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ON THE GROWTH OF HOMOTOPY GROUPS

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For a 1-connected space X of finite type and a prime p we define $R_{\pi_*(X;Z/p)}$ and $R_{H_*(\Omega X;Z/p)}$ to be the radius of convergence of the power series

$\Sigma \dim_{Z/p}((\pi_n(X;Z/p) \otimes Z/p)\cdot t^n$ and

$\Sigma \dim_{Z/p}(H_n(\Omega X;Z/p)\cdot t^n$ respectively.

We prove that $R_{\pi_*(X;Z/p)} \geq \min\{R_{H_*(\Omega X;Z/p)}, C_p\}$ where C_p is a constant depending only on p and $C_p \geq \frac{1}{2}$ for all p. We conjecture that $\min\{1,R_{\pi_*(X;Z/p)}\} = \min\{1,R_{H_*(\Omega X;Z/p)}\}$.

0. At present a complete computation of the homotopy groups $\pi_n(X)$ of a finite, 1-connected space X seems to be out of reach. Therefore it appears to be reasonable to ask for qualitative information about $\pi_*(X)$ instead of quantitative information. The basic results in this direction are due to Serre.

a) Let X be 1-connected of finite type (i.e. each $H_n(X)$ is finitely generated) then each $\pi_n(X)$ is finitely generated.

b) Let p be a prime and X a 1-connected finite complex with $\overline{H}_*(X) \otimes Z/p \neq 0$, then $\pi_n(X) \otimes Z/p \neq 0$ for infinitely many n. In fact we even know that $\mathrm{Tor}(\pi_n(X), Z/p) \neq 0$ for infinitely many n ([M-N]).

In this paper we concentrate on the mod-p size of $\pi_n(X)$ or rather the numbers $\dim_{Z/p}(\pi_n(X;Z/p) \otimes Z/p) =: a_n$. Here $\pi_n(X;Z/p) := [P^n(p), X]$, with $P^n(p) := S^{n-1} \cup_p e^n$, are the mod-p-homotopy groups of X $(n \geq 2)$. If p is an odd prime, they are Z/p-vector spaces; for $p=2$ they are $Z/4$-modules ([N]). The growth of the function a_n is measured by the radius of convergence of the power series $\Sigma a_n t^n$. We will use the following notation. For a nonnegatively graded abelian p-group M of finite type let R_M be the radius of convergence of $\Sigma \dim_{Z/p}(M_n \otimes Z/p) \cdot t^n$.

By (b) above we have $R_{\pi_*(X;Z/p)} \leq 1$ if X is a 1-connected finite complex and $\overline{H}_*(X) \otimes Z/p \neq 0$. We will relate $R_{\pi_*(X;Z/p)}$ and $R_{H_*(\Omega X;Z/p)}$ as follows.

<u>THEOREM 1</u>. <u>Let X be a 1-connected space of finite type and p be a prime.</u>
<u>Then</u> $R_{\pi_*(X;Z/p)} \geq \min\{R_{H_*(\Omega X;Z/p)}, C_p\}$ <u>where</u> C_p <u>is a constant depending only on</u> p <u>and</u> $C_p \geq 1/2$ <u>for all</u> p.

If X is a 1-connected finite complex then the Eilenberg-Moore spectral sequence shows that $R_{H_*(\Omega X;Z/p)} > 0$.
This implies the following

<u>COROLLARY</u>. <u>Let</u> X <u>be a 1-connected finite complex and</u> p <u>be a prime. Then there is a constant</u> d <u>such that</u>

$$\dim_{Z/p}(\pi_n(X) \otimes Z/p) \leq d^n \quad \text{for all } n. \qquad \square$$

This corollary is also known to K.Iriye [I 1]. In [H] we conjectured that $R_{H_*(\Omega X;Z/p)} \leq R_{\pi_*(X;Z/p)}$ for all 1-connected finite complexes X and asked whether $R_{H_*(\Omega X;Z/p)} = R_{\pi_*(X;Z/p)}$ for such X. In fact Iriye [I 2] has proved the following

THEOREM 2. $\min\{1, R_{\pi_*(X;Z/p)}\} \leq R_{H_*(\Omega X;Z/p)}$ *for all 1-connected spaces of finite type*. □

This leads us to the following

CONJECTURE. a) Let X be 1-connected of finite type and p be a prime. Then

$$\min\{1, R_{\pi_*(X;Z/p)}\} = \min\{1, R_{H_*(\Omega X;Z/p)}\} .$$

b) If X is a 1-connected finite complex then

$$R_{\pi_*(X;Z/p)} = R_{H_*(\Omega X;Z/p)} .$$

Remarks: 1. Conjecture (a) implies Conjecture (b).

