+ \eta \cdot \tfrac{K}{2} \int_{Q_r} |\nabla v|^2 dz + K/\eta \int_{Q_r} |\nabla u|^2 dz \ ,$$

whence, by Young's inequality and choosing η small enough,

$$\int_{Q_r} |\nabla v|^2 dz \le C \int_{Q_r} |\nabla u|^2 dz \ ,$$

which clearly implies (6.8).

<u>QED.</u>

We conclude the preliminaries with

<u>LEMMA 6.4.</u> <u>There exist</u> $q > 2$ <u>and constants</u> C <u>and</u> k <u>such that for every</u> $Q_r = Q_r(z_0)$ <u>with</u> $Q_{kr} \subset D$ <u>for the solution</u> w <u>of</u> (6.4) <u>we have</u>

$$\text{(6.10)} \qquad \int_{Q_r} |\nabla w|^q dz \le C \int_{Q_{kr}} |\nabla u|^q dz \ .$$

<u>Proof.</u> In the following we write $Q'_\rho \equiv Q_\rho(z')$, when in general $z' \neq z_0$.

1) Since also w is a minimal solution (satisfying (1.2) with F replaced by $\frac{1}{2}A^0 \cdot p \cdot p$) we have according to (3.10): for every Q'_ρ with $Q'_{4\rho} \subset Q_r$

$$⨍_{Q'} |\nabla w|^2 dz \le C_\theta \left(⨍_{Q'_{4\rho}} |\nabla w|^{2^+} dz\right)^{2/2^+} + \theta ⨍_{Q'_{4\rho}} |\nabla w|^2 dz \ ,$$

where $\theta > 0$ can be made as small as we please.
This implies for $v \equiv u - w$,

$$\text{(6.11)} \qquad \begin{aligned} ⨍_{Q'_\rho} |\nabla v|^2 dz &\le C_\theta \left(⨍_{Q'_{4\rho}} |\nabla v|^{2^+}\right)^{2/2^+} + \theta ⨍_{Q'_{4\rho}} |\nabla v|^2 dz \\ &+ C ⨍_{Q'_{4\rho}} |\nabla u|^2 dz \ . \end{aligned}$$

Our aim is to achieve a similar inequality without the restriction $Q'_{4\rho} \subset Q_r$, in order to apply the Giaquinta-Modica lemma as done for the proof of Theorem 3.5.

2) First take $z' = (x',t') \in \Sigma_r(z_0)$ and let τ be a cut-off function for $(Q'_\rho, Q'_{2\rho})$. Similarly to the proof of

Lemma 6.3, insert into (6.9)

$$\phi \equiv (\tau^2 v_\varepsilon \chi^h_{t^*+\sigma, t'-\sigma})_\varepsilon \cdot \eta$$

(η, $\sigma > 0$ small, $t^* \equiv \max\{t_0 - r^2, t' - (2\rho)^2\}$) , which vanishes on $\Sigma_r(z_0)$ since, in particular, $v|_{\Sigma_r} = 0$. Again dividing the resulting inequality (6.9) by η, this yields after passage to zero with ε, h and σ

$$\frac{1}{2}\int_{B_r} \tau^2(t')v^2(t')dx + \int_{Q_r} \tau^2 A^0 \cdot \nabla v \cdot \nabla v \, dz$$

$$\leq \int_{Q_r} v^2 \tau\tau' dz + \int_{Q_r} A^0 \cdot \nabla u \cdot \nabla(\tau^2 v) dz$$

$$- \int_{Q_r} A[u - \eta\tau^2 v] \cdot \nabla u \cdot \nabla(\tau^2 v) dz$$

$$+ \eta \cdot \frac{1}{2} \int_{Q_r} A[u - \eta\tau^2 v] \cdot \nabla(\tau^2 v) \cdot \nabla(\tau^2 v) dz$$

$$+ \frac{1}{2\eta} \int_{Q_r} (A[u - \eta\tau^2 v] - A[u]) \cdot \nabla u \cdot \nabla u \, dz$$

$$- \int_{Q_r} A^0 \cdot \nabla v \cdot \nabla\tau^2 \cdot v \, dz ,$$

which, by obvious manipulations using (3.1) and Young's inequality, implies

$$\lambda \int_{Q'_{2\rho}} \tau^2 |\nabla u|^2 dz \leq \frac{C(1+\eta)}{\rho^2} \int_{Q'_{2\rho}} v^2 \, dz + C/\eta \int_{Q'_{2\rho}} |\nabla u|^2 \, dz$$
$$+ \eta \cdot \frac{K}{2} \int_{Q'_{2\rho}} \tau^2 |\nabla v|^2 \, dz ,$$

where we have extended $v \equiv 0$ outside $Q_r(z_0)$.

