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THE CONFORMAL STRUCTURE OF RIEMANN SURFACES WITH BOUNDARY PARAMETRIZING MINIMAL SURFACES

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Let F denote a surface with boundary ∂F, being contained in a Riemann surface R, such that $R \setminus F$ is somedisk. If we vary the boundary curve ω_0 parametrizing ∂F, we will get a manifold Ω of real dimension 6g-3, such that any $\omega \in \Omega$ bounds some F_ω and any local deformation $\tilde{F}$ of F is conformally equivalent to just one F_ω for $\omega \in \Omega$.
This result also implies that none of the conformal invariants of R will be an invariant of this F, since its neighbors $\{F_\omega | \omega \in \Omega\}$ cover all possible deformations of F at all.

It is the purpose of the present paper to present a local model for classes of Riemann surfaces with boundary if conformal isomorphisms are factored out.

We want to imitate an approach of R. Courant [4], who studies parallel slit domains as a class of normal domains under the relation of the conformal equivalence among multiply connected subdomains of the complex sphere $\mathbb{P}^1$. Therefore we fix a compact Riemann surface R of genus g and study subdomains $F \subset R$, which have smooth boundary and $R \setminus F$ is a disk. Then conformal equivalence is a nontrivial relation.

Our result is a local one, we construct a manifold Ω of real dimension 6g-3 such that any $\tilde{F}$ near F is conformally equivalent to one point there. This may be surprising, it states that any $F \subset R$ has "forgotten" all conformal invariants which belong to the manifold R.

Namely, if $G = F \cup \bar{F}$ and $\tilde{G} = \tilde{F} \cup \tilde{\bar{F}}$ denote the Schottky doubles of F and $\tilde{F}$, then our model Ω can be implemented very formally into the Teichmüller space $T(2g)$ of all compact surfaces of genus 2g, which has complex dimension 6g-3 (see e.g. [7]). From the results of Douady and others (see [5]) it is clear, that the symmetric Riemann surfaces as for example $G = F \cup \bar{F}$, is a submanifold $S \subset T(2g)$ of real dimension 6g-3. But then any point of that manifold S will correspond to a point in Ω from our construction.

Naturally, our present construction is purely local, and if their would exist an elaborated version of the index theorem of Atiyah and Singer (as loosely announced in [2]) for manifolds with boundary, e.g. for $F \subset R$, then the dimension 6g-3 would be an easy consequence of this theorem. But here we try to give a very elementary and selfcontained version of how to compute the index of the Cauchy-Riemann operator under Plateau boundary conditions. This will easily allow an application to the index formula of Böhme and Tromba [3], not only for surfaces of the type of the disk but for arbitrary genus.

Our method of proof imitates the paper [3], obtaining a natural manifold structure in the orbit of the reparametrizing group of diffeomorphisms of ∂F, such that the solution set of the classical Plateau problem is a zero set M in a trivial bundle η with the base space Σ of admissible boundary curves.

We thank F. Tomi for his encouragement. A different approach to the present questions was given by Fischer and Tromba [10].

Definition 1:

If R is a compact Riemann surface of genus g, we denote $F \subset R$ any submanifold with boundary which has the following properties: ∂F is a real analytic curve and regularly diffeomorphic to S^1. The manifold $D = R \setminus F$ is conformally equivalent to the unit disc in $\mathbb{C}$.

For the study of smooth functions on F we define several Sobolev spaces fixing the regularity class. We fix two integers r and s, $r \gg s \gg o$. We take any metric on R which is compatible with its conformal structure for the definition of the space $L^2(F,\mathbb{C})$. Then we get

Definition 2:

We define holomorphic functions, holomorphic differential forms and holomorphic vector fields on F:

$A(F) := A^r(F,\mathbb{C}) := \{f : F \to \mathbb{C} \mid f \in H^r(F,\mathbb{C})$ up to the boundary$\}$

$E(F) := E^r(F) := \{\omega : F \to T^*F \mid \omega \in H^r$ up to the boundary$\}$

$V(F) := V^r(F) := \{v : F \to TF \mid v \in H^r$ up to the boundary$\}$.