2. Conjecture (a) holds in the case of rational coefficients ([F-T]).

3. If X is a finite complex, $\dim \bigoplus_n \pi_n(X) \otimes Q < \infty$ and p is large, then

$$\Omega X_{(p)} \simeq \prod_\alpha S_{(p)}^{2n_\alpha+1} \times \prod_\beta \Omega S_{(p)}^{2n_\beta+1} \quad [\text{M-W}]$$

and hence (b) holds if it holds for spheres, i.e. if $R_{\pi_*(S^n;Z/p)} = 1$ for all n.

4. Theorem 1 implies that $R_{\pi_*(S^n;Z/p)} \geq 1/2$ thus improving on [B-H], if p=2.

The remainder of this paper is devoted to the proof of Theorem 1. The proof is based on an estimate of the E_2-term of a Bousfield-Kan SS (BKSS). In section 1 we recall the BKSS and reduce Theorem 1 to such an estimate (Prop.1). In section 2 we prove Proposition 1 in case p=2 and in section 3 we indicate the necessary changes if p is odd.

The author would like to thank K.Iriye for several conversations. Part (a) of the conjecture above emerged during such a conversation.

1. With any pointed space Y Bousfield and Kan associate a pointed tower of fibrations $\ldots \to (Z/p)_s Y \to (Z/p)_{s-1} Y \to \ldots$, $s \geq 0$, whose inverse limit is the Bousfield-Kan Z/p-completion, denoted by $(Z/p)_\infty Y$ ([B-K, I.1.-I.4.]). By [M 1, Thm.1.5.] the completion map $Y \to (Z/p)_\infty Y$ induces a homotopy equivalence $map_*(P^2(p), Y) \to map_*(P^2(p), (Z/p)_\infty Y)$ provided Y is nilpotent, e.g. $Y = \Omega X$, X 1-connected. Furthermore applying $map_*(P^2(p), -)$ to the tower of fibrations above gives a new tower of fibrations $\ldots \to map_*(P^2(p), (Z/p)_s Y) \to map_*(P^2(p), (Z/p)_{s-1} Y) \to \ldots$ whose inverse limit is $map_*(P^2(p), (Z/p)_\infty Y)$. This tower gets pointed by taking the constant maps as basepoints.

If Y is connected and nilpotent then the BKSS for $\pi_*(Y; Z/p)$ is the (extended) spectral sequence associated with this pointed tower [BK, Chapter IX]. Its E_2-term is given by

$$E_2^{s,t}(Y) := \mathrm{Ext}^s_{\underline{\underline{CA}}}(\bar{H}_*(P^{t+2}(p)), \bar{H}_* Y) \quad , \quad t \geq s \geq 0.$$

(cp [M1, 1.12.]). Here and in the following $\bar{H}_*$ denotes reduced homology with Z/p-coefficients and $\underline{\underline{CA}}$ is the category of connected commutative unstable coalgebras without counit over the mod p Steenrod algebra.
For $i > 0$ the spectral sequence converges completely to

$$\pi_i \, map_*(P^2(p), (Z/p)_\infty Y) \cong \pi_i \, map_*(P^2(p), Y) \cong \pi_{i+2}(Y; Z/p)$$

provided $\lim\limits_{\leftarrow r}{}^1 E_r^{s,s+i} = \lim\limits_{\leftarrow r}{}^1 E_r^{s,s+i+1} = 0$. ([B-K, IX.5]).

We will use the spectral sequence in the case that $Y = \Omega X$, X 1-connected and of finite type.
Theorem 1 will follow easily from the following

<u>PROPOSITION 1</u>. <u>Let</u> E(X) <u>be the graded</u> Z/p-<u>vector space with</u> $E_n(X) := \bigoplus_{s \geq 0} E_2^{s,s+n}(\Omega X)$. <u>Then</u> $R_{E(X)} \geq \min\{R_{H_*(\Omega X)}, C_p\}$ <u>where</u> C_p <u>is the same constant as in Theorem</u> 1.

<u>Remark</u>: Our proof of Proposition 1 can actually be used to give a somewhat involved explicit estimate for $\dim_{Z/p} E_n(X)$

and thus for $\pi_n(X;Z/p)$. We leave this to the reader.

