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REAL SPECTRA OF COMPLETE LOCAL RINGS

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We show that : (a) closures of a constructible subsets of real spectrum ($\mathrm{Spec}_r A$) of complete noetherian local ring A with formally real residue field R are constructible.
(b) The connected components of constructible subsets of $\mathrm{Spec}_r A$ are constructible if and only if R has finitely many ordernigs We define also "semialgebroid" subsets and we obtain for them similar properties to those of semianalytic germs subsets.

1. INTRODUCTION.

The pursuit of this paper is to study the real spectrum of a complete local noetherian ring A ($\mathrm{spec}_r A$), whose residue field R has "some hypothesis of reality". Namely, our main results are:

a) If R is formally real, the closure of a constructible subset of $\mathrm{Spec}_r A$ is constructible

b) If R has only finitely many orderings, every constructible subset of $\mathrm{Spec}_r A$ has a finite number of connected components, which are also constructible.

(see 3.3, 3.5 , 3.16).

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In the case A is a polynomial ring over a real closed field or the ring of a real analytic series, analogous results are well known (see [1], [3] and [6], [8]) and our method to prove (a) and (b) gives an alternative way to get them. Indeed in the quoted papers the results are obtained, (as far as we know) from the corresponding geometric ones for semialgebraic and semianalytic germ sets respectively. Here we work backwards: we keep ourselves in the real spectrum, we work out a "saucissonage" there (see 2.5) and at the end we drop the geometric results from the algebraic ones, 2.9 , 4.6 , 4.7.

The paper is organized as follows : in section 2 following closely Coste's "saucissonage", ([1] , theorem 2.3.1), we state a kind of "abstract version" for real spectra which allows us to study the behaviour of some topological notions in the real spectrum under finitely generated extensions (see 2.7). In Section 3 we prove the main results. Claims a) and b) before are easily reduced to the case $A = R[[x_1,\ldots,x_n]] = F_n$. Roughly, the proofs of them run by studying the relationship between $\mathrm{Spec}_r F_{n-1}[X_n]$ and $\mathrm{Spec}_r F_n$ For b) we need to introduce a "constructible notion" of path-connectedness.

Finally in Section 4 we introduce the semialgebroid sets, which can be regarded as the "geometric support" of constructible sets of $\mathrm{Spec}_r F_n$, (R real closed field), in the same way that semialgebraic sets and semianalytic germs are the geometric support of constructible subsets of the real spectrum of finitely generated R-algebras or the ring of real analytic series. Then we get the basic properties of semialgebroid subsets, namely ; their closures are constructible too, they have a finite number of connected-semialgebroid components and these are again semialgebroid and we get also a finiteness theorem (see 4.7).

2. ABSTRACT "SAUCISSONAGE".

We start by recalling some basic facts about real spectra and setting some notations (see [1] chap. VII and [3]).

(2.1) Let A be a commutative ring with identity and let $\mathrm{Spec}_r A$ stand for its real spectrum. Given $\alpha \in \mathrm{Spec}_r A$, supp α denotes the support of α , i.e. $\alpha \cap -\alpha$; $\leq_\alpha$ the ordering induced by α in the quotient field of $A/\mathrm{supp}\,\alpha$ and $K(\alpha)$ its real closure with respect to $\leq\alpha$. We set π_α for the cannonical map $\pi_\alpha : A \to K(\alpha)$ and we write $\pi_\alpha(f) = f(\alpha)$, for any $f \in A$. Also we define $\{f > 0\} := \{\alpha \in \mathrm{Spec}_r A \mid f(\alpha) > 0\}$, and $\{f \geq 0\}$, $V(f) := \{f = 0\}$ have thus the obvious meaning. The sets

$$D_{f_1,\ldots,f_r} = \bigcap_{i=1}^{r} \{f_i > 0\}$$

are a basis for a topology in $\mathrm{Spec}_r A$ called Harrison Topology. We shall always consider the real spectrum with this topology.

(2.2) For $\alpha, \beta \in \mathrm{Spec}_r A$ we write $\beta \to \alpha$ if $\beta \subset \alpha$, that is, β is a generization of α or α is a specialization of β. If $\beta \to \alpha$, then $I := \mathrm{supp}\,\alpha/\mathrm{supp}\,\beta$ is a prime convex ideal of the ring $(A/\mathrm{supp}\,\beta) \subset k(\beta)$. Thus , the convex hull of $(A/\mathrm{supp}\,\beta)_I$ in $k(\beta)$ is a convex valuation ring of $k(\beta)$, which we denote by V_α^β. Then V_α^β is henselian and its residue field, which we call $k(\alpha)^*$, is real closed. Moreover , $k(\alpha)^*$ is an archimedean ordered extension of $k(\alpha)$, and the associated place $\lambda_\alpha^\beta : k(\beta) \to k(\alpha)^*$ is order preserving.

(2.3) We recall that a subset $X \subset \mathrm{Spec}_r A$ is constructible iff there is a sentence with parameters in A, $\psi_X(a_1,\ldots,a_n)$ such that

$$X = \{\alpha \in \mathrm{Spec}_r A \mid k(\alpha) \models \psi_X(a_1(\alpha),\ldots, a_n(\alpha))\}$$

Constructible subsets of $\mathrm{Spec}_r A$ are the open and closed sets of the constructible topology which is compact and finer than

Harrison's topology. As a consequence a constructible subset is closed (rep. open) iff it is stable under specialization (resp. generization).

(2.4) Let T be an indeterminate and consider

$$p : \mathrm{Spec}_r\, A[T] \to \mathrm{Spec}_r\, A\ ,$$

the natural map. Then, p is open an onto ([3], 6.3). Given $f(T) \in A[T]$, $\beta \in \mathrm{Spec}_r A[T]$ and $\alpha = p(\beta)$, we shall write $f(\beta)$ for $\pi_\beta(f(T)) = f(T)(\beta)$, and $f^\alpha(T)$ for the canonical image of $f(T)$ in $K(\alpha)[T]$ via the map $\pi_\alpha[T] : A[T] \to k(\alpha)[T]$. Since the fiber $p^{-1}(\alpha)$ is identified with $\mathrm{Spec}_r k(\alpha)[T]$, ([3], 4.3), for any $\beta \in p^{-1}(\alpha)$ using the notation above we have: $f^\alpha(\beta) = f(\beta)$.

Finally let $X \subset \mathrm{Spec}_r A$ be a constructible subset. A semialgebraic section over X is a section of p, $\xi : X \to \mathrm{Spec}_r A[T]$ such that $\xi(X)$ is constructible. In particular, for $\beta \in X$, $\xi(\beta) = p^{-1}(\beta) \cap \xi(X)$ is a constructible subset of $p^{-1}(\beta) = \mathrm{Spec}_r\, k(\beta)[T]$. Therefore $k(\xi(\beta)) = k(\beta)$ and $\xi(\beta)$ is uniquely determined by the image $T(\xi(\beta))$ of T in $k(\beta)$. In the following we will identify freely $\xi(\beta)$ with $T(\xi(\beta))$. A continuous semialgebraic function over X is a semialgebraic section over X which verifies the following continuity condition due to Schwartz [12]: for $\alpha, \beta \in X$ with $\beta \to \alpha$ we have $T(\xi(\beta)) \in V_\alpha^\beta$ and $\lambda_\alpha^\beta(T(\xi(\beta))) = T(\xi(\alpha))$ (or in short, making use of the identification just mentioned $\lambda_\alpha^\beta(\xi(\beta)) = \xi(\alpha)$). See also [4] for equivalent definitions.

