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QUASILINEAR PROBLEMS WITH SINGULARITIES

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We solve here some quasilinear problems with a sum of Dirac masses at the right-hand side. For that purpose, we prove a regularity theorem for nonlinear systems of the Hodge-de Rham type, and we generalize de Giorgi's notion of perimeter to subsets of compacts manifolds.

1. INTRODUCTION:

1.1: Main Result:

This paper is devoted to the study of

$$(1) \qquad -\mathrm{div}(|\nabla u|^{p-2}\nabla u) := Au = \sum_{i=1}^{m} \gamma_i\, \delta(x-a_i) \ ; \ u \to 0 \text{ as } |x| \to \infty$$

where $1<p<\infty$, $a_i \in \mathbb{R}^N$, $m \geq 1$, $N \geq 2$, $\gamma_i \in \mathbb{R}$ and $\sum_{i=1}^{m} \gamma_i = 0$

(see Rem. 1 3).

We say that u solves (1) if $|\nabla u|^{p-1} \in L^1_{loc}(\mathbb{R}^N)$, and (1) holds in the sense of distributions. We build solutions which moreover satisfy

$$(2) \qquad \begin{cases} u \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_m\}) \\ u - \sum_{i=1}^{m} \gamma_i \, \varphi(x-a_i) \in L^\infty(\mathbb{R}^N) \end{cases}$$

where φ is a radial solution of $A\varphi = \delta$ in $\mathbb{R}^N$ namely

$$(3) \qquad \varphi(x) = \begin{cases} C(N,p) \; |x|^{(p-N)/(p-1)} & \text{if } p \neq N \\ C(N,N) \; \mathrm{Log}(1/|x|) & \text{if } p = N \end{cases}$$

$$(4) \qquad \begin{cases} C(N,p) = (p-1)(N-p)^{-1}(N\omega_N)^{-1/(p-1)} \\ C(N,N) = (N\omega_N)^{-1/(p-1)} \qquad \omega_N = \mathrm{vol}(B^N) \end{cases}$$

It turns out that solutions of (1) may be "characterized" by their local behaviour. More precisely our main result reads

<u>THEOREM I:</u> (1) <u>has a unique solution satisfying</u> (2)

<u>Remark 1.1:</u> The methods of the paper may be adapted to the case of bounded domains with prescribed (bounded) boundary values

Remark 1.2: Equation (1) is related to some models of quark confinement discussed e.g. in Adler - Piran [1].

Remark 1.3. The condition " $\sum_i \gamma_i = 0$ " can be omitted when p<N (see Rem. 2.3).

1.2 Methods:

The proof of Theorem 1 will be broken up into 3 cases each of which has necessitated special adapted tools. The most delicate cases are those when p<N and p=N because the problem does not have a variational structure.

i) p<N: We consider some particular "approximate" versions of (1) for which solutions may be estimated by symmetrization methods. We then use regularity estimates (in $C^{1,\alpha}$) which are proved in Part 3. These regularity results were motivated by, and extend, earlier work of Uhlenbeck, Evans, Tolksdorf,Uraltseva and others ([6,14,13,15] and earlier references therein). The novelty of our results lies in that they pertain to systems on differential forms, which do not contain pure "power-like" nonlinearities.

ii) p=N: Using conformal invariance we shift to the corresponding problem on S^N. We then symmetrize functions defined on the sphere so as to recover estimates similar to those used for p<N. We shall need to

define the perimeter of a measurable subset of a manifold (here S^N), in the spirit of the work of de Giorgi [5]. This construction is detailed in the Appendix.

iii) <u>p>N:</u> Here the problem admits a variational structure, on the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ (: completion of $\mathcal{D}(\mathbb{R}^N)$ for $\|\nabla u\|_{L^p}$). We also indicate very briefly how to use a similar argument on more general nonlinearities.

1.3 Organization of the text:

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2. PROOF OF THEOREM 1:

2.1 First case : $p<N$:

The proof is broken into 4 steps.

Step 1: Approximate problem: Define u^ε for ε small enough by

$$(5)\qquad \begin{cases} Au^\varepsilon = \chi^\varepsilon \\ u^\varepsilon \in W_0^{1,p}(B(0,1/\varepsilon)) \end{cases}$$

where

$$\chi^\varepsilon = \sum_{1\leq i\leq m} \gamma_i(\omega_N\varepsilon^N)^{-1}\chi_{B(a_i,\varepsilon)}$$

(χ_E denotes the characteristic function of E for any measurable set E).

Define the radial function φ^ε by

$$(6)\qquad \varphi^\varepsilon(x) = \begin{cases} \varphi(x) & \text{for } |x| > \varepsilon \\ c_\varepsilon\, |x|^{p/(p-1)} + b_\varepsilon & \text{for } |x| \leq \varepsilon \end{cases}$$

the constants b_ε, c_ε being adjusted so that $\varphi^\varepsilon \in C^1(B(0,1/\varepsilon))$, φ being the function defined in the introduction.

It is readily seen that:

$$(7)\qquad \begin{cases} \chi^{\varepsilon} \longrightarrow \sum_{1\leqslant i\leqslant m} \gamma_i\, \delta(x-a_i) \\ \\ A\varphi^{\varepsilon} = (\omega_N \varepsilon^N)^{-1}\, \chi_{B(0,\varepsilon)} \longrightarrow \delta(x) \\ \\ \min(\varphi, \varphi(\varepsilon)) \leqslant \varphi^{\varepsilon} \leqslant \varphi \qquad \text{over } B(0,1/\varepsilon) \end{cases}$$

Thus, u^{ε} is expected to "tend" to a solution of (1).

Step 2: Limiting process: Extend u^{ε} to $\mathbb{R}^N$ by setting $u^{\varepsilon} = 0$ for $|x| \geqslant 1/\varepsilon$. We need the following estimate:

LEMMA 2.1: $\exists \alpha$; $\forall K \subset\subset \mathbb{R}^N \setminus \{a_1, \dots, a_m\}$, $\exists C_K$

$$\| u^{\varepsilon} \|_{C^{1,\alpha}(K)} \leqslant C_K.$$

Assume for the moment that this lemma has been proved. In that case, u^{ε} is uniformly bounded on every annular domain around any a_i. Therefore, by the maximum principle on small balls around the points a_i, $u^{\varepsilon} - \gamma_i\, \varphi^{\varepsilon}(x-a_i)$ is bounded (independently of ε) on such balls. On the other hand there is a sequence $\varepsilon_k \to 0$ such that u^{ε_k} tends pointwise on $\mathbb{R}^N \setminus \{a_1, \dots, a_m\}$ to an element u of $C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_m\})$. This u thus satisfies

$$(8)\qquad \begin{cases} Au = 0 \quad \text{in } \mathbb{R}^N\setminus\{a_1,\dots,a_m\}; \quad u(x)=O(\varphi) \text{ as } |x|\to\infty \\ \text{for every } i, \quad u(x) - \gamma_i\,\varphi(x-a_i) \text{ is bounded near } a_i. \end{cases}$$

Indeed, the maximum principle on the exterior of a large ball gives $u = O(\varphi)$ (thus, in particular, u tends to zero at infinity).