Put the last term to the left by choosing η small enough. Then dividing the inequality by $|Q'_{2\rho}|$ and again using that $v|_{\Sigma_r \cap Q'_{2\rho}} = 0$ we get by the Sobolev-Poincaré inequality (valid for v)

$$\text{(6.12)} \qquad ⨍_{Q_\rho} |\nabla v|^2 dz \leq C \left(⨍_{Q'_{2\rho}} |\nabla v|^{2^+} dz \right)^{2/2^+} + C \, ⨍_{Q'_{2\rho}} |\nabla u|^2 dz$$

where, we recall, z' was assumed to lie on Σ_r . (Note that (6.12) is trivial if $t' = t_0 - r^2$, since then $v \equiv 0$ in Q'_ρ .)

3) We are left with the cases that either $Q'_\rho \subset Q_r$ but $Q'_{4\rho} \not\subset Q_r$, or that $Q'_\rho \cap Q_r \neq \emptyset$, but $z' \notin Q_r$. These, however can be handled by simple covering arguments using (6.12).

In fact, as to the first case, let $\hat{z} \in \Sigma_r(z_0)$ minimize the distance of $\Sigma_r(z_0)$ to the point z'. Then clearly $Q' = Q_\rho(z') \subset Q_{5\rho}(\hat{z}) \subset Q_{14\rho}(z')$, hence from (6.12)

$$\fint_{Q'_\rho} |\nabla v|^2 dz \le C \fint_{Q_{5\rho}(\hat{z})} |\nabla v|^2 dz$$

$$\le C \left(\fint_{Q_{10\rho}(\hat{z})} |\nabla v|^{2^+} dz\right)^{2/2^+} + C \fint_{Q_{10\rho}(\hat{z})} |\nabla u|^2 dz$$

$$\le C \left(\fint_{Q'_{14\rho}} |\nabla v|^{2^+} dz\right)^{2/2^+} + C \fint_{Q'_{14\rho}} |\nabla u|^2 dz .$$

The other case is treated in the same way.

4) In conclusion, choosing some ball $U \subset D$ such that $Q_{14r} \subset U$, we get from (6.11), (6.12), and last inequality: for every Q'_ρ with $Q'_{14\rho} \subset U$

$$\fint_{Q_\rho} |\nabla v|^2 dz \le C\left(\fint_{Q'_{k\rho}} |\nabla v|^{2^+} dz\right)^{2/2^+} + \theta \fint_{Q'_{k\rho}} |\nabla v|^2 dz$$

$$+ \fint_{Q'_{k\rho}} |\nabla u|^2 dz, \quad \theta \in (0,1) ,$$

where $k = 14$, which perhaps is not optimal, but suffices for our purpose).

Applying the Giaquinta-Modica lemma as in the proof of Theorem 3.5, and specializing to $Q'_\rho = Q_r(z_0)$, this yields for some $q \in (2,p]$ (p from Theorem 3.5),

$$\left(\fint_{Q_r} |\nabla v|^q dz\right)^{1/q} \le \left(\fint_{Q_r} |\nabla v|^2 dz\right)^{1/2} + C \left(\fint_{Q_{kr}} |\nabla u|^q dz\right)^{1/q}$$

$$\le C \left(\fint_{Q_r} |\nabla u|^2 dz\right)^{1/2} + C \left(\fint_{Q_{kr}} |\nabla u|^q dz\right)^{1/q} , \text{ by Lemma 6.3}$$

$$\le C \left(\fint_{Q_{kr}} |\nabla u|^q dz\right)^{1/q} ,$$

and thereby (6.10).

QED.

6.2. Proof of Theorem 6.1.

The principle of proof is the same as in [11], except that we have to prepare the steps differently.