Now, we state a "saucissonage" in the abstract setting of real spectra following closely the version that appears in [1] Chap. II.

<u>LEMMA 2.5.</u> Let $f_1,\dots,f_s \in A[T]$ be a family of polynomials stable under derivation. Then there is a partition $S_1,\dots,S_m$ of $\mathrm{Spec}_r A$ into constructible sets, and for every $i = 1,\dots,m$ continuous semialgebraic functions $\xi_{i1} < \dots < \xi_{il_i}$, $\xi_{ij} : S_i \to \mathrm{Spec}_r A[T]$ such that :

(1) For every $\alpha \in S_i$, $\xi_{i1}(\alpha)$,..., $\xi_{il_i}(\alpha)$ is the set of roots of the non-zero polynomials among $f_1^{\alpha}(T),\ldots,f_s^{\alpha}(T) \in k(\alpha)[T]$.

(2) For each $k = 1,\ldots,s$, the sign of $f_k^{\alpha}(T)$ at $t \in K(\alpha)$, $\alpha \in S_i$, depends only on the signs of $t - \xi_{i1}(\alpha),\ldots,t - \xi_{il_i}(\alpha)$.

Proof.- Every but the continuity of the ξ_{ij}'s follows at once from theorem 2.3,1 of [1] and the remark (2.3) above.

Now fix i. We prove that $\xi_{i1},\ldots,\xi_{il_i} : S_i \to \mathrm{Spec}_r A[T]$ are continuous. Let $\alpha,\beta \in S_i$, $\beta \to \alpha$. Since $\{f_1,\ldots,f_s\}$ is stable under derivation, each $\xi_{ij}(\alpha)$ $(j = 1,\ldots,l_i)$ is a simple root in $k(\alpha)$ of some $f_m^{\alpha}(T) \in k(\alpha)[T]$. Let V_{α}^{β} and λ_{α}^{β} be as in (2.2), and notice that $f_{m_j}^{\beta}(T) \in V_{\alpha}^{\beta}[T]$ and $\lambda_{\alpha}^{\beta}(f_{m_j}(T)) = f_{m_j}^{\alpha}(T)$, for $j = 1,\ldots,l_i$. Since V_{α}^{β} is henselian, it follows that there exist $t_j \in V_{\alpha}^{\beta}$ $k(\beta)$ such that $f_{m_j}^{\beta}(t_j) = 0$ $j = 1,\ldots,l_i$ and $\lambda_{\alpha}^{\beta}(t_j) = \xi_{ij}(\alpha)$. Since λ_{α}^{β} is order preserving we have $t_1<\ldots<t_{l_i}$. Therefore, from part (1) we have $\xi_{ij}(\beta) = t_j$, $j = 1,\ldots,l_i$, and we are done.

REMARK 2.6 With the notations of the lemma, we define the following subsets of $\mathrm{Spec}_r A[T]$:

$$\Gamma_{ij}^{(1)}=\{\beta \in p^{-1}(S_i) \mid T(\beta) = \xi_{i,j}(p(\beta))\} \text{ for } i = 1,\ldots,s,\ j = 1,\ldots,l_i,$$

and

$$\Gamma_{ij}^{(2)}=\{\beta \in p^{-1}(S_i) \mid \xi_{i,j}(p(\beta))<T(\beta)<\xi_{i,j+1}(p(\beta)) \text{ in } k(\beta) \supset k(p(\beta))\},$$

for $i = 1,\ldots,s$, $j = 0,\ldots,l_i$ where we set $\xi_{i,0} = -\infty$ and $\xi_{i,l_i+1} = +\infty$.

Clearly these sets form a partition of $p^{-1}(S_i)$. The sets $\Gamma_{ij}^{(1)}$ will be called <u>graphs</u>, while the sets $\Gamma_{ij}^{(2)}$ will be refered to as <u>slices.</u>

For any $i \in \{1,\ldots,s\}$ and any choice of sign $?_k$(with $?_k$

equal to > , < , =) , we define the constructible subsets $M = \{\beta \in p^{-1}(S_i) \mid f_k(T)(\beta) ?_k 0,\ k = 1,\dots,s\}$, which also build a partition of $p^{-1}(S_i)$. On the other hand, condition (2) of 2.5 implies that each f_k takes the same sign (positive, negative or zero) on $\Gamma_{ij}^{(1)}$ for all i,j and l. Hence, every $\Gamma_{ij}^{(1)}$ coincide with some M , so they are constructible. Moreover the $\Gamma_{ij}^{(2)}$ are open because they are stable under generization. Hence $p : \Gamma_{ij}^{(2)} \to S_i$ is open and surjective for any i and j.

Throughout this paper we will use the notation of Remark 1.6 and <u>we will refer to</u> $(S_i,(\xi_{ij}))$ <u>as a "saucissonage" of the family</u> $\{f_1,\dots,f_s\}$.

As in the geometrical case, lemma 2.5 allows us to study the behaviour of certain topological notions under finitely generated extensions as our next result shows. We point out first that since constructible sets are compact in the constructible topology (see (2.3)), if they have a finite number of connected components (in Harrison's topology), then each of them must be constructible.

<u>THEOREM 2.7.</u> Let A be a ring.

(i) If every constructible subset of $\mathrm{Spec}_r A$ has a finite number of connected components (hence constructible), the same holds for every constructible subset of $\mathrm{Spec}_r A[T]$.

(ii) If the closure of every constructible subset of $\mathrm{Spec}_r A$ is again constructible, the same holds for every constructible subset of $\mathrm{Spec}_r A[T]$ described by monic polynomials.

<u>Proof</u>.- (i) Let $C \subset \mathrm{Spec}_r A[T]$ be constructible and let $\{f_1,\dots,f_s\}$ be a set of polynomials describing C and their derivatives. We apply lemma 2.5 to have a "saucissonage" $(S_i,(\xi_{ij}))$ of $\{f_1,\dots,f_s\}$. By our hypothesis, up to a new partition of S_i's we may assume that they are connected. Moreover, after remark (2.6), C is union of some $\Gamma_{ij}^{(l)}$'s, so it is enough to prove that the $\Gamma_{ij}^{(l)}$'s are connected. We consider the case l = 2 , since case l = 1 is obvious.

By (2.6), $p : \Gamma_{ij}^{(2)} \to S_i$ is open and surjective; moreover, for every $\alpha \in S_i$, $p^{-1}(\alpha) \cap \Gamma_{ij}^{(2)}$ is connected (it is an "open interval" in $\mathrm{Spec}_r k(\alpha)[T]$). Hence the $\Gamma_{ij}^{(2)}$'s are connected.