That u satisfying (8) gives indeed a solution to (1) is proved in Step 3

Let us now prove Lemma 2.1:

<u>Proof of Lemma 2.1:</u> a) Let $(u^\varepsilon)^*$ be the decreasing rearrangement of $|u^\varepsilon|$. As the rearrangement of χ^ε is a multiple of $A\varphi^\varepsilon$ the estimates of Talenti [9] imply that

$$(9)\qquad (u^\varepsilon)^* \leqslant C\varphi.$$

Now there is a $q > p-1$ such that $\varphi \in L^q(B(0,1))$. Therefore

$$\int_{\mathbb{R}^N} e^{-|x|^2}\,|\varphi(x)|^q\,dx \leqslant C.$$

By a theorem of Hardy-Littlewood,

$$\int_{\mathbb{R}^N} e^{-|x|^2}\,|u^\varepsilon|^q\,dx \leqslant C.$$

This proves that for every $K \subset\subset \mathbb{R}^N$

(10) $$\| u^{\varepsilon} \|_{L^q(K)} \leq C_K.$$

b) Let us now take for K a ball B(a,R) (which does not contain any of the a_i's). We now have Au = 0 on K. Let us estimate

(11) $$\| u^{\varepsilon} \|_{W^{1,p}(B(a,R/4))}$$

First note that it suffices to estimate $\| u^{\varepsilon} \|_{L^p(B(a,R/2))}$. Indeed it is known that, multiplying Au = 0 by $\zeta^p u$, with ζ a suitable smooth function, one can estimate (11).

We now use a classical trick [8]: let β, $k > 0$, $l > k$, $\hat{u} = |u|+k$, $r = q/p > 1/p'$ with $\beta = 1+p(r-1)$, and let

(12) $$\begin{cases} F(\hat{u}) = \begin{cases} \hat{u}^r & \text{if } k \leq \hat{u} \leq l \\ r l^{r-1} \hat{u} - (r-1) l^r & \text{if } \hat{u} \geq l \end{cases} \\ G(u) = \operatorname{sgn}(u) \{F(\hat{u})F'(\hat{u})^{p-1} - r^{p-1}k^{\beta}\} \end{cases}$$

so that $G'(u) = (\beta/r)F'^{p}(\hat{u})$ if $|u| < l-k$, F'^{p} otherwise. We then pick $\zeta \in \mathcal{D}(B(a,R))$ and compute $(Au, \zeta^p G(u))$. This gives, after use of Hölder's inequality,

(13) $$\int_{B(a,R)} \zeta^p |\nabla F(\hat{u})|^p \, dx \leq C \int_{B(a,R)} |\nabla \zeta|^p F(\hat{u})^p \, dx.$$

Now (10) means that $F(\hat{u}) \in L^p(B(a,R))$. So we let $l \to \infty$ and then

$k \to 0$ to obtain (by Sobolev)

(14) $$|u|^r \in L^{Np/(N-p)}(B(a,R)).$$

As $N/(N-p) > 1$ we may iterate this process t times with $p \leqslant q\,[\,N/(N-p)\,]^t$.

Remark 2.1: This lemma remains true for $p \geqslant N$ because (14) then becomes: $|u|^r \in L^s_{loc}(B(a,R))$ for any $s > 1$.

c) The regularity estimates of Part 3 now prove that u^ε is locally bounded in some space $C^{1+\alpha}$.

Lemma 2.1 is proved.

Step 3: Proof of existence completed: We now have to show that u satisfying (8) solves (1). As $\varphi \to 0$ at infinity it will follow that $u - \sum_i \gamma_i \varphi(x-a_i)$ is bounded. It therefore suffices to prove

LEMMA 2.2: If $u \in C^1(B(0,1)\setminus\{0\})$, $Au = 0$ on $B(0,1)\setminus\{0\}$ and $u - \varphi \in L^\infty(B(0,1))$, then

$$Au = \delta \quad \text{in} \quad \mathcal{D}'(B(0,1))$$

Proof of Lemma 2.2: Let $u_\sigma(x) = u(\sigma x)/\sigma^{(p-N)/(p-1)}$, $1/2<|x|<1$. $Au_\sigma = 0$, and u_σ is bounded so that by Lemma 1, $|\nabla u_\sigma|$ is locally bounded in

$1/2<|x|<1$. This proves that

(15) $$(\nabla u - \nabla\varphi)(x) = o(\nabla\varphi(x)) \quad \text{as } x\to 0.$$

Then compute for ζ, $\eta_r \in C_0^{\infty}(B(0,1))$, $\eta_r = 0$ on $\{|x|< r\}$, 1 on $\{|x| \geq 2r\}$ and $|\nabla\eta_r| \leq C/r$,

$$\int_{B(0,1)} |\nabla u|^{p-2}\nabla u.\nabla\zeta\, dx = \lim_{r\to 0}\int_{B(0,1)} |\nabla u|^{p-2}\nabla u.\eta_r\nabla\zeta\, dx$$

$$= \lim_{r\to 0}\int_{B(0,1)} -|\nabla u|^{p-2}\nabla u.\zeta\nabla\eta_r\, dx$$

$$= \lim_{r\to 0}\int_{B(0,1)} -|\nabla\varphi|^{p-2}\nabla\varphi.\zeta\nabla\eta_r\, dx$$

$$= \zeta(0). \qquad \text{q.e.d.}$$

Step 4: Proof of uniqueness: Let u, v be solutions of (1), (3). We shall prove that $\forall\lambda > 0$, $u+\lambda \geq v$, which clearly proves $u \geq v$, and by symmetry $u\equiv v$.