In principle, we will need a finite atlas of F in order to make all H^r-norms explicitely well defined, but the following argument does the same too.

Lemma 1:

There exist $v_* \in V(F)$ and $\omega_* \in E(F)$ such that $V(F) = A(F) \cdot v_*$ and $E(F) = A(F) \cdot \omega_*$, since v_* and ω_* do vanish nowhere.

Proof:

The manifold with boundary F is contained in an open surface $\tilde{F} \subset R$ and the bundles $T\tilde{F}$ and $T^*\tilde{F}$ are necessarily trivial,[6].

Definition 3:

We define the following spaces of boundary values for functions or vector fields on F:

$$H(\partial F) = H(\partial F,\mathbb{C}) = H^{r-1/2}(\partial F,\mathbb{C})$$

and $$V(\partial F) = V(\partial F,TR) = \{u : \partial F \to TR_{|\partial F} | u \in H^{r-1/2}(\partial F)\} .$$

If $n \in V(\partial F)$ vanishes nowhere and is orthogonal to ∂F in all its points, then the normal bundle N on ∂F can be identified pointwise with $\mathbb{R} \cdot n(p)$, $p \in \partial F$.

If $\Gamma(N)$ denotes the space of $H^{r-1/2}$-sections in the bundle N, then we get $\Gamma(N) \subset V(\partial F,TF)$ and $\Gamma(N) \cong H^{r-1/2}(\partial F,\mathbb{R})$.

Lemma 2:

There exists a continuous well defined linear mapping ext: $H(\partial F,\mathbb{C}) \to H^r(F,\mathbb{C})$ such that $\text{ext}(f) = g$ if $g_{|\partial F} = f$ and $\Delta g \equiv 0$, g harmonic on F.

Proof:

This follows trivially from Dirichlet's principle and regularity theory.

Lemma 3:

There exists a continuous well defined linear mapping ext: $V(\partial F) \to H^r(F,TF)$ with $\text{ext}(u) = v_o \cdot \text{ext}(\frac{u}{v_o})$.

ext(u) is holomorphic if and only if $\text{ext}(\frac{u}{v_o})$ is holomorphic.

Proof:

A multiplier on a holomorphic function gives a holomorphic result nontrivially only if the multiplier is itself holomorphic.

Definition 4:

We define the subspace $V_1 = V_1(\partial F) \subset V(\partial F)$ by the property $\mathrm{ext}(V_1) = \mathcal{V}(F)$. We have a natural projection $r_o : V(\partial F) \to \Gamma(N)$, being an orthogonal projection in any fibre.

Lemma 4:

The projection $r_o : \mathcal{V}(F) \to \Gamma(N)$ is a Fredholm operator.

Proof:

This follows easily from the estimates of Agmon, Douglis and Nirenberg (II) [1]. Namely, near the boundary, the operator r_o is equivalent to an equation for u in a half space $\{z \mid \mathrm{Im}\ z \geq 0\}$, and we get $\frac{d}{d\bar{z}} u \equiv 0$, $\mathrm{Rea}(u) \equiv g$.

This is an elliptic system. We can compute the index of r_o step by step. If for any complex function f the function Rea(f) denotes its real part, we have:

Lemma 5:

The mapping Rea: $A(F) \to H(\partial F, \mathbb{R})$ is Fredholm of $\mathrm{index}_{\mathbb{R}}\ (\mathrm{Rea}) = 1-2g$.

Proof:

The dimension of the kernel of this mapping is 1. The dimension of its cokernel is 2g. Namely the dimension over the reals of the vector space of holomorphic differentials on F which have purely imaginary periods, equals 2g.