(ii) Let C, $\{f_1,\dots,f_s\}$ and $(S_i,(\xi_{ij}))$ be as above. Where now the f_i's are assumed to be monics. Again after (2.6) and with the notation there, it is enough to prove that every $\overline{\Gamma_{ij}^{(1)}}$ is constructible.

But, take

$$\Gamma_{ij}^{(1)} = \{\beta \in p^{-1}(S_i) \mid f_k(T)(\beta) ?_k 0 \ , \ k=1,\dots,s\}$$

with $?_k$ equal to $>$, $<$, $=$, (see (2.6)) and let us set

$$Y = \{\beta \in p^{-1}(\overline{S}_i) \mid f_k(T)(\beta) \ \overline{?}_k \ 0 \ , \ k=1,\dots,s\}$$

with $\overline{?}_k$ equal to $\geq$, $\leq$, $=$ respectively. Notice that, by our hypothesis on $\mathrm{Spec}_r A$ Y is constructible. We write for simplicity Γ, S and ξ for $\Gamma_{ij}^{(1)}$, S_i, ξ_{ij} (i.e. either Γ is the graph of ξ or a slice bordered by ξ, see (2.6)). Obviously we have $\overline{\Gamma} \subset Y$, and we want to prove the equality. Let $\beta_0 \in Y$ and set $\alpha_0 = p(\beta_0) \in \overline{S}$. By Thom's lemma, ([1], Prop 2.5.4), $p^{-1}(\alpha_0) \cap Y$ is either a single point (namely β_0) or a closed interval, which is the closure of $p^{-1}(\alpha_0) \cap \Gamma$.

Assume first we are in the second case. Since the f_k's are monic the $f_k^{\alpha_0}$'s do not vanish identically, and so, the $?_k$ are strict inequalities ($>$ or $<$). Consequently there exists $\beta_1' \to \beta_0$, s.t. $p(\beta_1') = \alpha_0$, β_1' is an ordering in $k(\alpha_0)[T]$ and $f_k^{\alpha_0}(\beta_1') ?_k 0$ for all $k=1,\dots,s$. Now, take $\alpha_1 \in S$, $\alpha_1 \to \alpha_0$ and denote by $\lambda : k(\alpha_1) \to k(\alpha_0)^*, \infty$ the place associated to this specialization, (2.2). Since $k(\alpha_0)^*$ is an ordered extension of $k(\alpha_0)$, the ordering defined by β_1' in $k(\alpha_0)(T)$ extends to an ordering in $K(\alpha_0)^*(T)$ which we still denote by β_1'. Now, let $\tilde{\lambda} : k(\alpha_1)(T) \to k(\alpha_0)^*(T)$ be an extension of λ with $\tilde{\lambda}(T) = T$ (see [2]), and let β_1 be an ordering in $k(\alpha_1)(T)$, compatible with $\tilde{\lambda}$. We still denote by β_1 the point in $\mathrm{Spec}_r k(\alpha_1)[T]$ defined by β_1. So, we have $\beta_1 \to \beta_1' \to \beta_0$. Since by our assumption $f_k^{\alpha_0}(T) \neq 0$ for every k and $\tilde{\lambda}\,(f_k^{\alpha_1}(T)) = f_k^{\alpha_0}(T)$, we get:

$$\text{sgn of } f_k^{\alpha_1}(T) \quad \text{in} \quad (k(\alpha_1)(T),\beta_1) =$$

$$\text{sgn of } f_k^{\alpha_0}(0) \quad \text{in} \quad (k(\alpha_0)(T),\beta_1')$$

(for all k). Hence, $f_k^{\alpha_1}(\beta_1) ?_k\, 0$ and so $\beta_1 \in \Gamma$.

Finally suppose $p^{-1}(\alpha_0) \cap Y = \beta_0$. So $k(\beta_0) = k(\alpha_0)$. We take $\alpha_0 \in S$, $\alpha_1 \to \alpha_0$ and consider $\phi^{-1}(\alpha_1) \cap Y$. We have (see 2,6) $\xi(\alpha_1) \in \overline{p^{-1}(\alpha_1) \cap \Gamma} = p^{-1}(\alpha_1) \cap Y$ by Thom's lemma. In particular $\xi(\alpha_1) \in \overline{\Gamma}$. On the other hand all roots of all $f_k^{\alpha_1}(T)$ go via $\lambda : k(\alpha_1) \to k(\alpha_0)^*, \infty$ to roots of $f_k^{\alpha_0}(T)$. Hence, we can extend ξ to α_0 by setting $\xi(\alpha_0) = \lambda(\xi(\alpha_1))$. Thus, since λ is order preserving we have $\xi(\alpha_1) \to \xi(\alpha_0)$. Hence $\xi(\alpha_0) \in p^{-1}(\alpha_0) \cap Y = \beta_0$. It follows that $\beta_0 = \xi(\alpha_0) \in \overline{\{\xi(\alpha_1)\}} \subset \overline{\Gamma}$, and 2.7 is proved.

<u>COROLLARY 2.8.</u> Let A be a ring such that every constructible subset of $\mathrm{Spec}_r A$ has a finite number of connected components. Then $\mathrm{Spec}_r A[T]$ is locally connected.

<u>Proof.-</u> Let $\alpha \in \mathrm{Spec}_r A[T]$, and take an open neighbourhood basis of α of constructible subsets. Let D be one of these neighbourhoods. Then D has a finite number of connected components. Let D' be the connected componet of D with $\alpha \in D'$. Then D' is an open neighbourhood of α, since D' is open in D. Obviously the family of these D' forms an open neighbourhood basis of α .

<u>COROLLARY 2.9.</u> Let A be a finitely generated R-algebra , R a formally real field. Then the closure of a constructible subset of $\mathrm{Spec}_r A$ is constructible. Moreover if R has only finitely many orderings every constructible subset of $\mathrm{Spec}_r A$ has a finite number of connected components.

Proof. We may assume A be a polynomial ring over R, say in n indeterminates. Then the proof is done by induction on n by using 2.7.

3. REAL SPECTRA OF COMPLETE LOCAL RINGS .

Let us denote by F_n the formal power series ring in n indeterminates over a formally real field R.

In this paragraph we prove our main results, theorems 3.3 and 3.6. Both of their proofs work word by word for the ring of analytic series over $\mathbb{R}$. Shortly, the proofs of these results are done by induction on n , using Weierstrass Preparation theorem , studing carefully the relationship between $\mathrm{Spec}_r F_{n-1}[X_n]$ and $\mathrm{Spec}_r\, F_n$. The first result we state goes in that direction:

THEOREM 3.1. ("Going-down for real spectra") [10]

Let $A \to B$ be a regular morphism between excellent rings. Then the associated map $f : \mathrm{Spec}_r\, B \to \mathrm{Spec}_r\, A$ has the going-down property , i.e. if $\alpha_1 \to \alpha_0 = f(\beta_0)$, $\beta_0 \in \mathrm{Spec}_r B$, there exists $\beta_1 \in \mathrm{Spec}_r\, B$, $\beta_1 \to \beta_0$ and $f(\beta_1) = \alpha_1$

REMARK 3.2. With the assumption of 3.1., if C is a constructible subset of $\mathrm{Spec}_r A$, we have $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$. For recall that $\alpha \in \overline{C}$ implies that there exists $\beta \in C$, with $\beta \to \alpha$.