Let λ, $\rho > 0$, η smooth such that

$$\eta = \begin{cases} 1 & \text{if} \quad \min_{1\leq i\leq m} |x-a_i| > 2\rho \\ 0 & \text{if} \quad \min_{1\leq i\leq m} |x-a_i| < \rho \end{cases} \quad ; \ |\nabla\eta| < C/\rho.$$

Then $\zeta = \eta(v-u-\lambda)^+ \in W^{1,p}(\mathbb{R}^N)$ and is compactly supported. As $\zeta = 0$ near $\{a_1,\dots, a_m\}$, $(\zeta,Au) = 0$ and

$$\int_{\{v\geq u+\lambda\}} \eta(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\ ,\ \nabla u - \nabla v)\ dx\ +$$

(16)

$$+ \sum_{1\leq i\leq m} \int_{\{\rho\leq|x-a_i|\leq 2\rho;v\geq u+\lambda\}} (|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\ ,\ \nabla\eta)(u-v)\ dx = 0.$$

For every i, (15) proves that $|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v = o(|\nabla\varphi|^{p-1}(x-a_i))$ as $x\to a_i$. Therefore as $\rho\searrow 0$ the second term of (16) goes to 0 (recall that u - v is bounded). Thus $\nabla u = \nabla v$ a.e. on $\{u+\lambda\leq v\}$. In other words, $v\leq u+\lambda$.

Theorem 1 is proved (for $p < N$).

<u>Remark 2.2:</u> We never used the property $\sum_i \gamma_i = 0$. Theorem 1 is thus also true for $p<N$ and $\sum_i \gamma_i \neq 0$.

<u>2.2: Second case: p=N:</u>

<u>a)Conformal invariance :</u> In this case, as $|\varphi|\to\infty$ at both 0 and infinity, an estimate such as (9) is clearly impossible ($(u^{\varepsilon})^* \geq 0$, $\varphi(x) = C \operatorname{Log}(1/|x|)$). On the other hand, the problem is conformally invariant and thus equivalent to its analogue on the N-sphere.

More precisely, let $\pi: S^N \to \mathbb{R}^N$ denote the stereographic projection from the North pole. It is easily seen that (1) is equivalent to (S^N being endowed with its customary Riemannian structure)

$$(17)\quad \begin{cases} -\mathrm{div}_{S^N}\left(|\nabla \hat{u}|_{S^N}^{N-2}\nabla \hat{u}\right) = \sum_{1\leq i\leq m} \gamma_i\, \delta(x - \pi^{-1}(a_i)) \\ \hat{u}\,(\text{North}) = 0 \quad ; \quad \hat{u} = u_\circ\pi \end{cases}$$

Remark 2.3: One might think that there is a new singularity at the North pole. This does not occur because if $\hat{A}\,\hat{u}$ is the l.h.s. of (17), $\hat{A}\,\hat{u} = 0$ on $V\setminus\{\text{North}\}$ where V is any neighborhood of the pole, and we know that u is bounded (because $u\to 0$). Therefore $\nabla u \in L^N_{loc}(V)$ if V is small enough and the singularity is removable: u is of class C^1 near the pole (cf. Lemma 2.1 above).

b) Symmetrization on S^N and end of proof: Let us solve (17). We define $\hat{\varphi} = \varphi_\circ\pi$, $\hat{\varphi}^\varepsilon = \varphi^\varepsilon{}_\circ\pi$ so that

$\hat{A}\,\hat{\varphi}^\varepsilon = \Lambda(A\varphi^\varepsilon)_\circ\pi$ where $\Lambda(x)$ is the Jacobian of π. It is therefore sufficient to prove that solutions of

$$\hat{A}\,u = f \quad \text{on } S^N \quad ; \quad \|f\|_{L^1} \leq C$$

with $f \in C^\infty$ satisfy an L^q bound for some $q > N-1$. Indeed such a bound will enable us to obtain a $C^{1+\alpha}$ local bound on solutions of approximate problems, just as in the case $p<N$.

Now it is enough to consider solutions of

$$(18)\qquad A_\varepsilon\, u := -\mathrm{div}_{S^N}\left((\varepsilon^2+|\nabla u|^2)^{(N-2)/2}\nabla u\right) = f \quad ;$$

indeed, if u_ε , solution of (18), satisfies an L^q bound (uniform in ε), we may pass to the limit as $\varepsilon \to 0$ (using the fact that u_ε is bounded in $C^{1+\alpha}$ in terms of, say, the L^∞ norm of f)-similarly, one may approximate functions such as χ^ε by smooth ones. Let us therefore shift our attention to (18). Standard results now show that u is itself smooth. Its regular values are therefore dense, and for t regular value of u,

$$H^{N-1}(\{u=t\}) = P(\{u>t\})$$

by equation (33) of the appendix). Federer's co-area formula gives.

(19) $$-d/dt(\int_{u>t} |\nabla u|\, dx) = P(\{u>t\}) \qquad \text{a.e.}$$

(moreover, the r.h.s. is in $L^1(\mathbb{R})$). Let

$$\Phi(t) := \int_{u>t} (\varepsilon^2 + |\nabla u|^2)^{(N-2)/2} |\nabla u|^2 = (A_\varepsilon u , (u-t)^+)$$

Φ is nonincreasing and one has a.e.

(20) $$-\Phi'(t) \leq \int_{u>t} f \leq \|f\|_{L^1}$$

(To prove (20), just consider $(\Phi(t+h)-\Phi(t))/h$).

Let us now define the symmetrization of u as follows.

$$(21)\quad \begin{cases} \text{if } \mu(t) = \text{mes}(u>t) \quad \text{and} \quad 0 \leqslant s \leqslant |S^N| = \int_{S^N} 1\, dV, \text{ let} \\ \\ u^*(s) := \inf\{t \in \mathbb{R}\,;\, \mu(t) \leqslant s\}. \end{cases}$$

Note that u is not assumed to be nonnegative. u and u^* are equimeasurable,and for F Borel, $\int_{S^N} F(u)\, dV = \int_0^{|S^N|} F(u^*)\, ds$ (adapt the proof of the corresponding result in $\mathbb{R}^N$).