Lemma 6:

Let $q : F \to \mathbb{P}^1$ denote a meromorphic function without poles or zeroes on ∂F, and $\mathfrak{q}$ denotes its multiplication operator. Then $\text{Rea} \circ \mathfrak{q} : A(F) \to H(\partial F)$ is well defined and a Fredholm operator with index of $1-2g + 2\#(P-Z)(q)$, where $\#(P-Z)(q)$ denotes the number of poles of q minus the number of zeroes of q.

Proof:

We simply have to check that any pole of q increases the source space with 2 dimensions and any zero of q decreases this dimension again with 2.

Theorem 1:

The index of the projection $r_o : V(F) \to \Gamma(N)$ equals 3-6g. The proof depends on several propositions.

Proposition 1:

If $F \subset R$ is sufficiently large and $\partial F \subset R$ sufficiently close to a circle around a point ζ_∞ in $R \setminus F$, then there exists a meromorphic vector field v_1 on R, which has no poles along ∂F and satisfies $\langle v_1, n \rangle \geq \varepsilon_* > 0$ in any point of ∂F, n being the normal field of definition 3.

Proof:

It is easy to construct a differential $d\chi$ on $\mathcal{R}$, which is meromorphic, and has exactly at ζ_∞ a pole of order 1 and no zeroes or other poles nearby. Then the vector field $v_1 := d\chi^{-1}$ satisfies our proposition, if ∂F is a circle around ζ_∞ and the leading term of the Taylor series of v_1 at ζ_∞ satisfies the inequality well.

Corollary 1:

The vector field $v_1 : F \to TF$ satisfies that its number P of poles minus its number Z of zeroes in F is

$\#(P-Z)(v_1)(F) = 2g-1$.

Proof:

The differential $d\chi$ satisfies trivially $\#(Z-P)(d\chi)(F) = 2g-1$, since one pole of $d\chi$ is outside F.

Proposition 2:

If $\rho_o : V \to H(\partial F, \mathbb{R})$ is defined by $\rho_o(v) := \langle v, v_1 \rangle |_{\partial F}$, then the index of $\rho_o : V \to H(\partial F, \mathbb{R})$ and the index of $r_o : V \to \Gamma(N)$ coincide.

Proof:

Using proposition 1 and the properties of v_1 there, it is easy to construct a homotopy between both operators.

The product $\langle v, v_1 \rangle$ is a real scalar product and can be evaluated equivalently as $\text{Rea}(v \cdot \bar{v}_1)$ in the complex notation for functions along ∂F.

Proposition 3:

If $\rho_1 : V(F) \to H(\partial F, \mathbb{R})$ is defined by $\rho_1(v) := \text{Rea } d\chi(v)$, with $d\chi$ as in proposition 1, then the index of ρ_1 and the index of ρ_o coincide.

Proof:

Since $d\chi = (v_1)^{-1}$, this proposition just rewrites the former one.

Proposition 4:

The index of $\rho_1 : V(F) \to H(\partial F, \mathbb{R})$ equals 3-6g.

Proof:

The index is additive under the composition of Fredholm operators. In general, $\text{Rea } d\chi(v) = \text{Rea } d\chi(fv_o)$, where $v \in V(F)$ and $f \in A(F)$ are not restricted. If we define $q := d\chi(v_o)$, then we know the number of poles and zeroes of q on F.

$\#(P-Z)(q)(F) = 2g-1$. Therefore the index of the operator $\text{Rea} \circ q$ is $1-2g + 2(1-2g) = 3-6g$, from lemma 6.

Proof of Theorem 1:

Under the assumptions of proposition 1 we have constructed a homotopy between the mapping $r_o : V(F) \to \Gamma(N)$ and the mapping $\rho_1 : V(F) \to H(\partial F, \mathbb{R})$, if $\Gamma(N)$ and $H(\partial F)$ are identified. Therefore proposition 4 implies theorem 1 under the assumptions of proposition 1. We only have to generalize its assumptions in the theorem itself. It is easy to construct a homotopy of domains, i.e. a one-parameter family $F(t)$ of domains in R, continuous in t, such that $F(0)$ satisfies proposition 1 and $F(1)$ is arbitrary. Only the definition of the spaces $V_1(\partial F(t))$ is necessary, since these are not always conformally equivalent. But using a family of quasiconformal isomorphisms near identity between these domains $F(t)$, one can construct isomorphisms between these spaces $V_1(\partial F(t))$, depending continuously on t.