Basically, one has to produce an open set $Q' \subset D$ such that to every $z_0 \in Q'$ belongs a neighbourhood $U(z_0)$ in which for some $R > 0$ and all $\rho \in (0,R)$ there holds

$$\rho^{-n} \int_{Q_\rho(\hat{z})} |\nabla u|^2 dz \le C\, \rho^{2\alpha} \ , \quad \hat{z} \in U(z_0) \ , \tag{6.13}$$

where $\alpha \in (0,1)$ is arbitrary and C as well as R may depend on α and z_0.

Then by Proposition 3.6 we get

$$\int_{Q_{\frac{\rho}{2}}(\hat{z})} |u-u_{\rho/2}|^2 dz \le C\rho^{2\alpha}$$

which, for some possibly smaller neighbourhood $\tilde{U}(z_0)$, implies $u \in C^\alpha(\tilde{U}(z_0);\delta)$, and hence $u \in C^\alpha(Q';\delta)$, on account of Campanato's integral characterization of Hölder continuous functions (in the refined version for general metrics of Da Prato [3]).

Secondly, as will be seen in the course of proving (6.13), for the closed set $S \equiv D\backslash Q'$ we have the inclusion

$$S \subset \{z_0 \in D \,|\, \overline{\lim_{R\to 0+}}\ R^{-(n+2-p)} \int_{Q_R(z_0)} |\nabla u|^p dz > 0\} \tag{6.14}$$

with $p > 2$ as in Theorem 3.5.

As is known (e.g. from [7], Theorem 2.2, p. 101), the (n+2-p)-dimensional Hausdorff measure of the latter set vanishes, hence also

$$\mathcal{H}_\delta^{n+2-p}(S) = 0 \ . \tag{6.15}$$

To verify (6.13) and (6.14) we proceed in three steps.

1) For arbitrary $z_0 \in D$ choose $r > 0$ such that $Q_{4kr} \equiv Q_{4kr}(z_0) \subset D$, k as in Lemma 6.4. Let w be the solution of the corresponding linear system (6.4) (defined on Q_r). Writing $u = w+(u-w)$ we get from (6.6) and (6.8)

for every $\rho \in (0,r)$

$$(6.16) \qquad \int_{Q_\rho} |\nabla u|^2 dz \le C\left(\frac{\rho}{r}\right)^{n+2} \int_{Q_r} |\nabla u|^2 dz + C \int_{Q_r} |\nabla(u-w)|^2 dz$$

2) To estimate the second summand observe that, setting for brevity $\phi_h \equiv ((u-w)_\varepsilon \chi^h_{t_0+\sigma-r^2, t_0-\sigma})_\varepsilon \quad (\sigma, h, \varepsilon > 0)$,

$$\lambda \int_{Q_r} |\nabla(u-w)|^2 dz \le \int_{Q_r} A^0 \cdot \nabla(u-w) \cdot \nabla(u-w) dz - 2 \int_{Q_r} w\phi_h' dz$$

$$- 2 \int_{Q_r} A^0 \cdot \nabla w \cdot \nabla((u-w) - \phi_h) dz = (*) ,$$

where intermediately we used (6.4). Noting that the last term is $o(1)$ as $\varepsilon, h, \sigma \to 0$, and smuggling in some terms, we get further,

$$(*) = -2 \int_{Q_r} w\phi_h' dz + o(1) + \int_{Q_r} (A^0 - A[u]) \cdot \nabla(u+w) \cdot \nabla(u-w) dz$$

$$+ \int_{Q_r} (A[u] \cdot \nabla u \cdot \nabla u - A[u-\phi_h] \cdot \nabla(u-\phi_h) \cdot \nabla(u-\phi_h)) dz$$

$$- \int_{Q_r} A[u] \cdot (\nabla u \cdot \nabla w - \nabla w \cdot \nabla u) dz$$

$$+ \int_{Q_r} (A[u-\phi_h] \cdot \nabla(u-\phi_h) \cdot \nabla(u-\phi_h) - A[u] \cdot \nabla w \cdot \nabla w) dz .$$

Thanks to u being a minimal solution, the forth term is less than $2\int_{Q_r} u\phi_h' dz$ which, added to the first term, gives a negative contribution (in the limit ε, h, $\sigma \to 0$). Since by symmetry of A, (6.2), the fifth term vanishes also, passing to zero with ε, h, σ we are left with

$$\lambda \int_{Q_r} |\nabla(u-w)|^2 dz \le \int_{Q_r} |A^0 - A[u]| |\nabla u|^2 dz$$

$$+ \int_{Q_r} |A^0 - A[u]| |\nabla w|^2 dz + \int_{Q_r} |A[w] - A[u]| |\nabla w|^2 dz .$$