We are now going to estimate u^*. But first of all, we assume, as we may, that u is added a constant in (18) so that

$$\text{mes}(u>0) \text{ and } \text{mes}(u<0) \text{ are both } \leqslant |S^N|/2.$$

Define $C_\varepsilon : \mathbb{R}_+^* \to \mathbb{R}_+^*$, decreasing, by

$$C_\varepsilon((\varepsilon^2 + t^2)^{(N-2)/2}\, t) = 1/t.$$

We check that $C_\varepsilon(s) \geqslant s^{-1/(N-1)}$ and that C_ε is convex on $\mathbb{R}_+^*$

Jensen's inequality now gives

$$(22)\quad C_\varepsilon\left[\frac{d/dt(\int_{u>t} (\varepsilon^2+|\nabla u|^2)^{(N-2)/2}|\nabla u|^2\, dV}{d/dt(\int_{u>t} |\nabla u|\, dV)}\right] \leqslant \frac{\mu'(t)}{d/dt(\int_{u>t} |\nabla u|\, dV)} \text{ a.e.}$$

[Proof: replace d/dt by a difference quotient and use Jensen's inequality].

Inserting (19) and (20) into (22) now gives:

(23) $$-\mu' \geq C\mu^{1-1/N} C_\varepsilon(C/\mu^{1-1/N}) \geq C\mu.$$

μ is nonincreasing, so is its singular part w.r. to the Lebesgue measure dt. We may thus integrate (23) for $t \geq 0$:

If $t \geq 0$, $\quad t \leq C\, \mathrm{Log}(\mu(0)/\mu(t)) \leq C\, \mathrm{Log}\,(|S^N|/2\mu(t))$

so that if $s < |S^N|/2$,

$$0 \leq u^* \leq C\, \mathrm{Log}\,(|S^N|/2s).$$

Similarly, one obtains a bound on the negative part of u^*. Finally, we see that $\|u\|_{L^q(S^N)}$ is bounded for any $q>1$, in terms of $\|f\|_{L^1}$ only, uniformly in ε. q.e.d.

The proof of Theorem in this case is now completed as before.

Remark 2.3: We may treat the case when $\sum_i \gamma_i \neq 0$ as well: performing an inversion,one may create a new singularity with $-\sum_i \gamma_i$ as coefficient. This brings us back to the case we have just treated. We may thus consider that if $\sum_i \gamma_i \neq 0$, there is one more singularity, "at infinity".

2.3: Third case: p>N:

a) Existence: We first solve

(24) $$\begin{cases} Au_R = \sum_{1 \leq i \leq m} \gamma_i\, \delta(x-a_i) & \text{for} \quad |x| < R \\ u_R = 0 & \text{for} \quad |x| = R \end{cases}$$

by minimization of $p^{-1}[\int_{B_R}|\nabla u_R|^p\,dx - \sum_{1\leq i\leq m}\gamma_i u_R(a_i)]$ over $W_0^{1,p}(B_R)$.

Let $\eta\in\mathcal{D}(\mathbb{R}^N)$, $\eta = 1$ near $\{a_1,\ldots,a_m\}$, supported by B_{R_0}, with $R_0 \leq R$, fixed. Let also $\hat{u}_R = |B_{R_0}|^{-1}\int_{B_R} u_R(x)\,dx$.

$$\int_{B_R}|\nabla u_R(x)|^p\,dx = \langle \mu\ ,\ \eta(u_R-\hat{u}_R)\rangle$$

because $\sum_{1\leq i\leq m}\gamma_i = 0$ (this is the only place where we use this assumption). This quantity is estimated by Poincaré's inequality on B_{R_0}:

$$\int_{B_R}|\nabla u_R|^p\,dx \leq C\,\|\,\eta(u_R-\hat{u}_R)\,\|_{W_0^{1,p}(B_R)}$$

$$\leq C\,(\|\nabla u_R\|_{L^p(B_R)} + \|\,u_R-\hat{u}_R\,\|_{L^p(B_R)})$$

$$\leq C\,\|\nabla u_R\|_{L^p(B_R)}$$

Therefore, $\|\nabla(u_R-\hat{u}_R)\|_{L^p(B_R)}$ is uniformly bounded. Now, by regularity, we obtain a bound on the modulus of continuity of ∇u_R on every ball which does not contain any of the points a_i. We can then pass to the limit $R\to\infty$, to obtain a solution u of $Au = 0$ on $\mathbb{R}^N\setminus\{a_1,\ldots,a_m\}$, and as u converges in C^1 on a large sphere containing all singularities, u_R can be compared to the solution of $Au = \sum_{1\leq i\leq m}\gamma_i\,\delta(x-a_i)$ with u (restricted to that sphere) as boundary value. Thus u satisfies (1). That u tends to a limit at infinity (which we can assume to be 0) follows from Harnack's inequality (as in [10] for instance).

b) Uniqueness: If u and v are solutions of (1),(3), then for every $\lambda>0$, $(u-v-\lambda)^+$ is compactly supported. One then argues as in the case $p<N$.

Remark 2.4: The reason why singularities are characterized by their "growth" may be seen in various different ways: let u satisfy $Au = 0$ in $B(1)\setminus\{0\}$.

- If $u = O(\varphi)$ as $x\to 0$, then $|\nabla u|^{p-1} \in L^1$ and there is a constant c such that $Au = c\delta$. Indeed, it is classically shown (see e.g. [3]) that if D is a vector field in L^1, and if $\operatorname{div} D = 0$ in $B(1)\setminus\{0\}$, then $\exists$ c such that $\operatorname{div} D = c\delta$.

- If $u = O(\varphi)$, finer scalings prove that for some c, $u \sim c\varphi$, so that $u-c\varphi \in L_{loc}^{\infty}$, $Au = c\delta$. The argument runs as follows: let $u_\sigma(x) = u(\sigma x)/\sigma^{(p-N)/(p-1)}$, $c = \limsup_{x\to 0} u/\varphi$. There is $\sigma_n\to 0$, and a sequence (x_n) of unit vectors such that $x_n\to x_\infty$ $(|x_\infty|=1)$, $u_{\sigma_n}(x_n)\to c\, C(N,p)$. Restrict u_σ to an annulus; modulo extraction, $u_{\sigma_n} \to v \leq c\varphi$ with equality at x_∞. By the strong maximum principle, $v\equiv c\varphi$ (see [10]); and the maximum principle on annular domains gives $u/\varphi \to c$ as $x\to 0$. Therefore, $\forall\varepsilon>0$, $u\leq(\gamma+\varepsilon)\varphi+C$ and thus $u-c\varphi \in L_{loc}^{\infty}$.