Theorem 2:

The kernel of the linear mapping $r_o : V_1(\partial F) \to \Gamma(N)$ has dimension 0, if $g \geq 1$, and dimension 3, if $g = 0$.

Proof:

Any $v \in \ker r_o$, $v \neq 0$, is a holomorphic vector field on F, which is tangential to ∂F at any point of ∂F. If $F \cup \bar{F}$ denotes the Schottky double of F, then v easily can be extended to a vector field on $F \cup \bar{F}$, which cannot exist if $g > 0$. If $g = 0$, these fields are well known.

Definition 5:

We define $K = K^r(F) := \{f : F \to R \mid f$ is holomorphic in F, and is in the class H^r up to the boundary$\}$.

For the explicit study of $K(F)$ we use the following construction.

Lemma 7:

There exists a holomorphic flow on F, but such that Φ maps $[0,1] \times TF$ into F with the properties that $\Phi(0;p,v) = p$, $\dot{\Phi}(0;p,v) = v$ and $v \in T_pF$.

Proof:

We fix some vector field $Y : F \to TF$ without zeroes on F and will study its multiples $Z = \alpha \cdot Y$, $\alpha \in \mathbb{C}$. Naturally, there exists a flow line $\phi(t)$ in R, which solves the differential equation $\dot{\phi} = Y \circ \Phi$ with the initial values $\Phi(0) = p$, $\dot{\Phi}(0) = Y(p)$.

Similarly for any $v \in T_pF$, we have $v = \alpha \cdot Y(p) = Z(p)$, and there is a flow line $\Phi(t) = \Phi(t;p,Z(p))$ solving the differential equation $\dot{\Phi} = Z \circ \Phi$ with the initial values $\Phi(0) = p$, $\dot{\Phi}(0) = Z(p) = v$. This flow is holomorphic on F and on any T_pF.

Proposition 5:

The flow Φ has the following property: for any $p,q \in F$, if dist(p,q) is sufficiently small, there exists a unique $v_q \in T_pF$ such that $\phi(1;,p,v_q) = q$ and a unique $v_p \in T_qF$ such that $\phi(-1;q,v_p) = p$.

The flow line $\phi(t,p,v_q)$ depends holomorphically on p and q.

Proof:

This proposition is a consequence of the implicit function theorem. Or we can apply the construction of [9], chap. 18.3, where a "champ rabatteur" is studied and constructed for the same purpose. If we simply introduce coordinates on F, such that the field Y is represented locally as a constant field, we can interpret our proposition giving a flow for the equation $\ddot{\Phi} \equiv 0$. Clearly we can use the theorem I.6.A in [11]: The family of graphs of solutions of any second-order homogeneous linear equation is locally diffeomorphic to the family of graphs of solutions of the simplest equation $\frac{d^2y}{dx^2} \equiv 0$.

Corollary 2:

For any $v \in \tilde{V}(F)$, denoting an open neighborhood of the zero vector field there is well defined a unique $f \in K(F)$, such that $f = \Phi(1;id,v)$, or $f(p) = \Phi(1;,p,v(p))$ for $p \in F$.

Proof:

If v(p) is sufficiently small, then f is only a perturbation of the identity on R at p, and f is clearly holomorphic in p.

Corollary 3:

For any $f \in K(F)$, if $\sup_{p \in F} \text{dist}(p,f(p))$ is sufficiently small, there exists a unique $v \in V(F)$ such that $f = \Phi(1;\ id,v)$.

Proof:

In appropriate coordinates we can study the flow Φ as a flow $\dot{\Phi}$ = constant or even $\ddot{\Phi} \equiv 0$. But the two-point boundary value problem $\Phi(0) = p$, $\Phi(1) = q$ is trivially solvable and has the stated properties. That f, being holomorphic, produces a vector field v, which is holomorphic again, follows easily from the properties of the holomorphic vector field Y, having no zeroes.