Now we prove:

THEOREM 3.3. The closure of a constructible subset of $\mathrm{Spec}_r\, F_n$ is also constructible.

Proof. For $n = 0$ the result is obvious since every constructible subset of $\mathrm{Spec}_r R$ is closed (and open) . We assume that it holds true for $n-1$ and let C $\mathrm{Spec}_r F_n$ be constructible. By Weierstrass Preparation theorem [13,th.2.1.3] , we may assume C be described by distinguished or Weierstrass polynomials in X_n [13,def.2.1.1] . This follows at once from the easy fact that given $\alpha \in \mathrm{Spec}_r F_n$ and $u \in F_n$ a unity with $u(0) = a \in R \setminus \{0\}$, the signs of u and a in α coincide. Now let C^* be the constructible subset of $\mathrm{Spec}_r F_{n-1}[X_n]$ described by the same inequalities that C, and let

$$\pi : \mathrm{Spec}_r F_n \to \mathrm{Spec}_r F_{n-1}[X_n]$$

be the natural map. We have $\pi^{-1}(C^*) = C$. On the other hand, by the induction hypothesis and 2.7, $\overline{C^*}$ is constructible. Thus since by 3.2 $\pi^{-1}(\overline{C^*}) = \overline{\pi^{-1}(C^*)} = \overline{C}$, $\overline{C}$ is constructible.

COROLLARY 3.4. Let A be a local complete noetherian ring with formally real residue field. Then the closure of a constructible subset of $\mathrm{Spec}_r A$ is also constructible.

Proof.- Since A is equicharacteristic , by Cohen's theorem , $A \simeq R[[X_1,\dots,X_n]]/I$. So the result follows readily from 3.3.

Recently, J.Ruiz has extended theorem 3.4 to any excellent ring. On the other hand the result is false in general: H.Delfs and J.M.Gamboa have found , [5] , a constructible subset of the real spectrum of the ring of continuous functions whose closure is not constructible.

We turn, now our attention to connectedness properties of constructible subsets of $\mathrm{Spec}_r F_n$. We assume first that R is real closed. Since the maximal point (i.e. the point with support the maximal ideal of F_n) is in the closure of any constructible set, it is inmediate that any closed constructible subset is connected (in particular $\mathrm{Spec}_r F_n$). More in general we have:

THEOREM 3.5. Let $F_n = R[[X_1,\ldots,X_n]]$ be the ring of formal power series over a real closed field R. Then, every constructible subset of $\mathrm{Spec}_r\, F_n$ has a finite number of connected components (hence also constructible).

We devote the rest of this section to the proof of theorem 3.5. To do it we need to mix up several ingredients which we separate in different lemmata. The first thing we do is to take a closer look at the natural map $\pi : \mathrm{Spec}_r F_n \to \mathrm{Spec}_r\, F_{n-1}[X_n]$

LEMMA 3.6. (a) For each $r \in R^+$ let
$[-r,r] = \{\alpha \in \mathrm{Spec}_r F_{n-1}[X_n] \mid -r \leq X_n(\alpha) \leq r \quad \text{in} \quad k(\alpha)\}$. Then
$$\mathrm{im}\,\pi = \bigcap_{r \in R^+} [-r,r] \quad .$$
(b) Let $f \in F_{n-1}[X_n]$ be a Weierstrass polynomial in X_n. Then $V(f) \subset \mathrm{im}\,\pi$.

Proof. (a) Let us call $M = \bigcap_{r \in R^+} [-r,r]$. It is enough to show that for every $\alpha' \in M$ there exists $\alpha \in \mathrm{Spec}_r F_n$ with $\pi(\alpha) = \alpha'$. Let $m = m_{n-1} + (y_n)$, where m_{n-1} is the maximal ideal of F_{n-1}, m is a maximal ideal of $F_{n-1}[X_n]$ and $F_n = F_{n-1}[X_n]^{\wedge}_{m}$, the completion of $F_{n-1}[X_n]_m$.

On the other hand, by the very definition of M, we have supp α' m. Hence α' is "central" in the terminology of [9] and it extends to $\alpha \in \mathrm{Spec}_r F_n$ (loc. cit.).

(b) After (a), we only have to prove that for any $\alpha \in \mathrm{Spec}_r F_{n-1}$, any root ξ of $f^{\alpha}(X_n) \in k(\alpha)[X_n]$ in $k(\alpha)$ is infinitesimal with respect to R (i.e. $-r < \xi < r$ in $k(\alpha)$ for every $r \in R^+$).

Let $f = X_n^p + a_1 X_n^{p-1} + \ldots + a_p$, with $a_i \in F_{n-1}$ and $a_i(0) = 0$. Then, $\xi(\xi^{p-1} + a_1 \xi^{p-2} + \ldots + a_{p-1}) = -a_p$ which is infinitesimal w.r.t. R. Since ξ is bounded by a polynomial in the

a_i's , either ξ or $\xi^{p-1} + \dots + a_{p-1}$ has to be infinitesimal w.r.t. R. So , after repeating the argument p times , we get that ξ is infinitesimal w.r.t. R.

Our second step towards the proof of 3.5 is to introduce a notion of "path" -connectedness in $\mathrm{Spec}_r F_n$ (definition 3.9 below). This itself is based on the existence of "enough" constructible points in $\mathrm{Spec}_r F_n$. To explain what this means we point out first that every one-dimensional point $\alpha \in \mathrm{Spec}_r F_n$ (i.e. ht(supp(α)= n-1) is defined by a morphism $\xi : F_n \to R[[t]]$ where in $R[[t]]$ we fix from now on the unique order with $t > 0$. Indeed , let $p_\alpha = \mathrm{supp}\,\alpha = (g_1,\dots,g_s)\,F_n$, $B = F_n/p_\alpha$ and set B^ν for the normalization of B. Then $B^\nu \overset{\phi}{\cong} R[[t]]$ and we may take $\phi = \phi|_B$ (up to a possible automorphism $t \mapsto -t$). Moreover let (t^e) be the conductor of B in B^ν . Then there exists $f \in F_n$ such that $\phi(f + p_\alpha) = t^{2e+1}$. Consequently $\{\alpha\} = \{g_1 = \dots = g_s = 0\} \cap \{f > 0\}$, what shows that α is constructible. Summarizing, in order to have a precise reference to this fact we state :

PROPOSITION 3.7. Every one-dimensional point $\alpha \in \mathrm{Spec}_r F_n$ is constructible.

Now the term "enough" constructible points means that we can find constructible points in every constructible subset of $\mathrm{Spec}_r F_n$. This is an immediate consequence of the following well known Lasalle's specialization theorem:

THEOREM 3.8. ([7]) .- Let K be an ordered field and R its real closure. Let $f_1,\dots,f_r \in F_n$ and $p \in \mathrm{Spec}\, F_n$. Then if there exists an ordering in F_n/p making all $\bar{f}_i$'s positive, there is a k-algebra local homomorphism $\phi : F_n/p \to R[[t]]$ such that $\phi(\bar{f}_i) > 0$ for all i.