The case $p=N$ requires a second scaling of the type $\hat{u}_\sigma = u - C \operatorname{Log} \sigma$. At an isolated singularity, $u=c\varphi + \gamma + o(1)$ (γ=const.).

- That $u-c\varphi \in L_{loc}^{\infty} \Rightarrow Au = c\delta$ can be given yet another proof based on capacity estimates [7,8]. Tracing the constants in Serrin's results gives the conclusion (without using regularity estimates).

Remark 2.5: One might also have minimized directly $[\int |\nabla u|^p/p - \langle \mu,u \rangle]$

with $\mu=\sum_i \gamma_i\, \delta(x-a_i)$ over $\mathcal{D}^{1,p}(\mathbb{R}^N)$. The point is that we are then dealing with functions modulo constants. A similar approach is valid in more general cases, replacing Sobolev by Orlicz-Sobolev spaces.

Remark 2.6: After this work was completed, we learned that Boccardo and Gallouët have obtained some existence results for $Au = \mu$ bounded measure, on bounded domains [4]. Their result differs from ours in that they do not obtain sharp regularity results and do not study the precise behaviour of solutions at a singularity.

3. REGULARITY RESULTS:

The purpose of this part is to extend some known regularity results. We consider the following type of equation:

$$(P)_0 \qquad\qquad - \operatorname{div}\,(\rho(|\nabla u|^2)\, \nabla u) = 0$$

on, say, the unit ball. $\rho: \mathbb{R}_+ \to \mathbb{R}_+$ is not assumed to satisfy any homogeneity condition but rather some growth condition. For the sake of simplicity, we shall restrict ourselves to $\rho(t) = \mathrm{Log}(t+1)$ but one might have considerd much more general nonlinearities with $\rho(0+) = 0$. The argument works in particular for $\rho(t) = t^{\sigma}$, $\sigma > 0$ (see Uhlenbeck [14]). For $-1/2 < \sigma < 0$, the same result holds, and we prove it by a duality argument, given at the and of this section.

We rewrite the equation as a system on differential forms: $\omega = du$ solves:

$$(P) \qquad d\omega = 0 \quad ; \quad \delta(\rho(|\omega|^2)\omega) = 0 \qquad \text{in } B(0,1),$$

where d and δ denote exterior differentiation and codifferentiation.

THEOREM 2: *There exists an* $\alpha > 0$ *such that if*

$$\int_{B(1)} \rho(|\omega|^2)|\omega|^2 \, dV \leqslant a$$

and ω *solves* (P), *then*

$$\|\omega\|_{C^{0,\alpha}(B(0,1/2))} \leqslant C(a).$$

In other words, u *solution of* $(P)_0$ *is locally of class* $C^{1,\alpha}$.

N.B. We shall never use the fact that the degree of the differential form ω is 1.

Proof of Theorem 2: The argument breaks into two steps. The first is an L^∞ estimate, the second is the Hölder estimate. We are adapting here the argument of [14].

First Step: L^∞ estimate:

Let $Q = |\omega|^2$, and define H by $H(0) = 0$ and $H'(t) = \rho(t) + 2t\rho'(t)$. Note that $\rho/(\rho+2t\rho')$ is bounded above and below by positive constants. Write ρ for $\rho(Q)$ and similarly for H, ρ',... .

LEMMA 3.1: i) There is a uniformly elliptic operator with bounded coefficients, $L = -\partial_\alpha(a^{\alpha\beta}(x)\,\partial_\beta\ .\ )$, such that

$$L(H(Q)) \leqslant -C\rho\,|\nabla\omega|^2$$

ii) $\sqrt{H} \in H^1_{loc}(B(1))$.

This lemma, together with the weak Harnack inequality, shows that H (and thus Q, and ω) is bounded.

Proof of Lemma 3.1: (Sketch) i) Compute $(\omega, \partial_i\partial^i(\rho\omega))$ and recall that $-\partial_i\partial^i = d\delta+\delta d$. One obtains

i) with

$$a^{\alpha\beta} = \delta^{\alpha\beta} - (\rho'/H')b^{\alpha\beta}$$

where b is quadratic in ω and does not depend on ρ

ii) The idea is of course to "differentiate the equation" We shall use a differential quotient method. But we must first show that $d\omega = 0$ implies (locally) that $\omega = d\varphi$ where φ has weak derivatives which may be estimated in terms of $\int_{B(1)} \mathrm{Log}\,(|\omega|+1)|\omega|^2\,dx$. This in turn necessitates the extension of the Calderon-Zygmund inequality to the relevant Orlicz space, which is straightforward in this example. If $\Delta_{h,i}$ denotes the i^{th} differential quotient of step h, one then obtains ii) by writing (η being smooth, compactly supported in B(1))

$$(\eta^2\Delta_{h,i}\varphi, \Delta_{h,i}\,\delta(\rho\omega)) = 0$$

in which one lets $h \to 0$.

Second step: Hölder estimate.

Let us first introduce some notation:

$F(\omega) = \rho(Q)\omega \quad ; \qquad K(\omega) = \sqrt{\rho(Q)}\, \omega.$

We now prove a lemma which asserts, grossly speaking that either $\omega(0) = 0$ and ω is Hölder continuous at 0, or ω is close to a nonzero constant form - which by Lemma 3.3 below, again implies Hölder continuity.

<u>LEMMA 3.2</u> <u>Let</u> $\lambda \in (0,1)$, $Q \leqslant M$, $M(r) := \sup_{|x|\leqslant r} |\omega(x)|^2$. <u>Then, there is a</u> $C > 0$ <u>such that for every</u> $r > 1/4$, <u>one of the following holds</u>:

i) $M(r) \leqslant (1-\lambda)\, M(4r)$,

ii) $\exists\, \omega_0 \,;\, |\omega_0|^2 \leqslant M$,

$$\int_{B(r)} \rho(|\omega_0|^2)\, |\omega-\omega_0|\, dx \leqslant Cr^N \lambda\, M(4r)\, \rho(M(4r))$$

<u>and if</u> $Q_0 = \rho(|\omega_0|^2)$, $\rho(Q_0)Q_0 \geqslant \nu(\lambda)\, M(4r)\, \rho(M(4r))$ <u>with</u> $\nu(0+) = 1$.