Proof of Theorem 1 (assuming Proposition 1). We may assume that $R_{H_*(\Omega X)} > 0$ and hence $R_{E(X)} > 0$ by the proposition. In particular we deduce $\dim_{Z/p} E_n(X) < \infty$, i.e. $E_2^{s,s+n}(\Omega X)$ is finite dimensional for all s,n and $E_2^{s,s+n}(\Omega X) = 0$ if $s > s(n)$. This implies that the BKSS converges completely to $\pi_{n+2}(\Omega X;Z/p) \cong \pi_{n+3}(X;Z/p)$ if $n > 0$ and furthermore that

$$\dim_{Z/p} \pi_{n+2}(\Omega X;Z/p) \leq \sum_{s\geq 0} \dim_{Z/p} E_2^{s,s+n}(\Omega X) \leq \dim_{Z/p} E_n(X).$$

It follows that $R_{\pi_*(\Omega X;Z/p)} \geq R_{E(X)}$. □

2. In this section we will give a proof of Proposition 1 in case p=2.
It is possible to give a proof by a modification of the work of Bousfield-Curtis [B-C] (for p=2) and Wellington [W] (for p odd), i.e. by constructing an explicit algebraic E_1-term for the spectral sequence and estimating this E_1-term (cp [I1]). However, we choose to proceed in a different manner which we feel is more conceptual.

In a first step we reduce the study of $\mathrm{Ext}_{\underline{\underline{CA}}}$ to the study of $\mathrm{Ext}_{\underline{\underline{U}}}$ where $\underline{\underline{U}}$ is the category of right modules over the mod 2 Steenrod algebra with unstability condition $xSq^n = 0$ if $|x| < 2n$.

H.Miller [M 1,Thm.2.5] has constructed a convergent cohomological spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\underline{\underline{U}}}(M,\Sigma^{-1}R^tP(C)) \Rightarrow \mathrm{Ext}^{s+t}_{\underline{\underline{CA}}}(\Sigma M,C), \text{ natural in}$$

$M \in \underline{\underline{U}}$, $C \in \underline{\underline{CA}}$. Here $P : \underline{\underline{CA}} \to \underline{\underline{U}}$ is the primitive element functor, R^tP are the right derived functors of P and Σ^{-1} denotes desuspension.

Furthermore $H^*(\Omega X)$ is a Hopf algebra with commutative multiplication and therefore it is isomorphic (as an algebra) to a tensor product of a free commutative algebra and truncated polynomial algebras,

$$H^*(\Omega X) \cong S(V) \otimes \bigotimes_\alpha P(y_\alpha)/(y_\alpha^{k_\alpha}) \quad \text{with} \quad k_\alpha = 2^{n_\alpha} \quad .$$

From this we get an injective extension sequence of homology coalgebras ([B])

$$Z/2 \to H_*(\Omega X) \to S(V)^* \otimes \bigotimes_\alpha P(y_\alpha)^* \to \bigotimes_\alpha P(y_\alpha^{k_\alpha})^* \to Z/2.$$

Now [M1,Thm 2.5.] and [B,§3] imply that

$$R^0 P\overline{H}_*(\Omega X) \cong P\overline{H}_*(\Omega X)$$

$R^1 P\overline{H}_*(\Omega X) \cong W^*$ if W is the graded $Z/2$ vector space with basis $\{y_\alpha^{k_\alpha}\}$ and $R^t P\overline{H}_*(\Omega X) = 0$ if $t > 1$.

From the description of $R^1 P$ given above we see that

$$R_{H_*(\Omega X)} \leq R_{\Sigma^{-1} R^1 P\overline{H}_*(\Omega X)} \quad ;$$

$$R_{H_*(\Omega X)} \leq R_{\Sigma^{-1} P\overline{H}_*(\Omega X)} \quad \text{holds trivially.}$$

Proposition 1 will therefore follow from

<u>PROPOSITION 2</u>. <u>Let</u> $U \in \underline{\underline{U}}$ <u>and</u> $\overline{E}(U)$ <u>be the graded</u> $Z/2$-<u>vector space given by</u>

$$\overline{E}_n(U) = \bigoplus_{s\geq 0} \operatorname{Ext}^s_{\underline{\underline{U}}}(\overline{H}_*(P^{s+n+2}(2)), U) \quad .$$

<u>Then</u> $R_{\overline{E}(U)} \geq \min\{R_U, 1/2\}$.

We will deduce Proposition 2 from the following

<u>LEMMA 3</u>. Let $a_2(n,k) = \dim_{Z/2} \overline{E}_n(\overline{H}_*(S^k))$.
<u>Then</u> $a_2(n,k) \leq 2^{n-k+1}$.