In what follows we shall denote by 0_n the maximal point of $\mathrm{Spec}_r\, F_n$ (i.e. 0_n is defined by the natural morphism $F_n \to R$). 0_n is a specialization of any point of $\mathrm{Spec}_r F_n$. Finally we shall write 0_n^+ (resp. 0_n^-) for the constructible point defined by $\phi : F_n \to R[[t]]$, $\phi(x_i) = 0$, $i = 1,\dots,n-1$, $\phi(x_n) = t$ (resp. $\phi(x_n) = -t$).

Here comes the "path"-connectedness definition:

<u>DEFINITION 3.9.</u> Let X $\mathrm{Spec}_r\, F_n$ be a constructible set.

(a) Let α, $\beta \in X$, $\dim \alpha = \dim \beta = 1$. A constructible "path" <u>joining α and β</u> is a connected constructible subset Y $\mathrm{Spec}_r\, F_n$ such that α, $\beta \in Y$ and $\overline{Y} = Y \cup \{0_n\}$ (where $\overline{Y}$ is the closure of Y in $\mathrm{Spec}_r\, F_n$).

(b) We say that X is constructible - "path" - connected if for any one-dimensional α, $\beta \in X$ there is a constructible "path" Y joining α and β with $Y \subset X$.

As one should hope it holds

<u>LEMMA 3.10.</u> If X $\mathrm{Spec}_r\, F_n$ is constructible - "path" - connected, it is connected.

<u>Proof.-</u> Straighforward from 3.7.

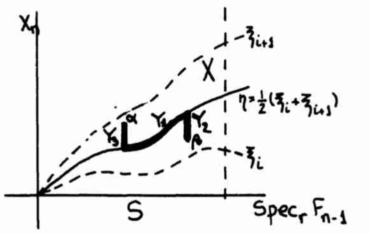
With this lemma, theorem 3.5 follows at once from the following proposition:

<u>PROPOSITION 3.11</u>. Let $X \subset \mathrm{Spec}_r F_n$ be a constructible subset. Then there exists a finite partition of X into constructible subsets which are constructible - "path" - connected.

Our strategy to prove 3.11 is the following: we consider the following diagram of natural maps

$$\begin{array}{ccc} \mathrm{Spec}_r\, F_n & \xrightarrow{\ \pi\ } & \mathrm{Spec}_r\, F_{n-1}[X_n] \\ & & \downarrow p \\ & & \mathrm{Spec}_r\, F_{n-1} \end{array}$$

We use induction and the intermediate space $\mathrm{Spec}_r\, F_{n-1}[X_n]$. Indeed we work here, joining any two one-dimensional points by means of suitable constructible sets: their inverse images under π are "paths".



These constructible sets we are to use are vertical segments and graphs of continuous semialgebraic functions (see figure). We start with the latters:

LEMMA 3.12. Let Z $\mathrm{Spec}_r\, F_{n-1}$ be a connected constructible set and $\xi : Z \to \mathrm{Spec}_r\, F_{n-1}[X_n]$ a continuous semialgebraic function such that $\xi(Z) \subset V(f)$, the zero set of some Weierstrass polynomial $f \in F_{n-1}[X_n]$. Then, $Y = \pi^{-1}(\xi(Z))$ is a connected constructible subset of $\mathrm{Spec}_r\, F_n$. Furthermore if $\overline{Z} = Z \cup \{0_{n-1}\}$ then $\overline{Y} = Y \cup \{0_n\}$.

Proof.- Let $g \in F_n$ and $D_g = \{g > 0\}$. Notice that, if g is not regular in X_n, taking $g^* = g - f^2$, we have $D_g \cap Y = D_{g^*} \cap Y$, and g^* regular in X_n. So we may assume g be a Weierstrass polynomial (in short W-p) in X_n.

Now, assume that there are open subsets, $U_i \subset \mathrm{Spec}_r\, F_n$, $i = 1,2$, such that $Y \subset U_1 \cup U_2$ and $Y \cap U_1 \cap U_2 = \emptyset$. We want to show that $Y \subset U_1$ or $Y \subset U_2$. By compactness of the constructible topology we may assume that each U_i is constructible (so it is a finite union of sets of the form $D_{g_1,\dots,g_r} = \bigcap_{i=1}^{r} D_{g_i}$). Moreover, as above we may assume the U_i defined by W-p in X_n.

Let us denote by U_i^* the constructible subsets of $\mathrm{Spec}_r\, F_{n-1}[X_n]$ defined by the same inequalities than U_i in $\mathrm{Spec}_r F_n$. So, we have $\pi(U_i) = U_i^* \cap \mathrm{im}(\pi)$. Also, we have $\xi(Z) \subset U_1^* \cup U_2^*$ and $U_1^* \cap U_2^* \cap \xi(Z) = \emptyset$, by 3.6(b). Since $\xi(Z)$ is connected, it is $\xi(Z) \subset U_i^*$ for some i, so $Y \subset U_i$.

For the second part notice that $0_n \in \overline{Y}$ and that the points of $\overline{Y} \setminus Y$ lie over $\overline{Z} \setminus Z = \{0_{n-1}\}$. So, in $\mathrm{Spec}_r F_{n-1}[X_n]$

they lie over $\pi(0_n^+)$ or $\pi(0_n^-)$. Since f is $W-p$, neither $\pi(0_n^+)$, nor $\pi(0_n^-)$ is in $V(f)$ and we are done.

Next, let $\alpha, \alpha_1 \in \mathrm{Spec}_r F_n$, $\alpha'' \in \mathrm{Spec}_r F_{n-1}$ be one dimensional points , with $p \circ \pi(\alpha) = p \circ \pi(\alpha_1) = \alpha''$. Let $\alpha' = \pi(\alpha)$, $\alpha_1' = \pi(\alpha_1)$. Then α' , $\alpha_1' \in p^{-1}(\alpha'') = \mathrm{Spec}_r K(\alpha'')[X_n]$ Moreover , since all points are of dimension one all the inclusions

$$F_{n-1}/\mathrm{supp}\,\alpha'' \subset F_{n-1}[X_n]/\mathrm{supp}\,\alpha' \subset F_n/\mathrm{supp}\,\alpha \subset (F_n/\mathrm{supp}\,\alpha)^\vee = R[[t]]$$

are finite. Hence $k(\alpha'') = k(\alpha') = k(\alpha)$, and analogously $k(\alpha'') = k(\alpha_1') = k(\alpha_1)$. Therefore α' and α_1' correspond to (i.e. are defined by) some values $X_n(\alpha')$, $X_n(\alpha_1') \in k(\alpha'')$, say a' and a_1'. Assume w.l.o.g. that $0 < a' < a_1'$. Then we define the interval joining α' and α_1' as

$$S_{\alpha'\alpha_1'} = \{\gamma \in p^{-1}(\alpha'') = \mathrm{Spec}_r K(\alpha'')[X_n] \mid a' \le X_n(\gamma) \le a_1'\} .$$

We have:

<u>LEMMA 3.13.</u> $Y = \pi^{-1}(S_{\alpha'\alpha_1'})$ is a constructible "path" joining α and α_1 .