The further estimation goes by routine arguments using Theorem 3.5, Proposition 3.6, and Lemma 6.3 and 6.4. Namely, since by (6.2) A is uniformly continuous, for every (x,v), $(\tilde{x},\tilde{v}) \in \overline{\Omega} \times \mathbb{R}^N$ we have

$$|A(x,v) - A(\tilde{x},\tilde{v})| \le \omega(|x-\tilde{x}|^2, |v-\tilde{v}|^2) ,$$

where $\omega(s,t) \ge 0$ is a bounded continuous function, $\omega(0,0) = 0$, which we may assume concave in t and increasing in both variables. Hence from the last inequality,

$$\int_{Q_r} |\nabla(u-w)|^2 dz \le C \int_{Q_r} \omega(r^2, |u-u_r|^2)|\nabla u|^2 dz$$

$$+ C \int_{Q_r} \{\omega(r^2, |u-u_r|^2) + \omega(0, |u-w|^2\}|\nabla u|^2 dz$$

$$\equiv I + II .$$

Now, with $q > 2$ from Lemma 6.4,

$$I \le C\, r^{n+2}\left(⨍_{Q_r} \omega(r^2, |u-u_r|^2)^{q/(q-2)} dz\right)^{(q-2)/q}\left(⨍_{Q_r} |\nabla u|^q dz\right)^{2/q}$$

which by the boundeness of ω , Jensen's inequality, and (3.11), is

$$\le C\, \omega(r^2, ⨍_{Q_r} |u-u_r|^2 dz)^{(q-2)/q} \int_{Q_{4r}} |\nabla u|^2 dz$$

$$\le C\, \omega(r^2, C\, r^{-n} \int_{Q_r} |\nabla u|^2 dz)^{(q-2)/q} \int_{Q_{4r}} |\nabla u|^2 dz .$$

by Proposition 3.6. By the same reasoning,

$$II \le C\, r^{n+2}\left[\omega(r^2, ⨍_{Q_r} |u-u_r|^2\, dz)^{(q-2)/q} + \omega(0, ⨍_{Q_r} |u-w|^2 dz^{(q-2)/q}\right]$$

$$\cdot \left(⨍_{Q_r} |\nabla w|^q dz\right)^{2/q}$$

$$\le C\, r^{n+2}\Big[\omega(r^2, C\, r^{-n} \int_{Q_{2r}} |\nabla u|^2 dz)^{(q-2)/q}$$

$$+ \omega(0, C\, r^{-n} \int_{Q_r} |\nabla(u-w)|^2 dz)^{(q-2)/q}\Big] \cdot \left(⨍_{Q_r} |\nabla u|^q dz\right)^{2/q}$$

(by Proposition 3.6, the Poincaré inequality for $u - w$, and Lemma 6.4),

$$\le C\, \omega(r^2, C\, r^{-n} \int_{Q_{2r}} |\nabla u|^2 dz)^{(q-2)/q} \int_{Q_{4kr}} |\nabla u|^2 dz$$

(by Lemma 6.3 and Theorem 3.5).

3) Insert the last two estimates into (6.16). Then, setting $R \equiv 4kr$

$$\int_{Q_\rho(z_0)} |\nabla u|^2 dz \le C\left[\left(\frac{\rho}{R}\right)^{n+2} + \psi(z_0,R)\right] \int_{Q_R(z_0)} |\nabla u|^2 dz .$$

where $\rho \in (0,R/(4k))$ and

$$\psi(z_0,R) \equiv \omega(R^2, C\, R^{-n} \int_{Q_R(z_0)} |\nabla u|^2 dz)^{(q-2)/q} .$$

Since trivially the last inequality also holds for $\rho \in [R/(4k),R]$ we have, setting $\varphi(z_0,\rho) \equiv \rho^{-n}\int_{Q_\rho(z_0)} |\nabla u|^2 dz$:

For every $R > 0$ with $Q_R(z_0) \subset D$ and all $\tau \in (0,1)$,

$$(6.17) \qquad \varphi(z_0,\tau R) \le C_1\tau^2(1+\tau^{-(n+2)}\psi(z_0,R))\,\varphi(z_0,R) .$$