<u>Proof of Lemma 3.2:</u> Assume that i) is false. By change of variables, we may assume $r=1/4$. Let $M_1 = M(1/4)$. For the first part of ii), use the fact that $H(M)-H(Q)$ is a <u>supersolution</u> of an elliptic operator, so that its inf, viz. $H(M)-H(M_1)$, is bounded below by $C\int_{B(3/4)} (H(M)-H)\, dx$. On the other hand, by Poincaré, if $⨍_{B(1/4)} K(\omega) = K(\omega_0)$,

(25) $$\int_{B(1/4)} |K(\omega)-K(\omega_0)|^2\, dx \leqslant C\int_{B(1/4)} |\nabla K(\omega)|^2\, dx$$

and $|\nabla K(\omega)|^2 \leqslant C\rho\, |\nabla\omega|^2$. Now, $\rho\, |\nabla\omega|^2 \leqslant -C\, L(H(Q))$. Multiply this inequality by η^2 (η smooth, compactly supported) and replace terms such as $\int H\Delta\eta^2$ by $\int (H-H(M))\, \Delta\eta^2$ After some tedious calculations, we obtain

(26) $$\int_{B(1/4)} \rho\, |\nabla\omega|^2\, dx \leqslant C \int_{B(1/2)} (H(M)-H(Q))\, dx,$$

and we thus bound

(27) $$\int_{B(1/4)} |K(\omega) - K(\omega_0)|^2\, dx$$

by $C(H(M)-H(M_1)) \leqslant C(M-M_1)\rho(M)$.

One then shows that $\rho(Q_0) \leqslant |K(\omega)-K(\omega_0)|^2 / |\omega-\omega_0|^2$ and as i) is false by assumption, $M-M_1 \leqslant \lambda M$.

As for the <u>last part of ii)</u>, we have:

$$\begin{aligned}
&\int_{B(1/4)} (\sqrt{(M\rho(M))} - \sqrt{(Q_0\rho(Q_0))})\, dx \\
&\qquad \leqslant \int_{B(1/4)} [(\sqrt{(M\rho(M))} - \sqrt{(Q\rho(Q))}) + (\sqrt{(Q\rho(Q))} - \sqrt{(Q_0\rho(Q_0))})]dx \\
&\qquad \leqslant \int_{B(3/4)} [C(H(M)-H(Q))/\sqrt{(M\rho(M))} + |K(\omega)-K(\omega_0)|]dx \\
&\qquad \leqslant C\, H(M)\, (\lambda+\sqrt{\lambda}) \,/\, \sqrt{(M\rho(M))} \;\leqslant\; C(\lambda+\sqrt{\lambda})\sqrt{(M\rho(M))}
\end{aligned}$$

by the preceding estimates. Therefore, $\sqrt{(Q_0\rho(Q_0))}/\sqrt{(M\rho(M))} \to 1$ as $\lambda \to 0+$.

The Lemma is proved.

Now take λ with $\lambda/\nu(\lambda) < \varepsilon$ and such that

$$\rho(Q_0)Q_0 > \nu(\lambda)M\rho(M) \;\Rightarrow\; Q_0 > \eta M,$$

where η and ε are defined in the next Lemma. Then apply the preceding lemma to $r = 4^{-i}$. Either i) is true for every integer i, and $\omega(0) = 0$, $|\omega(x)| \leqslant C|x|^\theta$ for some $\theta > 0$, or ii) is true for $i = i_0$ and the following can be appealed to:

LEMMA 3.3: There exists $\varepsilon, \nu > 0$ such that if ω is a solution of (P) on $B(1)$ and if ω_0 is a constant form with $|\omega_0|^2 < M = \sup_{B(1)} |\omega|^2$, then

i) $\int_{B(1)} |\omega-\omega_0|^2\,dx \leqslant \varepsilon M$ and

ii) $|\omega_0|^2 \geqslant \nu M$

imply that $|\omega(x) - \omega(0)| \leqslant C\sqrt{M}\,|x|^{1/2}$ on $B(1/2)$.

This clearly ends the proof of Theorem 2.

Proof of Lemma 3.3: The idea is similar to that of [14], but one must check each estimate in the present setting. We just outline the construction. We write as in Step 1, $\omega = d\varphi$; $\omega_0 = d\varphi_0$ and define ψ as the solution of the linearization of

$$\begin{cases} \delta(\rho d\varphi) + d\delta\varphi = 0 & \text{on } B(1), \\ \quad \varphi \text{ prescribed on } \partial B(1) \end{cases}$$

at φ_0. (This is relevant to our problem because we may choose $\delta\varphi = 0$). We ask that $\psi = \varphi-\varphi_0$ on $\partial B(1)$. Then let $\widetilde{\omega} = d\psi$ and $w = \widetilde{\omega}-(\omega-\omega_0)$. One proves that

$$(28) \qquad \int_{B(1)} w^2\,dx \leqslant C\nu^{-2} \left(\int_{B(1)} |\omega-\omega_0|^2\,dx\right)^{1+\beta}$$

for some positive β. This is the basic estimate.

Now let $\omega_1 = \omega_0 + \fint_{B(1/2)} \widetilde{\omega}(x)\,dx$ and define recursively ω_{i+1} from ω_i, as ω_1 was defined from ω_0, but replacing $x \mapsto \omega(x)$ by $x \mapsto \omega(r^i x)$ We then observe that ω_i tends to $\omega(0)$. On the other hand, constructing similar sequences $\omega_{i,x}$, $\omega_{i,y}$ for any x,y in B(1), one gets, if r is fixed and if x, y, satisfy $r^{i+1} \leqslant |x-y| \leqslant r^i$,

$$|\omega(x)-\omega(y)| \leqslant Cr^i \left(\int_{B(1)} |\omega-\omega_0|^2\right)^{1/2}$$

as desired.

This ends the proof of our regularity theorem.

Let us now give a duality theorem which will enable us to extend the preceding results in a very significant way. Indeed, assume tht J(t) is a convex function such that J(0) = 0, $J'(t) = t\rho(t^2)$. Then, if $\bar{J}$ is its conjugate, and $\bar{J}' = t\,\bar{\rho}\,(t^2)$, we see that to any solution of (P), we may associate a solution of ($\bar{P}$): $d\theta = 0$; $\delta(\bar{\rho}(|\theta|^2)\theta) = 0$ by setting

$$\theta = {}^*\, \rho(|\omega|^2)\omega \qquad ({}^* = \text{Hodge duality}).$$

If ($\bar{P}$) satisfies a regularity theorem, and if the assignement $\theta \mapsto \omega$ is (locally) Hölder continuous, one obtains a regularity result for (P) too. In particular,

<u>THEOREM 3:</u> <u>Let</u> p <u>be any real number</u> > 1. <u>If</u> $\int_{B(1)} |\omega|^p < \infty$, $d\omega = 0$, $\delta(|\omega|^{p-2}\omega) = 0$ <u>in</u> B(1), <u>then</u> ω <u>is locally Hölder continuous.</u>

<u>Remark 3.1:</u> In 2 dimensions we notice that this duality correspondence

associates to any "p-harmonic" function (i.e. solution of $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$) a "p/(p-1)-harmonic" function. For p=2, this is nothing but the correspondence between conjugate harmonic functions. Note also that the conjugate of a p-harmonic u of the form $r^{\sigma}g(x/|x|)$ is again of the same form.