<u>Proof</u>.- Firstly we show that $\overline{Y} = Y \cup \{0_n\}$. By (3.1) we have $\overline{Y} = \pi^{-1}(\overline{S}_{\alpha'\alpha_1'})$. Also $\overline{S}_{\alpha'\alpha_1'} \subset p^{-1}(\alpha'') \cup p^{-1}(0_{n-1})$. Hence we need only to show that neither $\pi(0_n^+)$ nor $\pi(0_n^-)$ lie in $\overline{S}_{\alpha'\alpha_1'}$. Since α'' is one-dimensional, arguing as in the proof of 3.7, $k(\alpha'')$ is a field of Puiseux series over R, say $R((t))^*$. Moreover $a_1' \in R[[t]]^*$ and is infinitesimal w.r.t. R, 3.6. Using a conductor argument as in 3.7 we find $p \in \mathbb{N}$ <u>even</u> and $f \in F_{n-1}$ with $f(0)=0$ such that $a_1'^p < f(\alpha'')$. It follows that $S_{\alpha'\alpha_1'} \subset \{\gamma \in \mathrm{Spec}_r F_{n-1}[X_n] \mid X_n^p(\gamma) < f(\gamma)\}$, since if $\gamma \in S_{\alpha'\alpha_1'}$, $f(\gamma) = f(\alpha'')$. Hence , for $\gamma \in \overline{S}_{\alpha'\alpha_1'}$ it is $X_n^p(\gamma) \le f(\gamma)$, but this inequality does hold neither in $\pi(0_n^+)$ nor in $\pi(0_n^-)$ (Geometrically speaking we find a " parabola" $X_n^p - f(X_1,\dots,X_{n-1})$ which separates $S_{\alpha'\alpha_1'}$ from

the X_n - axis). Thus $\overline{Y} = Y \cup \{0_n\}$.

To see that Y is connected, as in the proof of 3.12, we need only to show that given $g \in F_n$ there is $g^* \in F_{n-1}[X_n]$ such that $D_g \cap Y = D_{g^*} \cap Y$. We use once more a conductor argument as in 3.7. Let $\tau : F_{n-1} \to R[[t]]$ be the morphism defining α'' . Set $\tau(X_j) = x_j(t)$ $j = 1,\dots,n-1$. Thus we have $g(x_1(t),\dots,x_{n-1}(t),X_n) = \varepsilon\, t^r\, u(t,X_n)$ where $\varepsilon = \pm 1$, $u(0,0) \in R^+ \setminus \{0\}$ and $P(t,X_n) = X_n^n + \sum_{j=0}^{m-1} a_j(t)\, X_n^j$, with $a_j(0) = 0$. Now if (t^e) is the conductor of $F_{n-1}/p_{\alpha''}$ in $R[[t]]$, let $s \in \mathbb{N}$ even , $s \geq e$. Thus , for each $j = 1,\dots,m-1$, there is $h_j \in F_{n-1}$ such that $\tau(h_j) = t^s\, a_j(t)$. We claim that $g^* = (X_n^m + \sum_{j=1}^{m-1} h_j\, X_n^j)$ (which is a W-p) is the polynomial we sought. Indeed , $C = D_g \cap Y$ and $C^* = D_{g^*} \cap Y$ are constructible. Therefore , by 3.8 , to show that $\overline{C} = \overline{C^*}$ it is enough to see that $C \setminus C^*$ and $C^* \setminus C$ do not contain points of dimension one, what is obvious by construction.

Now we are finally ready to prove the proposition.

<u>Proof of 3.11</u>.- We proceed by induction on n. For $n = 0$ it is obvious since R is real closed. We assume it for F_{n-1} . Let X $\mathrm{Spec}_r\, F_n$. As mentioned in (3.3), up to a linear change of coordinates we may assume X described by W - p in X_n or constants (if $X = \mathrm{Spec}_r\, F_n$) . As above, we denote by X^* the constructible subset of $\mathrm{Spec}_r\, F_{n-1}\, [X_n]$ defined by the same formula that X.

Let $\{f_1,\dots,f_m\}$ be a set of elements describing X , together with their derivatives with respect to X_n. We consider a "saucissonage" $(S_i, (\xi_{ij}))$ of the family $\{f_1,\dots,f_m,X_n\}$. Up to a refinement of the partition S_i we may assume that :
(a) $\{0_{n-1}\}$ is a member of it, and (b) they are constructible - "path" - connected.

Let now B' be a graph or a slice over some S_i. Set $B = \pi^{-1}(B')$. Since $X^* \setminus p^{-1}(0_{n-1})$ is a union of some B' ,

$X \setminus \{0_n^-, 0_n, 0_n^+\}$ is a union of some B. Therefore it is enough to show that each one if these B' is constructible - "path" - connected. Let $\alpha, \beta \in B$ be one-dimensional points. We set $\alpha' = \pi(\alpha)$, $\beta' = \pi(\beta)$, $\alpha'' = p(\alpha')$, $\beta'' = p(\beta')$. Since $0_{n-1} \notin p(B')$ (we have taken it out!) it is $\dim \alpha'' = \dim \beta'' = \dim \alpha' = \dim \beta' = 1$. By the induction hypothesis there is a connected constructible "path" Z in $S_i = p(B'')$ joining α'' and β''. Now we distinguish two cases:

(i) B' is a graph of some $\xi_{ij} : S_i \to \mathrm{Spec}_r F_{n-1}[X_n]$. Then by lemma 3.12, $Y = \pi^{-1}(\xi_{ij}(Z))$ B is a constructible "path" joining α and β.

(ii) B' is a slice $]\xi_{ij}, \xi_{ij+1}[$, over S_i (see (2.6)) Assume first, $\xi_{ij} \neq -\infty$, $\xi_{ij+1} \neq \infty$. Let α', β', α'', β'' and we define $\eta = \frac{1}{2}(\xi_{i,j} + \xi_{i,j+1})$. Then, η is a semialgebraic continuous function [12] and we call $Y_1' = \eta(Z)$, $\alpha_1' = (\alpha'')$ and $\beta_1' = \eta(\beta'')$. Since $\xi_{i,j}$ and $\xi_{i,j+1}$ verify a Weierstrass polynomial they are in particular integer over F_{n-1}. Therefore η is integer over F_{n-1}, and verifies a monic polynomial in $F_{n-1}[X_n]$. Applying Weierstrass preparation theorem it follows that η is a root of a Weierstrass polynomial. Let Y_2' (resp. Y_3') be the segments joining α' with α_1' (resp. β' with β_1') (see 3.13). Finally we set $Y_j = \pi^{-1}(Y_j')$, $j = 1,2,3$, and $Y = Y_1 \cup Y_2 \cup Y_3$. Hence Y B and by the lemmas before Y is a constructible "path" joining α and β.

Notice that, since X_i belongs to the family that we have used to make the "saucissonage" it can not be both $\xi_{ij} = -\infty$, $\xi_{ij+1} = +\infty$. By the same reason, ξ_{ij}, ξ_{ij+1} do not change their signs on S_i. Finally, if for instance is $\xi_{ij} = -\infty$ and $\xi_{ij+1} > 0$ (resp. < 0) the proof goes as above taking $\eta = \frac{1}{2}\xi_{ij+1}$ (resp. $2\xi_{ij+1}$).