Proposition 6:

The mapping $\Phi := \Phi(1;id,.) : V(F) \to K(F)$ is an isomorphism between a neighborhood $\tilde{V}$ of the zero section in $V(F)$ and a neighborhood $\tilde{K}$ of the identity map in $K(F)$.

Proof:

From proposition 5, we get a flow line $\Phi(t;p,v_q)$ such that $\Phi(0;p,v_q) = p$ and $\Phi(1;p,v_q) = q$, if for given $q = f(p)$ the vector $v_q \in T_pF$ is choosen appropriately. If $q = f(p)$, the vector $v_q \in T_pF$ depends holomorphically not only on q, but on p itself.

Therefore to any $f \in K(F)$ near $id: F \to F$ there corresponds a unique $v \in V(F)$ such that $\Phi(1;id,v) = f$.

Conversely it is clear, that any $v \in V(F)$ produces $f = \Phi(1;id,v) \in K(F)$.

Corollary 4:

$\tilde{K}(F)$ is a manifold with model $\tilde{V}(F)$.

Proof:

The regularity classes are the correct ones and $\pi \circ \Phi(1;id,.)$ produces an isomorphism and induces a manifold structure, since Φ is differentiable. Trivially $(\pi \circ \Phi)_{*,v=0} : V \to V$ is the identity and therefore $T_{id}K(F) = V(F)$.

Definition 6:

We denote $\Sigma = \Sigma^r$ the space $Emb^r(S^1,R)$ of embeddings S^1 into R, which are of class $H^{r-1/2}$ and have a disk as exterior and some F as interior in R.

We denote $\mathcal{D} = \mathcal{D}^s$ the space of diffeomorphisms of S^1, $0 \ll s \ll r$. For all $\sigma \in \Sigma$ we define $\mathfrak{d}\mathfrak{I}(\sigma) = \{\sigma \circ u | u \in \mathcal{D}\}$ and $\eta := \bigcup_{\sigma \in \Sigma} \mathfrak{d}\mathfrak{I}(\sigma)$, such that η is a fiber bundle over Σ.

Lemma 8:

$\mathcal{D}$ is a Hilbert manifold, Σ is a Hilbert manifold, and η is a C^{r-s}-smooth fiber bundle over Σ.

Proof:

All results are very classical, essentially equivalent to results in [3], and we can quote Penot for example.[8].

Lemma 9:

For all $v \in V(\partial F)$ there is well defined $\phi(v) =: \sigma \in \Sigma$, if v is near the zero section.

For all $\sigma \in \Sigma$ there exists a unique $w \in V(\partial F)$, such that $\phi(w) = \sigma$, if σ is not too far away from $\omega_o : S^1 \to \mathbb{R}$, which parametrizes ∂F.

Proof:

We apply proposition 6 and its corollaries. The regularity classes are correct, since the flow is C^∞ up to the boundary.

Corollary 5:

If $\sigma \in \Sigma$ bounds an F_1 such that F_1 is conformally equivalent to F, then σ can be reparametrized to $\sigma \circ u_\sigma, u_\sigma \in \mathrm{Diff}(S^1)$, such that $\sigma \circ u_\sigma$ is boundary value of $f \in K(F)$ and $\phi^{inv}(\sigma \circ u_\sigma) = v \circ u_\sigma \in V_1(\partial F)$.

Proof:

If $\sigma \circ u_\sigma$ is boundary value of $f \in K(F)$, then $\phi^{inv}(\sigma \circ u_\sigma)$ is a holomorphic vector field, or its boundary values, by proposition 7 The action of diffeomorphisms $u : S^1 \to S^1$ on the data does not influence the construction of the flow Φ .

There is a natural uniqueness result for the construction above.