APPENDIX: PERIMETER ON MANIFOLDS.

In this section, we propose a method of construction of the perimeter of a general measurable subset of a compact oriented Riemannian manifold $\mathcal{M}$. It generalizes a work of E. de Giorgi [5]. We shall use freely notions of calculus on manifolds, for which we refer the reader to [2] for instance. We just recall that if $\Delta = d\delta+\delta d$ is the Laplacean on k-forms over $\mathcal{M}$, there exists a heat kernel $e_k(x,y,t)$ which is a smooth double k-form (see e.g. Patodi [7]) and satisfies $d_x e_k(x,y,t) = \delta_y e_{k+1}(x,y,t)$, where d_x denotes exterior differentiation w.r. to x and δ_y codifferentiation w.r. to y. Thus, for any bounded k-form θ,

(29) $$\theta(x,t) = \int_{\mathcal{M}} e_k(x,y,t) \wedge * \theta(y)$$

satisfies

$$\theta_t + \Delta\theta = 0 \qquad \text{for } t > 0$$

and, if θ is smooth, $\theta(x,t) \to \theta(x)$ uniformly as $t \searrow 0$

a) Definition of the perimeter:

We first need a lemma.

LEMMA A.1: There is C, depending only on $\mathcal{M}$ such that if $u_0 \in L^\infty(\mathcal{M})$ ("initial value"), $u(t) := e_0 * u_0$, then the function

$$f_{u_0} : \quad t \to \int_{\mathcal{M}} e^{-Ct} |du(t)| \, dV$$

is nonincreasing over $\mathbb{R}+^*$

DEFINITION: $\lim_{t \searrow 0} f_{\chi_E}(t) := P(E) \leq \infty$ is the perimeter of the measurable subset E of

(We more generally call P(u) the limit of $f_{u_0}(t)$ as $t \searrow 0$ for any $u_0 \in L^\infty(\mathcal{M})$.)

Remark A.1: This is exactly de Giorgi's perimeter if $\mathcal{M}$ is replaced by $\mathbb{R}^N$. In that case C can be taken equal to 0.

Proof of Lemma A.1: We may assume $u_0 \in C^\infty(\mathcal{M})$ because the heat kernel is smoothing. Let $du = \omega$.

$$d/dt \; u(t) + \Delta u(t) = 0$$

$$d/dt \; \omega(t) + \Delta \omega(t) = 0.$$

Let J be a smooth convex even function, and assume that $J(t) = t\rho(t^2)$ with ρ smooth. Let $Q = |\omega|^2$, H defined by $H(0)=0$ and $H'(t) = \rho(t) + 2t\rho'(t)$. We need the following lemma: (from now on, the letter C stands for a generic constant).

LEMMA A.2: In each local chart (with local coordinates (x^α)), one has, if $\rho=\rho(|\omega|^2)$, $Q=|\omega|^2$

(30) $$C(\omega,-\nabla_\alpha\nabla^\alpha(\rho\omega)) \geqslant -\nabla_\alpha\nabla^\alpha H(Q) + C\min(\rho,\rho+2Q\rho')|\nabla\omega|^2$$

Proof of Lemma A.2: Write $\omega = k!^{-1}\omega_I dx^I$ with $\omega_I = \omega_{i_1 \dots i_k}$, antisymmetric, $|\nabla\omega|^2 = k!^{-1}\omega_{I;\alpha}\omega^{I}{}_{;}{}^{\alpha}$ where ; denotes covariant differentiation.

$$(\omega,\nabla_\alpha\nabla^\alpha(\rho\omega)) =$$

$$= k!^{-1}\{\nabla^\alpha[(\rho\nabla^\alpha\omega_I + 2\rho'(k!)^{-1}\omega^J\omega_{J;}{}^{\alpha}\omega_I)\omega^I] -$$

$$-(\nabla_\alpha\omega^I)(\rho\nabla^\alpha\omega_I + 2\rho'(k!)^{-1}\omega^J\omega_{J;}{}^{\alpha}\omega_I)\}$$

$$= k!^{-1}\nabla_\alpha[(\rho+2Q\rho')\omega^I\nabla^\alpha\omega_I] - \rho\,|\nabla\omega|^2 - 2\rho'(\omega,\nabla_\alpha\omega)(\omega,\nabla^\alpha\omega).$$

Now, taking normal coordinates at x_0, we note that

$$0 \leqslant (\omega,\nabla_\alpha\omega)(\omega,\nabla^\alpha\omega) \leqslant Q\,|\nabla\omega|^2.$$

As $2\omega^I\nabla^\alpha\omega_I = k!\,Q_{;}{}^{\alpha}$, (30) follows, by distinguishing the cases when $\rho' \geqslant 0$ and $\rho' \leqslant 0$.