Note that $\psi(z_0,R)$ is small when $\varphi(z_0,R)$ is. Now the conclusion of (6.13) from (6.17) is a standard procedure which we repeat for the reader's convenience (compare [7]). Given $\alpha \in (0,1)$, take $\tau_0 \in (0,1)$ such that $2\,C_1\tau_0^{2-2\alpha}=1$. Then choose R and some small $\varepsilon_0 > 0$ in such a way that

$$(6.18) \qquad \varphi(z_0,R) < \varepsilon_0$$

and

$$\psi(z_0,R)(= \omega(R^2,\ C\,\varphi(z_0,R))^{(q-2)/q}) < \tau_0^{n+2} ,$$

Then from (6.17),

$$\varphi(z_0,\tau_0 R) \le \tau_0^{2\alpha}\varphi(z_0,R) < \varepsilon_0 ,$$

hence also $\psi(z_0,\tau_0 R) < \tau_0^{n+2}$.

Therefore we obtain by induction

$$\varphi(z_0,\tau_0^k R) \le \tau_0^{2k\alpha}\varphi(z_0,R) , \quad k \in \mathbb{N} ,$$

which easily implies for arbitrary $\rho \in (0,R)$,

$$\varphi(z_0,\rho) \le C\,\rho^{2\alpha} ,$$

where $C = \varepsilon_0\tau_0^{-(n+2\alpha)}R^{-2\alpha}$.

By continuity of $\varphi(z_0,R)$ w.r.t. $z_0 \in D$, if (6.18) holds then also $\varphi(\hat{z},R) < \varepsilon_0$ for all $\hat{z}$ in some neighbourhood $U(z_0)$ of z_0, and repeating the arguments we conclude

$$\varphi(\hat{z},\rho) \le C\,\rho^{2\alpha} \quad \text{for all } \hat{z} \in U(z_0) ,$$

thereby proving (6.13).

Hence, in view of the remarks made at the beginning of the proof, the set Q' where u is Hölder continuous is open and clearly contains the set

$$\{z_0 \in Q \mid \overline{\lim_{R\to 0+}}\, R^{-n} \int_{Q_R(z_0)} |\nabla u|^2 dz = 0\} ,$$

i.e. for the complement $S \equiv D\backslash Q'$,

$$S \subset \{z_0 \in D \mid \overline{\lim_{R\to 0}}\, R^{-n} \int_{Q_R(z_0)} |\nabla u|^2 dz > 0\} .$$

Thus the L^p-estimate (3.11) yields (6.14), which concludes the proof.

QED.

6.3. Proof of Theorem 6.2.

As in [11], this is more or less a by-product of the preceding proof. Again let $Q' \subset D$ be the regular set, and let $Q'' \subset\subset Q'$ be an arbitrary compactly contained subdomain. Then for $z_0 \in Q''$ and $Q_{4kr} = Q_{4kr}(z_0) \subset Q''$ and $\alpha \in (0,1)$ we have from the proof of Theorem 6.1 (k as in Lemma 6.4),

$$r^{-n} \int_{Q_{4kr}} |\nabla u|^2 dz \le C\, r^{2\alpha} , \tag{6.19}$$

where C depends now on α and dist $(Q'', \partial Q')$ but not on z_0 . Let w be the solution of (6.4) in $Q_r(z_0)$. This time using (6.7), compare u with w to obtain for $\rho \in (0,r)$

$$\begin{aligned} \int_{Q_\rho} |\nabla u-(\nabla u)_\rho|^2 dz &\le C \left(\frac{\rho}{r}\right)^{n+4} \int_{Q_r} |\nabla u-(\nabla u)_r|^2 dz \\ &\quad + C \int_{Q_r} |\nabla(u-w)|^2 dz \\ &\le C \left(\frac{\rho}{r}\right)^{n+4} \int_{Q_r} |\nabla u-(\nabla u)_r|^2 dz + \\ &\quad C\,\omega(r^2, C\, r^{-n} \int_{Q_{2r}} |\nabla u|^2 dz)^{(q-2)/q} \int_{Q_{4kr}} |\nabla u|^2 dz , \end{aligned}$$

by step 2 of the last proof $(q > 2)$,

$$\equiv (*)$$

By the C^β assumption on the coefficients $A_{ij}^{\alpha\beta}$, $\omega(s,t) \le (s^\beta + t^\beta)$, which together with (6.19) gives

$$(*) \leq C \left(\frac{\rho}{r}\right)^{n+4} \int_{Q_r} |\nabla u-(\nabla u)_r|^2 dz + C\, r^{n+\alpha(2+\beta\frac{q-2}{q})}$$