<u>Proof of Proposition 2</u> (assuming the Lemma). By using the skeletal filtration of U we see that

$$\dim_{Z/2}\overline{E}_n(U) \leq \sum_k a_2(n,k)\cdot \dim_{Z/2} U_k$$
$$\leq \sum_k 2^{n-k+1}\cdot \dim_{Z/2} U_k \quad .$$

The right hand side is the n-th coefficient of the power series $Q(t) = (\Sigma\, 2^{r+1} t^r)\cdot(\Sigma\, \dim_{Z/2} U_s \cdot t^s)$. The result follows now from comparing the radius of convergence of $Q(t)$ with that of $\Sigma \dim_{Z/2} \overline{E}_n(U)\cdot t^n$. □

It remains to give the

Proof of Lemma 3. We will use the following abbreviations: S^k instead of $\overline{H}_*(S^k)$, P^k instead of $\overline{H}_*(P^k(2))$ and $\mathrm{Ext}_{\underline{\underline{U}}}^{s,t}(U)$ instead of $\mathrm{Ext}_{\underline{\underline{U}}}^{s}(P^{t+2},U)$.

There is a wellknown long exact "EHP-sequence"

$$\ldots \to \mathrm{Ext}_{\underline{\underline{U}}}^{s,t-1}(\Omega U) \to \mathrm{Ext}_{\underline{\underline{U}}}^{s,t}(U) \to \mathrm{Ext}_{\underline{\underline{U}}}^{s-1,t-1}(\Omega_1 U) \to \ldots \quad (t>0)$$

where Ω is the loop functor in $\underline{\underline{U}}$ and Ω_1 is its first right derived functor (cp [M1,§8]).

Furthermore, if $k>0$, then
$$\Omega S^k = S^{k-1}$$
$$\Omega_1 S^k = S^{2k-1} .$$

Hence the EHP-sequence implies

(1) $a_2(n+1,k) \leq a_2(n,k-1)+a_2(n+1,2k-1)$, $n\geq 0$, $k\geq 1$.

In addition we have

(2) $a_2(n,k) = 0$ if $n<k-2$

(3) $a_2(n,1) = 0$ if $n\geq 0$

(4) $a_2(0,2) = 1$.

We postpone the proof of (2)-(4) and continue with the proof of Lemma 4.

Let $I_2 = \{(i_1,i_2,\ldots,i_e) \mid e\geq 0$ and $i_j>0$, $i_{j+1}\leq 2i_j$ for all $j\}$ and let

$$b_2(n,k) = \mathrm{card}\{(i_1,\ldots,i_e)\in I_2 \mid i_1<k \text{ and } \Sigma i_j = n+2-k\}.$$

Then it is easy to verify that the $b_2(n,k)$ satisfy

(1') $b_2(n+1,k) = b_2(n,k-1)+b_2(n+1,2k-1)$

(2') $b_2(n,k) = 0$ if $n<k-2$

(3') $b_2(n,1) = 0$ if $n \geq 0$

(4') $b_2(0,2) = 1$

By induction we see that $a_2(n,k) \leq b_2(n,k)$ for all n,k, in particular $a_2(n,k) \leq c_2(n-k+2)$ if

$$c_2(m) = \text{card}\{(i_1,\dots,i_e) \in I_2 \mid \Sigma i_j = m\} .$$

Furthermore $c_2(m) \leq d(m)$ if

$$d(m) = \text{card}\{(i_1,\dots,i_e) \mid e \geq 0,\ i_j > 0 \text{ for all } j \text{ and } \Sigma i_j = m\}.$$

Finally it is clear that $d(0) = 1$ and easy to see that

$$d(m) = \sum_{j=1}^{m} d(m-j) \quad \text{if} \quad m>0.$$

Now induction on m shows that $d(m) = 2^{m-1}$ if $m>0$ and therefore $a_2(n,k) \leq 2^{n-k+1}$.

It remains to prove statements (2)-(4).

For (2) consider a minimal projective resolution $\underline{G} = \{G_r, d_r\}$ of P^{s+n+2} by direct sums of $G(m)$'s (cp [M1, §§6,7]). Then G_r has dimension $\leq s+n+2-r$ and (2) follows.

For (3) consider the long exact EHP-sequence

$$\dots \to \text{Ext}_{\underline{\underline{U}}}^{s,s+n-1}(S^0) \to \text{Ext}_{\underline{\underline{U}}}^{s,s+n}(S^1) \to \text{Ext}_{\underline{\underline{U}}}^{s-1,s+n-1}(S^1) \to \dots$$

Because $S^0 \in \underline{\underline{U}}$ is injective we get for $s>0$

$$\text{Ext}_{\underline{\underline{U}}}^{s,s+n}(S^1) \cong \text{Ext}_{\underline{\underline{U}}}^{s-1,s+n-1}(S^1) \cong \dots \cong \text{Hom}_{\underline{\underline{U}}}(P^{n+2},S^1) = 0$$

if $n \geq 0$.