End of the proof of Lemma A.1: By Weizenböck's formulae,

(31) $$\Delta\omega = -\nabla^\alpha\nabla_\alpha\omega + \mathcal{R}(\omega)$$

where $|\mathcal{R}(\omega)| \leq \gamma |\omega|$, with a constant γ depending only on curvature. As $\omega_t + \Delta\omega = 0$, for $t > 0$ one has

$$\int_{\mathcal{M}} ((\rho\omega,\omega_t)+(-\nabla^{\alpha}\nabla_{\alpha}(\rho\omega),\omega)+(\mathcal{R}(\rho\omega),\omega))\, dV = 0.$$

Taking (30) into account yields:

$$\int_{\mathcal{M}} C \min(\rho,\rho+2Q\rho')\, |\nabla\omega|^2\, dV + d/dt \int_{\mathcal{M}} J(|\omega|)\, dV \leq C \int_{\mathcal{M}} \rho\, |\omega|^2\, dV.$$

Assume now that $\rho Q \leq CJ(|\omega|)$. We then have

(32) $$d/dt \left(e^{-Ct} \int_{\mathcal{M}} J(|\omega|)\, dV \right) \leq 0$$

We want (32) with "$J(t) = |t|$". Therefore, we set $J_{\varepsilon}(t) = (\varepsilon^2+t^2)^{1/2}$ $\rho_{\varepsilon}(t) = (\varepsilon^2+t^2)^{-1/2}$ and $t^2\rho_{\varepsilon}(t^2) \leq J_{\varepsilon}$. Letting $\varepsilon \searrow 0$ in equation (32) written for $J=J_{\varepsilon}$, we obtain that $t \to \int_{\mathcal{M}} |\omega|\, dV$ is nonincreasing. P is thus well defined.

b) Properties of P:

i) Characterization of functions with measures as derivatives: If u is smooth and if t is a regular value of u, $\{u=t\}$ is a smooth submanifold of $\mathcal{M}$, and one has, if φ^{α} is a smooth vector field,

$$\int_{u>t} \nabla_{\alpha}\varphi^{\alpha}\, dV = -\int_{u=t} \varphi^{\alpha}\nabla_{\alpha}u \,/\, |\nabla u|\, d\sigma$$

where $d\sigma$ is the measure on $\{u=t\}$ induced by the Riemannian structure of $\mathcal{M}$. Thus, $\chi_{\{u>t\}}$ has weak derivatives which are measures. One has:

$$(33)\qquad \|\nabla\chi_{\{u>t\}}\| = \sup_{\varphi_\alpha\varphi^\alpha \leqslant 1} \int_{\{u>t\}} -\nabla_\alpha\varphi^\alpha \, dV = H^{N-1}(\{u=t\}).$$

Thus, $P(\{u>t\})$ represents indeed the perimeter of $\{u>t\}$ in the ordinary sense. This fact is made more precise by the following result:

THEOREM 4: *Let* $u \in L^\infty(\mathcal{M})$. *The following properties are equivalent.*

1. $P(u) < \infty$
2. *All derivatives of first order of* u *are measures.*

In case one of these holds,

$$(34)\qquad \sup_{\varphi^\alpha\varphi_\alpha \leqslant 1} \int_{\mathcal{M}} -u\nabla^\alpha\varphi_\alpha dV = P(u).$$

Proof of Theorem 4: 1. ⇒ 2. It is enough to define $X(u)$ where X is a smooth vector field. Let $X = X^\alpha\partial_a$ in local coordinates $-\partial_\alpha(X^\alpha.)$ is its formal adjoint X^*.

Let φ be a smooth function on $\mathcal{M}$ and $v(t)$ the solution of the heat equation such that $v(0) = u$. Define the measure μ_t by

$$\mu_t \cdot \varphi \rightarrow \langle \varphi, Xv(t) \rangle = \int_{\mathcal{M}} \varphi Xu(t)\, dV.$$

As $P(u) < 0$, we know that $\|Xv(t)\|_{L^1} \leqslant C$. Thus $\exists\, t_n \rightarrow \infty$ such that $\mu_{t_n} \longrightarrow \mu$ (vaguely).

Now,

$$< \varphi , Xv(t_n) > \;\to\; \mu(\varphi)$$

and

$$< \varphi , Xv(t_n) > = < S(t_n)(-X^*\varphi) , u(x) >$$

where S denotes the semigroup of the heat equation. Now, as $t_n \to 0$, $S(t_n)(-X^*\varphi) \longrightarrow -X^*\varphi$ and we have proved that

(35) $$\mu(\varphi) = < -X^*\varphi , u > \qquad \text{q.e.d..}$$

Note that $\int|\mu| \leqslant P(u)$.

<u>2. ⇒ 1.</u> We know here that du is a 1-form with measures as coefficients (a "current of order 0").Let $v(t) = S(t)\,u$. If $t>0$,

$$v(t)(x) = \int_{\mathcal{M}} e_0(x,y,t) \wedge_y {}^* u(y)$$

$$dv(t)(x) = \int_{\mathcal{M}} de_0(x,y,t) \wedge_y {}^* u(y)$$

$$= \int_{\mathcal{M}} \delta e_1(x,y,t) \wedge_y {}^* u(y).$$

If $\varphi = \varphi_\alpha dx^\alpha$ is a 1-form such that $\varphi^\alpha\varphi_\alpha = 1$,

$$< dv(t),\varphi > = \int_{\mathcal{M}} (\varphi(x),\int_{\mathcal{M}} (\delta e_1(x,y,t),v(y))\, dV(y)\,)\, dV(x)$$

$$= < \delta e_1(x,y,t),\varphi\otimes u >$$

$$= < \delta < e_1(x,y,t),\varphi(x) > , u(y) > = < S(t)\varphi,du >$$

(u being L^∞,Fubini applies).

Thus,

$$\|dv(t)\|_{L^1} = \sup_{|\varphi| \leq 1} < dv(t),\varphi > \leq \|\varphi\|_{L^\infty} \int_{\mathcal{M}} |du|$$

(where $\int_{\mathcal{M}} |du|$ stands for $\sup_{|\varphi| \leq 1} < du,\varphi >$). This shows that $\|dv(t)\|_{L^1}$ is bounded and that its limit P(u) is $\leq \int_{\mathcal{M}} |du| < \infty$. q.e.d..

ii)Poincaré-type inequality.

THEOREM 5: There is a constant S such that if $u \in L^\infty$,($N'=N/(N-1)$)

(36) $$\inf_{c\in\mathbb{R}} (\int_{\mathcal{M}} |u-c|^{N'})^{1/N'} \leq S\, P(u).$$

Remark A.2: This gives an isoperimetric inequality if $u = \chi_E$.

Proof of Theorem 5: (36) is clear for smooth u. Let, for $u \in L^\infty$, $v(t) = S(t)\, u$. Take $(t_n)\searrow 0$. We have:

$$P(v(t_n)) \geq S\, (\int_{\mathcal{M}} |v(t_n)-c(t_n)|^{N'}\, dV)^{1/N'}.$$

u being bounded, we may assume that $c(t_n) \to c_\infty$. On the otherhand, $v(t_n) \to u$ in L^2 and also in $L^{N'}$ (because u is bounded). Moreover, by definition, $P(v(t_n))\nearrow P(u)$. This proves (36).

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