Lemma 10:

If $\sigma \in \Sigma$ is near $\omega_o \in \Sigma$, ω_o the standard parametrization of ∂F, and if u_1 and u_2 are sufficiently close to each other, then it is impossible that $\sigma \circ u_1$ and $\sigma \circ u_2$ are both boundary values of two different holomorphic functions $f_1 : F \to R$ and $f_2 : F \to R$.

Proof:

Otherwise $g := f_1^{inv} \circ f_2 : F \to F$ would be a holomorphic isomorphism of F close to id. Since the group of holomorphic isomorphisms of F is finite with bounded order, g cannot be close to identity.

In the space $\eta = \bigcup_{\sigma\in\Sigma} d\mathcal{I}(\sigma) \subset H^s(S^1,R)$ we study a subspace which gives all boundary curves of conformally equivalent domains.

Definition 7:

We define $M \subset \eta$, $M = \bigcup_{\sigma\in\Sigma_1} d\mathcal{I}(\sigma)$, where Σ_1 contains all boundary curves of domains $\tilde{F}$ near F, $\tilde{F}$ being conformally equivalent to F. Formally, $\Sigma_1 := \{\sigma \in \Sigma | \exists\, u_\sigma \in \mathcal{D}, \sigma \circ u_\sigma$ is the boundary value of some $f \in K(F)\}$.

Proposition 7:

M is a Hilbert submanifold of the bundle η.

Proof:

Obviously M is zero set of $\frac{d}{d\bar{z}} f \equiv 0$ in η under Plateau boundary conditions. Using the inverse exponential map ϕ^{inv} and the extension map ext: $H^{r-1/2}(\partial F,\mathbb{C}) \to H^r(F,\mathbb{C})$ one will be reduced to the boundary condition $\mathrm{Im} \circ (\frac{\partial}{\partial r} - i \frac{\partial}{\partial \phi}) : H^{r-1/2}(\partial F,\mathbb{C}) \to H^{r-3/2}(\partial F,\mathbb{R})$. This conformality operator is a Fredholm operator on any $\mathfrak{C}(\sigma)$ by classical arguments, like in [3]. Therefore its corank is at most finite dimensional. If we admit also variations of σ, the base point, it is easy to obtain a natural surjectivity of $\mathrm{Im} \circ (\frac{\partial}{\partial r} + i \frac{\partial}{\partial \phi})$. Its zero set therefore is a manifold, locally.

Proposition 8:

The projection $\pi : \eta \to \Sigma$ induces a Fredholm mapping $\pi : M \to \Sigma$.

Proof:

If $(\sigma,u_\sigma) \in \mathfrak{C}(\sigma) \subset M$ is such that $\sigma \circ u_\sigma$ is boundary value of a holomorphic mapping, then by regularity also $\tau := \sigma \circ u_\sigma$ is in Σ, and $(\tau,\mathrm{id}) \in M$.

It is enough to study $\pi_{*(\tau,\mathrm{id})} : TM \to T\Sigma$. Obviously, we are now in the situation of theorem 1.

From the construction of $M \subset \eta$ it is clear, that $\pi_{*(\tau,\mathrm{id})} : T_{(\tau,\mathrm{id})}M \to T_\tau\Sigma$ has the same range and corank as $r_o : V(\partial\tilde{F}) \to T(\tilde{N})$, where $\tilde{F}$ is the image of F under the mapping $\mathrm{ext}(\tau) \in K(F)$, and $\tilde{N}$ denotes its normal bundle.

Since to any point in $K(F)$ corresponds an orbit in $M \subset \eta$, all tangential variatios along τ are present trivially in the range of $\pi_{*(\tau,\mathrm{id})}$. And r_o as well as $\pi_{*(\tau,\mathrm{id})}$ produce exactly the same normal variations in the manifold $T_\tau\Sigma$. Since the kernel of r_o is always 0 only, and $\pi_{*(\tau,id)}$ is trivial on all tangential variations, the index of π_* and r_o must coincide.