REMARK 3.14. Let C $\mathrm{Spec}_r F_n$ be a constructible subset defined by Weierstrass polynomials in X_n. Then, if X^* is a connected component of C^*, $X = \pi^{-1}(X^*)$ is a connected component of C.

Proof.- Looking at the proof of 2.7, X^* is union of graphs and slices , which are given by sign conditions on some $W-p$. It is easy to carry out that we may assume X^* be a graph or a slice lying over a connected subset of $\mathrm{Spec}_r\, F_{n-1}$.

Assume X is not connected. Then X has a finite number of connected components , which, looking at the proof of 3.10 and 3.11 we know that are also constructible and defined by elements of $F_{n-1}[X_n]$. Hence, we may write $X = A \cup B$ where A and B are closed in X , $A,B \neq \emptyset$, $A \cap B = \emptyset$ and both defined by elements of $F_{n-1}[X_n]$. Let us call $M = \mathrm{im}(\pi)$. Thus $\pi(X) = X^* \cap M \subset A^* \cup B^* \subset \overline{A^*} \cup \overline{B^*}$. Moreover $\overline{A^*} \cap \overline{B^*} \cap \overline{X^* \cap M} = \emptyset$ since we have , by 3.1, $\pi^{-1}(\overline{A^*} \cap \overline{B^*} \cap \overline{X^*}) = \pi^{-1}(\overline{A^*}) \cap \pi^{-1}(\overline{B^*}) \cap \pi^{-1}(\overline{X^*})$ $= \overline{A} \cap \overline{B} \cap \overline{X} = A \cap B \cap X = \emptyset$. Now , by (3.6) (a), and the compacity of the constructible topology (2.3) there exists $r \in R^*$, with $X^* \cap [-r,r] \subset \overline{A^*} \cup \overline{B^*}$ and $\overline{A^*} \cap \overline{B^*} \cap X^* \cap [-r,r] = \emptyset$. Consequently , if X^* is a graph, by 3.6 , $X^* \subset M \subset [-r,r]$, and we get a contradiction since X^* is connected. If X^* is a slice bounded by two roots of some $W-p$, the same argument applies. Finally, if X^* is an "unbounded" slice , the intersection $X^* \cap [-r,r]$ is also a slice (lying over the same connected subset of $\mathrm{Spec}_r\, F_{n-1}$ that X^*) hence, by 2.6, is connected.

Our theorem 3.5 extends also to a more general situation.

COROLLARY 3.16 Let A be a local complete noetherian ring with formally real residue field having only finitely many orderings, then every constructible subset C $\mathrm{Spec}_r A$ has a finite number of connected components.

Proof.- By Cohen's theorem $A = R[[X_1,\dots,X_n]]/I$, for some ideal I . Hence we assume $A = R[[\underline{X}]] = R[[X_1,\dots,X_n]]$, where R has only finitely many orderings.

Let $\mathrm{Spec}_r R = \{\gamma_1,\dots,\gamma_m\}$, R_i the real closure of (R,γ_i) and $\pi : \mathrm{Spec}_r A \to \mathrm{Spec}_r R$, $\varepsilon_i : \mathrm{Spec}_r\, R_i[[\underline{X}]] \to \mathrm{Spec}_r\, A$ the natural restriction maps. Since, the corollary holds for $R_i[[\underline{X}]]$, 3.5, and $X_i = \pi^{-1}(\gamma_i)$ $(i = 1,\dots,m)$ are a partition of $\mathrm{Spec}_r\, A$ into constructible sets, it is enough to prove that $\operatorname{im}(\varepsilon_i) = X_i$. We work out the non trivial part $X_i \subset \operatorname{im}(\varepsilon_i)$. Take $\alpha \in X_i$ and let A^h be the strict henselization of A with respect to α and $(A^h)^\wedge$ its completion . Thus we have $\operatorname{Spec}_r(A^h)^\wedge \xrightarrow{\phi} \operatorname{Spec}_r A^h \xrightarrow{\psi} \operatorname{Spec}_r A$. Moreover $\alpha \in \operatorname{im}\psi$, ϕ is surjective , [9], and $R_i[[X]] \subset (A^h)^\wedge$. Hence $\alpha \in \operatorname{im}\psi \circ \phi \subset \operatorname{im}\varepsilon_i$ and the proof is complete.

<u>REMARK 3.17</u> Let $\pi : \operatorname{Spec}_r A \to \operatorname{Spec}_r R$ be , as above the canonical map induced by the inclusion $R \to A$. Since $\operatorname{Spec}_r R$ is totally disconnected , and π is onto it follows that if R has infinitely many orderings, $\operatorname{Spec}_r A$ has infinitely many connected components, what shows the converse of 3.16.

4. <u>SEMIALGEBROID SETS</u>.

(4.1) Through this paragraph R will be a real closed field and $F_n = R[[X_1,\dots,X_n]] = R[[\underline{X}]]$

We want to describe some "geometric support" for construc_ tible subsets of $\operatorname{Spec}_r F_n$ in the same way that semialgebraic sets and semianalytic germs are respectively , the support of construc_ tible subsets of the real spectrum of finitely generated R-algebras and the ring of analytic functions Θ_n. We point out that our ap_ proach (together with Artin-Lang theorem in each case: algebraic and analytic), allows us to obtain the basic properties of semi_ algebraic and semianalytic germs from those of their correspondying constructible sets, namely: that the number of connected components is finite; that they are semialgebraic or semianalytic and the finiteness theorem (see [3] and [6], where they work in the opposite direction).

(4.2) We start by considering <u>parametrizations</u>, that is R-algebra local homomorphisms, $\tau : F_n \to R[[t]]$, and we denote the set of all of them by $\operatorname{Hom}_{loc}(F_n, R[[t]])$. Given $E \subset F_n$ and

$S \subset \mathrm{Hom}_{loc}(F_n, R[[t]])$ we define:

$$V(E) = \{\tau \in \mathrm{Hom}_{loc}(F_n, R[[t]]) \mid \tau(f) = 0 \quad \forall f \in E\}$$

$$Y(S) = \{f \in F_n \mid \tau(f) = 0 \quad \forall \tau \in S\}.$$

We fix the ordering in $R[[t]]$ making $t > 0$ and we set for $f \in F_n$:

$$S(f) = \{\tau \in \mathrm{Hom}_{loc}(F_n\, R[[t]]) \mid \tau(f) > 0\}$$

The sets $S(f)$ are a subbasis of a topology in $\mathrm{Hom}_{loc}(F_n, R[[t]])$ which we call semialgebroid topology.

Our "geometric points" are going to be the "formal" half-branches. In order to identify two parametrizations τ, $\tau' \in \mathrm{Hom}_{loc}(F_n, R[[t]])$ describing the same half-branch, we define the following equivalence relation: $\tau \sim \tau'$ iff there is an isomorphism ϕ of $R[[t]]^{\omega}$ (the Puiseux formal power series ring) such that $\phi \circ \tau = \tau'$.

