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Jahr: 1986

PURL: https://resolver.sub.uni-goettingen.de/purl?365956996_0055|log9

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manuscripta math. 55, 93 - 99 (1986)

manuscripta
mathematica
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A SHARP BOUND FOR THE MINIMAL NUMBER OF GENERATORS OF PERFECT HEIGHT TWO IDEALS

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Let R a regular local ring of dimension N. In this paper we give a sharp bound for the minimal number of generators of perfect height two ideals I of R in terms of the multiplicity of R/I, and also some characterizations of the ideals which reach that bound. For a similar bound for the number of generators of height three ideals I of R such that R/I is Gorenstein see theorem 1 in [3].

1.- Notations. We will use the following notations:

k	the residue field of R.
M	maximal ideal of R.
I	a fixed ideal of A=R/I.
v(I)	minimal number of generators of I.
f_I	Hilbert function of A.
Gr(A)	associated graded ring of A with respect to its maximal ideal.
Gr(I)	homogeneous ideal of Gr(R) generated by the initial forms of elements of I, so that Gr(A)= Gr(R)/Gr(I)

Given a numerical function $f:\mathbb{N}\longrightarrow\mathbb{N}$, $\Delta f:\mathbb{N}\longrightarrow\mathbb{N}$ will be the numerical function defined by $(\Delta f)(0)=0$ and for

all $n \geq 1$ $(\Delta f)(n)=f(n)-f(n-1)$, and $\Delta^d f = \Delta(\Delta^{d-1} f)$. For all $x \in R$ we denote by $\bar{x}$ the coset of x in A and by $in(\bar{x}) \in Gr(A)$ the initial form of $\bar{x}$.

2.- The bound.

For the basic properties of minimal reduction see [6].

THEOREM 1. Let I be a perfect ideal of R such that ht(I)=h and $I \subset M^t$. If $\bar{x}_1,\ldots,\bar{x}_{N-h}$ is a minimal reduction of M/I then

$$e(A)= \binom{t-1+h}{h} + l\left(M^t/I+(x.)M^{t-1}\right)$$

where e(A) is the multiplicity of A.

Proof. Without loss of generality we can assume that k is infinite. Since $\bar{x}_1,\ldots,\bar{x}_{N-h}$ is a minimal reduction of M/I and A is Cohen-Macaulay we have that the set $\bar{x}.= \{\bar{x}_1,\ldots,\bar{x}_{N-h}\}$ is an A-regular sequence. Hence $e(\bar{x}.)$, multiplicity of the ideal $(\bar{x}.)$, is equals to $l(A/(\bar{x}.))$. From [6] §1 Theorem 1 we obtain that $e(A)=e(\bar{x}.)$, so that $e(A)=l(A/(\bar{x}.))$ and

$$e(A)= l\left(A/(\bar{x}.)+m^t\right) + l\left((\bar{x}.)+m^t/(\bar{x}.)\right) .$$

Since $\bar{x}.$ is a set analitically independent ([6]§4 Lemma 2) we get that the forms $in(x_i)$, i=1,...,N-h, have degree one and form a set linearly independent. From this we obtain that $\{in(x_i);\ i=1,\ldots,N-h\}$ is a Gr(R)- regular sequence and that R/(x.) is a local regular ring of dimension h. Since $I \subset M^t$ we get

$$l\left(R/(\bar{x}.)+m^t\right) = l\left(R/(x.)+M^t\right) = \binom{t-1+h}{h} .$$

We know that $\{in(x_i);i=1,\ldots,N-h\}$ is a Gr(R)-regular sequence so $(x.) \cap M^{s+1} = (x.)M^s$ for all $s \geq 0$ ([11] theorem 1.1). Hence we have $(\bar{x}.)+m^t/(\bar{x}.) = M^t/I+(x.)M^{t-1}$,

this completes the proof.

COROLLARY 2. If $e(A)=\binom{t-1+h}{h}$ then I has a standard basis of elements of order t.

Proof. For each $i \in \{1,\ldots,N-h\}$ consider the morphism

$$g_i^r : m^{t+r}/m^{t+r+1} \xrightarrow{\quad .\bar{x}_i \quad} m^{t+r+1}/m^{t+r+2} .$$

By Theorem 1 $I+M^s = I+(x.)M^{s-1}$ for all $s \geqslant t$, so that $\sum_{i=1}^{N-h} \mathrm{Im}(g_i^r) = m^{t+r+1}/m^{t+r+2}$ for all $r \geqslant 0$. Hence the ideal Gr(I) has a minimal basis of elements of degree t, so we obtain the claim.

From now we assume $N \geqslant 2$.

THEOREM 3. Let I be a perfect height two ideal of R. Then we have:

$$v(I)(v(I)-1) \leqslant 2.e(R/I).$$

Moreover: for all $r \geqslant 2$ there exists a perfect heigt two ideal I_r of R such that $v(I_r)=r$ and for which the multiplicity of R/I_r is $\frac{1}{2}r(r-1)$.