Theorem 3:

The projection $\pi : \eta \to \Sigma$ induces a nonlinear Fredholm operator $\pi : \mathcal{M} \to \Sigma$ of real index 3-6g. The kernel of π_* is always zero if $g \geq 1$.

Proof:

We only take together proposition 8 and theorem 2. In principle theorem 3 is a statement on the index of the Cauchy-Riemann-system $\frac{d}{d\bar{z}} \equiv 0$ on F with Plateau boundary conditions, and it is natural (see [1],p. 532) that we get a topological result. In the case g=0 holds the same too, classically considered with a three-point-condition.

Corollary 6:

If $(\sigma,id) \in \mathcal{M}$ such that $\sigma : \partial F \to R$ is the boundary value of a holomorphic function $f : F \to R$, there exists near σ in Σ a foliation of real parameter dimension (6g-3), such that any leaf of this foliation represents the orbit of conformal isomorphisms of some $\tilde{F}$ in R .

Proof:

Since $\pi_* : T\mathcal{M} \to T\Sigma$ is injective, and has index m_o , there exists a subspace W_σ in $T_\sigma\Sigma$, which complements the range of $\pi_* : T_{(\sigma,id)}\mathcal{M} \to T_\sigma\Sigma$.

$\Omega := \Phi \circ ext(W_\sigma)$ represents a m_o-dimensional manifold in Σ, such that no $\tau \in \Omega$ can be reparametrized by $\tau \circ u$, $u \in \mathcal{D}$, except the trivial one, for

to get then the boundary value of a mapping from $K(F)$.
On the other hand $\Phi \circ \text{ext} \circ \pi_*(T_{(\sigma,\text{id})}M)$ parametrizes $K(F)$ according to prop. 6 and the construction of M (Lemma 10 and proposition 7). So we get easily a smooth isomorphism between the Hilbert manifolds $K(F)$ and $M = M(F)$.
The manifold Ω is a transverse section of Σ_1 in Σ such that $T\Omega \oplus T\Sigma_1 \cong T\Sigma$. Clearly K, Ω and M admit the action of the group $\mathcal{D}^s$ of reparametrizations without any local change.
$\Sigma_1 \subset \Sigma$ gives the conformal equivalence class of F.

<u>Remark 2:</u>

Since all mappings $f : F \to R$ which are near identity will not cover conformal automorphisms of F, the space $K(F)$ will not forget the markings on F used in Teichmüller theory (see [4]).
Therefore Ω will be a local model for the Teichmüller space of surfaces with boundary near F in R.

<u>Definition 9:</u>

Let $\Theta(F,R)$ denote the quotient space of Σ near ω_o under the action of conformal isomorphisms near identity, where $\alpha \overset{\sim}{\sim} \omega_o$, if $\alpha \in \Sigma_1$, and F denotes the interior of ω_o in R. The symbol F means, that the conformal structure of F will not always be kept, but the conformal structure of R remains.

<u>Theorem 4:</u>

$\Theta(F,R)$ is a real manifold of real dimension 6g-3.

Proof:

By construction in theorem 3 and following remark 2 and corollary 2, the manifold Ω is actually a model for $\Theta(F,R)$.

Theorem 5:

The manifold $\Theta(F,R)$ is a local model for the Teichmüller space of all compact complex curves of genus $2g$, which are symmetric under an involution.

Proof:

If F is fixed, its Schottky double $G = F \cup \bar{F}$ is such a complex curve.

Any $\tilde{F}$ near F produces a Schottky double $\tilde{G}$ near G. The action of conformal isomorphisms will never destroy markings on the surface, if these isomorphisms are produced by $\tilde{K}(F)$ in some fixed R as above. On the other hand, the Teichmüller space T of all conformal equivalence classes of manifolds like G has complex dimension 6g-3. It is obvious from the work of Thurston and Douady [5] that the part $T_1 \subset T$, which gives all surfaces symmetric against an involution, as above, is locally a submanifold of real dimension 6g-3. Therefore T_1 must coincide locally with the space $\Theta(F,R)$ from theorem 4. This proves our statements at the beginning.

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