We consider $M_n = \mathrm{Hom}_{loc}(F_n\, R[[t]])^* / \sim$ where $\mathrm{Hom}_{loc}(F_n, R[[t]])^* = \mathrm{Hom}_{loc}(F_n, R[[t]]) \setminus \{0\}$ and $\mathrm{Ker}\ 0 = \mathcal{M}_n$, the maximal ideal of F_n. We also provide M_n with the quotient topology (of the semialgebroid topology) that we still call semi_ algebroid topology.

On the other hand, as we have remarked, see (3.7) every $\tau \in \mathrm{Hom}_{loc}(F_n, R[[t]])^*$ defines a prime cone $\alpha_\tau \in \mathrm{Spec}_r F_n$ with $\dim \alpha_\tau = 1$, and every one-dimensional point of $\mathrm{Spec}_r F_n$ can be described in this way. Consequently we have

<u>PROPOSITION 4.3.</u> The map $A : M_n \to \mathrm{Spec}_r F_n : \tau \to \alpha_\tau$ defines a homeomorphism of M_n onto the subset H of all one-dimensional points of $\mathrm{Spec}_r F_n$.

<u>Proof</u>.- By the very definition of $\sim$ and 3.7, A is a bijection Moreover for $f \in F_n$ it holds $A(S(f)) = D(f) \cap H$, so it is homeomorphism.

The space M_n is the "formal" analogue to the affine space R^n.

DEFINITION 4.4 A subset X M_n is semialgebroid if it belongs to the lattice generated by the sets $S(f)$ by a finite number of Boolean operations : union, intersection and complementary.

Now , let $S = \sum_{i=1}^{m} (S(f_{i_1},\dots,f_{i_r}) \cap V(g_i)$ be a semi_ algebroid subset of M_n. We define the constructible subset of $\mathrm{Spec}_r\, F_n$:

$$\tilde{S} = \bigcup_{i=1}^{m} \{f_{i_1} > 0 ,\dots, f_{i_r} > 0 , g_i = 0 \}$$

Then

THEOREM 4.5. The map $S \to \tilde{S}$ defines a one-to-one lattice homo_ morphism between the lattices of semialgebroid subsets of M_n and the constructible subsets of $\mathrm{Spec}_r\, F_n$. Moreover this map preserves openess.

Proof. For the first assertion it suffices to prove that $S \neq \emptyset$ if and only if $\tilde{S} \neq \emptyset$, and to do that we may assume that S is basic , i.e., $S = S(f_1) \cap \dots \cap S(f_r) \cap V(g)$. Let $\tau \in S$. Then if $\mathfrak{p} = \ker\tau$ we have $(g)\, F_n \subset \mathfrak{p}$, and $\tau(f_i) > 0$ for all $i = 1,\dots,r$. Then $\alpha_\tau \in \{f_1 > 0,\dots,f_r > 0 ,g = 0\} = \tilde{S}$. Conversely if $\alpha \in \tilde{S}$ then $g \in \mathrm{supp}\,\alpha$ and there is an order in the quotient field of $F_n/\mathrm{supp}\,\alpha$ in which $f_1,\dots,f_r$ are positive. Thus by 3.8 we have $\emptyset \neq S(f_1,\dots,f_r) \cap V(\mathrm{supp}\,\alpha) \subset S(f_1,\dots,f_r) \cap V(g) = S$.

Finally for the moreover part we prove that the map preserves closedness . Assume that S is closed. Then by 3.3 $\tilde{S}$ is constructible , so by the part just proved, to show that $S = \overline{\tilde{S}}$ it is enough to show that $\tilde{S} \cap M_n = \overline{\tilde{S}} \cap M_n$. Take any neighborhood $S(f_1,\dots,f_r)$ of τ in M_n. Then $\alpha_\tau \in D(f_1,\dots,f_r) \cap \overline{\tilde{S}}$, whence $D(f_1,\dots,f_r) \cap \tilde{S} \neq \emptyset$. Again by the first part of the theorem we get $S(f_1,\dots,f_r) \cap S \neq 0$, whence $\tau \in \overline{S} = S = \tilde{S} \cap M_n$ and we are done.

THEOREM 4.6.

(i) Every semialgebroid set has a finite number of semialgebroid connected components[1].

(ii) The closure of a semialgebroid set is again semialgebroid.

Proof.- Inmediate from 4.5 together with 3.5 and 3.3 respectively.

THEOREM 4.7. (Finiteness Theorem). If S is a closed (resp. open) semialgebroid subset of M_n then S is a finite union of sets of the form $\{f_1 \geq 0, \dots, f_r \geq 0\}$ (resp. $S(f_1, \dots, f_r)$).

Proof.- If S is open then $\tilde{S}$ is open by 4.5. By the compactness of $\tilde{S}$ in the constructible topology we have that $\tilde{S}$ is a finite union of sets of the form $D(f_1, \dots, f_r)$. Thus S is a union of sets of the form $S(f_1, \dots, f_r)$. The closed case follows by taking complementaries.

Finally we point out that all statements in this section remain true when we replace F_n by a quotient $A = F_n/I$ where I is an ideal of F_n.

REFERENCES.

[1] Bochnak, J., Coste M., Roy M.F. :"Géometrie algébrique réelle". To appear in Springer-Verlag

[2] Bourbaki, N. : "Algebre commmutative". Chap. VII, Herman, Paris (1961-65)

(1) We say that a semialgebroid subset X is semialgebroid connected if whenever $X \subset Y \cup Z$, with Y, Z open semialgebroid subsets and $Y \cap Z \cap X = \emptyset$, we have $X \subset Y$ or $X \subset Z$

[3] Coste,M., Roy,M.F., : "La topologie du spectre réel". Contemporary Math. 8 (1982) 27-59)

[4] Delfs,H. : "The homotopy axiom in semialgebraic cohomology". J. reine angew. Math. 355 (1985) , 108-128

[5] Delfs,H., Gamboa,J.M. : "Abstract semialgebraic functions". Preprint (1985) NWF-I Mathematik, Universitat Regensburg

[6] Fernandez,F ,Recio,T, Ruiz,J.: "Generalized Thom's lemma in semianalytic geometry". Bull. Pol. Ac. Math., to appear

[7] Lasalle,G. : "Sur le théoreme des zéros differentiables". Lect. Notes in Math 535 (1975) . Springer-Verlag

[8] Łojasiewicz,S. : "Ensembles semianalytiques". I.H.E.S. 1965

[9] Ruiz,J. : "Cones locaux et completions". C.R.Ac.Sc. París, t. 302 , Séries I, n°2 , 1986

[10] Ruiz,J. : "A going down theorem for real spectra". Preprint 1986. Dpto. de Algebra Univ. Complutense Madrid

[11] Ruiz,J. : "Constructibility of closures in real spectra". Preprint . 1986 Dpto. De Algebra. Univ. Complutense Madrid

[12] Schwartz,N.: "Real closed spaces". Habilitationsschrift,München 1984

[13] Tougeron,J.C.:" Ideaux des fonctions differentiables" Ergebnisse der Mathematik 71 (1972), Springer-Verlag

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