Choosing α close enough to 1 so that $\alpha(2+\beta\frac{q-2}{2}) > 2$, we obtain

$$\int_{Q_\rho} |\nabla u-(\nabla u)_\rho|^2 dz \leq C \left(\frac{\rho}{r}\right)^2 \int_{Q_r} |\nabla u-(\nabla u)_r|^2 dz + C\, r^{2\gamma} ,$$

where $2\gamma \equiv \alpha(2+\beta\frac{q-2}{p}) - 2 > 0$.

By [7], Lemma 2.1, p. 86, we conclude that

$$\int_{Q_\rho} |\nabla u-(\nabla u)_\rho|^2 dz \leq C \left(\frac{\rho}{r}\right)^{2\gamma} \int_{Q_r} |\nabla u-(\nabla u)_r|^2 dz + C\rho^{2\gamma} ,$$

which implies $\nabla u \in C^\gamma(\overline{Q''};\delta)$ for every $Q'' \subset\subset Q'$, and hence $\nabla u \in C^\gamma(Q';\delta)$.

QED.

(We were not yet able to conclude from the boundedness of ∇u (on every Q'') that of ∇w also, by which one could get that actually $\gamma = \beta$.)

References.

[1] BREZIS, H., Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Amsterdam: North-Holland, 1973

[2] CAMPANATO, S., Equazione paraboliche del secondo ordine e spaci $L^{2,\theta}(\Omega,\delta)$. Ann. Mat. Pura Appl. 73 55 - 102 (1966)

[3] DA PRATO, G., Spazi $L^{(p,\theta)}(\Omega,\delta)$ e loro proprietà, Ann. Mat. Pura Appl. 69, 383 - 392 (1965)

[4] DI BENEDETTO, E.; TRUDINGER, N.S., Harnack inequalities for quasi-minima of variational integrals, Ann. Inst. H. Poincaré Analyse non linéaire 1, 295 - 308 (1984)

[5] EKELAND, J.; TEMAM, R., Convex Analysis and Variational Problems, Amsterdam: North-Holland 1976.

[6] GEHRING, F.W., The L^p-integrability of the partial derivatives of a quasi conformal mapping, Acta Math. 130, 265 - 277 (1973)

[7] GIAQUINTA, M., Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of

Mathematical Studies 105, Princeton, 1983

[8] GIAQUINTA, M., The Regularity Problem of Extremals of Variational Integrals, in: J.M. Ball (Ed), Systems of Nonlinear PDE, Reidel Publ. Comp., Dordrecht 1983

[9] GIAQUINTA, M.; GIUSTI, E., Quasi-minima, Ann. Inst. Henri Poincaré, Analyse non linéaire 1. 79 - 107 (1984)

[10] GIAQUINTA, M.; MODICA, G., Regularity results for some classes of higher order non linear elliptic systems, J. Reine Angew. Math. 311/312, 145 - 169 (1979)

[11] GIAQUINTA, M.; STRUWE, M., On the Partial Regularity of Weak Solutions of Nonlinear Parabolic Systems, Math. Z. 179, 437 - 451 (1982)

[12] GIUSTI, E., Some aspects of the Regularity Theory for Nonlinear Elliptic Systems, in: J.M. Ball (Ed.) Systems of Nonlinear PDE, Reidel, Dordrecht (1983)

[13] LADYSHENSKAYA, O.A.; SOLONNIKOV, V.A.; URAL'CEVA, N,N., Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr. 23, AMS, Providence (1968)

[14] LIONS, J.L.; MAGENES, E., Non-Homogeneous Boundary Value Problems and Applications, Springer, Berlin, Heidelberg, New York (1972)

[15] STRUWE, M., Some regularity results for quasilinear parabolic systems, to appear in Commentat. Math. Univ. Carol

[16] TREVES, F., Basic Linear Partial Differential Equations, Acad. Press, New York (1975)

[17] WIESER, W., Partial regularity of parabolic quasiminimizers, Inst. Mittag-Leffler, Report No. 8 (1985)

[18] HILDEBRANDT, S., Über Flächen konstanter Krümmung, Math. Z. 112, 107 - 144 (1969)

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