Proof. Since the number of elements in a minimal basis of an ideal and the multiplicity are unchanged by flat extension, in order to prove the first part of the therem we may assume that k is infinite.
From [1] Corollary 2 to Theorem 5 we know that $I \subset M^{v(I)-1}$, by Theorem 1 we obtain the bound.
Let $X_1,\ldots,X_N$ be a minimal basis of M R. We define I_r as the ideal of R generated by $f_i=X_1^{r-1-i}X_2^i$ for $i=0,\ldots,r-1$. Notice that $\{f_i\}_{i=1,\ldots,N}$ are the maximal minors of the matrix

$$B = \begin{pmatrix} X_2 & & & & \\ X_1 & X_2 & & & \\ & X_1 & X_2 & & \\ & & \ddots & \ddots & \\ & & & X_1 & X_2 \\ & & & & X_1 \end{pmatrix} \quad \begin{matrix} \uparrow \\ r \\ \downarrow \end{matrix}$$

$\longleftarrow r-1 \longrightarrow$

Consider the Eagon-Northcott complex associated to B ([2] section 3):

$$(EN) \qquad 0 \longrightarrow R^{r-1} \xrightarrow[B]{} R^{r} \longrightarrow R \longrightarrow R/I_r \longrightarrow 0$$

Since $ht(I_r)=2$ by [2] Theorem 1 we get that (EN) is exact. From this it easy to see that I is perfect and $v(I_r)=r$. A straightforward computation give us $e(R/I_r)=\frac{1}{2}r(r-1)$.

In 1916 MACAULAY gave ([4], [7]) a prime ideal M_n of $\mathbb{C}[X,Y,Z]_{(X,Y,Z)}$ such that $v(M_n)=n$ for all $n \geqslant 2$. In 1974 MOH found ([5]), for all even integer $n \geqslant 2$, a prime ideal H_n of $k[X,Y,Z]$ such that $v(H_n)=n$, where k is a field of characteristic zero.

From Theorem 3 and the way the ideals M_n and H_n are defined we get:

COROLLARY 4. For all integers $n \geqslant 2$ (resp. even integers $n \geqslant 2$) the prime M_n (resp. H_n) has minimal mutiplicity among the ideals I such that $v(I)=n$.

REMARK. For other results on the same problem of Theorem 3 see [9], [10], [12] and the references one can find in these papers.

DEFINITION. Let v be an integer. A perfect height two ideal I of R will be called v-extremal if and only if $v=v(I)$ and $\frac{1}{2}v(v-1)$ is the multiplicity of R/I.

Notice that for all integers $n \geqslant 2$ (resp. even integers $n \geqslant 2$) the prime M_n (resp. H_n) is a n-extremal ideal.

3.- CHARACTERIZATIONS OF v-EXTREMAL IDEALS.

Let a,b integers, if $a < b$ we put $\binom{a}{b}=0$.

THEOREM 5. Let I be a height two perfect ideal of R. For all $v \in \mathbb{N}$ the following conditions are equivalent:

(i) I is v-extremal.

(ii) $v=v(I)$ and I has a standard basis of elements of order v-1.

(iii) $v=v(I)$, I has a standard basis of elements of order v-1 and Gr(R/I) is Cohen-Macaulay.

(iv) $\Delta f_I(t) = \binom{N-2+t}{N-1} - v\binom{N-v-1+t}{N-1} + (v-1)\binom{N-v-2+t}{N-1}$ for all $t \geqslant 0$.

Proof. Without loss of generality we can assume that k is infinite.

(i) $\Longrightarrow$ (ii). Corollary 2.

(ii) $\Longrightarrow$ (iii). [8] Corolllary 4.4.

(iii) $\Longrightarrow$ (iv). Let $\{f_i\}_{i=1,\ldots,v}$ be a standard basis of I and suppose that $\text{order}(f_i)=v-1$ for i=1,..,v. Since $\{f_i\}_{i=1,\ldots,v}$ is a minimal basis of I we have that $\{in(f_i)\}_{i=1,..,v}$ is a minimal basis of Gr(I). Consider a homogeneous resolution of Gr(R/I):

$$0 \longrightarrow (Gr(R)(-v))^{v-1} \longrightarrow (Gr(R)(-v+1))^{v} \longrightarrow$$

$$\longrightarrow Gr(R) \longrightarrow Gr(R/I) \longrightarrow 0$$

$$(in(f_i))$$

where Gr(R)(s) is the ring Gr(R)$\cong k[X_1,\dots,X_N]$ with the grading shifted by s.

Now (iv) is obtained by a straightforward computation on this resolution.

(iv)$\Longrightarrow$(i). A straightforward computation gives us $e(R/I)=\frac{1}{2}v(v-1)$. By Theorem 3 we have that $v(I)\leqslant v$ so that we only need to prove that $v\leqslant v(I)$. From (iv) we get that $\dim_k(m^{v-1}/m^v)=\dim_k(M^{v-1}/M^v)-v$ and $\dim_k(m^{n-1}/m^n)=\dim_k(M^{n-1}/M^n)$ for all $n\leqslant v-1$, so that $v\leqslant v(I)$.

ACKNOWLEDGMENT. I am greatly indebted to the referee for a number of suggestions which have improved this paper. Indeed, among other things Theorem 1 was suggested by him.

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(Received March 4, 1985;
in revised form August 16, 1985)