For (4) consider the long exact EHP-sequence

$$\dots \to \text{Ext}_{\underline{\underline{U}}}^{s,s-1}(S^1) \to \text{Ext}_{\underline{\underline{U}}}^{s,s}(S^2) \to \text{Ext}_{\underline{\underline{U}}}^{s-1,s-1}(S^3) \to \dots \quad .$$

Now $\text{Ext}_{\underline{\underline{U}}}^{s-1,s-1}(S^3) = 0$ by (2). If $s>0$ we get

$$\text{Ext}_{\underline{\underline{U}}}^{s,s}(S^2) \cong \text{Ext}_{\underline{\underline{U}}}^{s,s-1}(S^1) \cong \dots \cong \text{Ext}_{\underline{\underline{U}}}^{1}(P^2,S^1) = 0 \quad \text{because}$$

$P^2 = G(2)$ is projective.

If $s = 0$ then $\mathrm{Hom}_{\underline{\underline{U}}}(P^2, S^2) = \mathbb{Z}/2$ which proves (4).
This completes the proof of Lemma 3. □

Remark: The proofs of Proposition 2 and Lemma 3 show that the constant C_2 occuring in Theorem 1 and Proposition 1 can be taken to be the radius of convergence of $\Sigma c_2(m)t^m$.

3. The following modifications are necessary in case of an odd prime.

a) Instead of reducing to the category $\underline{\underline{U}}$ we reduce to the category $\underline{\underline{V}}$ of right modules over the mod p Steenrod algebra with unstability condition $xP^t = 0$ for $|x| \leq 2pt$ (cp. [M2]). For this we replace the spectral sequence of [M1, Thm.2.5.] by that of [M2,Thm.2.5.].

b) Proposition 2 and Lemma 3 read as before (with $\underline{\underline{U}}$ replaced by $\underline{\underline{V}}$). Also the proof of Proposition 2 remains unchanged while the proof of Lemma 3 makes use of the following exact EHP-sequence in $\underline{\underline{V}}$

$$\ldots \to \mathrm{Ext}_{\underline{\underline{V}}}^{s,t-1}(\Omega' V) \to \mathrm{Ext}_{\underline{\underline{V}}}^{s,t}(V) \to \mathrm{Ext}_{\underline{\underline{V}}}^{s-1,t-1}(\Omega_1' V) \to \ldots$$

where Ω' is the loop functor in $\underline{\underline{V}}$ and Ω_1' is its first derived functor ([M1,§8] and [M2]).

Furthermore
$$\Omega' S^k = \begin{cases} 0 & k = 1 \\ S^{k-1} & k = 0 \end{cases}$$

and
$$\Omega_1' S^k = \begin{cases} S^1 & k = 1 \\ P^{(k-1)p+1} & k \equiv 1 \bmod 2 \ , \ k>1 \\ 0 & k \equiv 0 \bmod 2 \ . \end{cases}$$

This implies (for $n \geq 0$, $k \geq 1$)

$$(1_p) \qquad a_p(n+1,2k) = a_p(n,2k-1)$$

$$a_p(n+1,2k+1) \leq a_p(n,2k) + a_p(n+1,2kp) + a_p(n+1,2kp+1) .$$

In addition (2_p)-(4_p) read exactly as before and are proved essentially as before.

Now define

$I_p = \{(i_1, i_2, \ldots, i_e) \mid e \geq 0, i_j > 0, i_{j+1} \leq p i_j$ and $i_j \equiv -1$ or $0 \bmod 2(p-1)\}$

and $\quad b_p(n,2k) = \text{card}\{(i_1, \ldots, i_e) \in I_p \mid i_1 < 2k(p-1)-1$ and $\Sigma i_j = n+2-2k\}$

$b_p(n,2k+1) = \text{card}\{(i_1, \ldots, i_e) \in I_p \mid i_1 \leq 2k(p-1)$ and $\Sigma i_j = n+2-2k-1\}$.

Then one checks easily that the $b_p(n,k)$ satisfy the obvious equations $(1_p')-(4_p')$.

As before we conclude that

$a_p(n,k) \leq b_p(n,k) \leq c_p(n-k+2)$ where

$c_p(m) = \text{card}\{(i_1, \ldots, i_e) \in I_p \mid \Sigma i_j = m\}$.

Then $c_p(m) \leq d(m) \leq 2^{m-1}$ which finishes the proof. □

Remark: The constant C_p occuring in Theorem 1 and Proposition 1 can be taken to be the radius of convergence of $\Sigma c_p(m) \cdot t^m$.

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