manuscripta math. 55, 373 - 384 (1986)

manuscripta mathematica

WEIGHTED-ADDITIVE DEVIATIONS WITH THE SUM PROPERTY

Wolfgang Sander

Herrn Hans-Joachim Kowalsky zum 65. Geburtstag gewidmet

We determine all weighted-additive deviations having the sum property by means of all weighted-additive entropies having the sum property.

1. Introduction

Let $\Gamma_n = \{P = (p_1,\ldots,p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$.

Entropies are functions $I'_n : \Gamma_n \to \mathbb{R}$ and deviations are functions $I_n : \Gamma_n \times \Gamma_n \to \mathbb{R}$ $(n \geq 2)$.

The characterization of weighted-additive entropies or deviations having the sum property leads to the functional equations (cf. [4], [6], [7], [8])

$$(1) \quad \sum_{i=1}^{n}\sum_{j=1}^{m} I'(p_i q_j) =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G(p_i) I'(q_j) \quad + \quad \sum_{i=1}^{n}\sum_{j=1}^{m} L(q_j) I'(p_i)$$

for all $P \in \Gamma_n$, $Q \in \Gamma_m$ and

(2) $$\sum_{i=1}^{n}\sum_{j=1}^{m} I(p_i q_j, v_i w_j) =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G(p_i) I(q_j, w_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} L(q_j) I(p_i, v_i)$$

for all $P,V \in \Gamma_n$, $Q,W \in \Gamma_m$, respectively.
Here $I', G, L : [0,1] \to \mathbb{R}$ and $I : J \to \mathbb{R}$ where

(3) $$J = [0,1] \times (0,1] \cup \{0,0\}.$$

In the following we suppose that the functions I', G, L or I, G, L satisfy the equations (1) and (2), respectively, for a fixed pair (n,m), $n \geq 3$, $m \geq 3$.
If in particular

$$G(p) = p^{\alpha} \text{ and } L(p) = p^{\beta} \qquad \alpha, \beta \in \mathbb{R}$$

and if I' or J is measurable then the functional equations (1) and (2), respectively, were solved in [3], [4], [5] and [6]. Here (and throughout the paper) the convention

$$0^{\alpha} = 0 \ , \ \alpha \in \mathbb{R}$$

is used. Generalizing the result in [4] all measurable triples (I', G, L) satisfying (1) were determined in [9] where moreover G and L were supposed to satisfy

(4) $$G(0) = L(0) = 0.$$

Note that in the special case $G(p) = p^{\alpha}$ and $L(p) = p^{\beta}$ the functions G and L also fulfill the condition (4) because of our convention.
In [8] all measurable triples (I, G, L) satisfying (2) and (4) were derived under the additional assumption

$$I(0,0) = 0.$$

In this case all solutions of equation (2) are independent on n and m, but it is known that there are solutions of (2) which are dependent upon the pair (n,m) (cf. [6])
So the question arises to derive all measurable solutions I,G,L satisfying (2) and (4). In this paper we give an answer to this question by showing that all weighted-additive deviations with the sum property can be determined with the aid of all weighted-additive entropies with the sum property. The method of proof is a refinement of considerations in [8]. Our result contains the above mentioned results as special cases.

2. Main results

We make use of the following result ([8]).

LEMMA 1. *Let* $A : J \times J \to \mathbb{R}$ *be measurable in all four variables and let* A *satisfy the functional equation*

$$(5)\qquad \sum_{i=1}^{n} \sum_{j=1}^{m} A(p_i, v_i, q_j, w_j) = 0 \;, \quad P,V \in \Gamma_n \;, \quad Q,W \in \Gamma_m$$

for some fixed pair (n,m), $n \geq 3$, $m \geq 3$. *Then* A *is given by*

$$(6)\qquad A(p,v,q,w) = A(0,0,q,w)(1-np) + A(p,v,0,0)(1-mq) -$$

$$- A(0,0,0,0)(1-np)(1-mq) +$$

$$+ (v-p)\left[A(0,1,q,w)-A(0,0,q,w)-(1-mq)(A(0,1,0,0)-A(0,0,0,0))\right] +$$

$$+ (w-q)\left[A(p,v,0,1)-A(p,v,0,0)-(1-np)(A(0,0,0,1)-A(0,0,0,0))\right] +$$

$$+ (v-p)(w-q)\left[A(0,1,0,0)-A(0,0,0,0)+A(0,0,0,1)-A(0,1,0,1)\right].$$

THEOREM 2. *Let* $G, L : [0,1] \to \mathbb{R}$ *be measurable and let* $I : J \to \mathbb{R}$ *be measurable in each of its two variables. Then* I, G, L *satisfy* (2) *and* (4) *for a fixed pair* (n,m) ($n \geq 3$, $m \geq 3$) *if, and only if they are of one of the following forms* :

(7) $\quad I(p,v) = a(p-v)+I(0,0)+p(nm-n-m)I(0,0)+p\log p^A v^B + v\log v^C,$

$\quad G(p) = L(p) = p,$

(8) $\quad I(p,v) = a(p-v) + p^A \log p^B v^C,$

$\quad G(p) = L(p) = p^A \ , \ A \neq 1$

(9) $\quad I(p,v) = a(p-v) + I'(p)$

where - for fixed n,m - (I',G,L) *is a measurable solution of* (1) *and* (4), *but where* G(p) *and* L(p) *are not of one of the forms given in* (7) *or* (8).

Here a,A,B,C *are constants; moreover we follow the convention*

$$0\log 0 = 0.$$

Proof. It is easy to check that the functions I,G,L given by (7), (8) or (9) have all properties stated in the theorem. To prove the converse statement we define

$$A(p,v,q,w) = I(pq,vw) - G(p)I(q,w) - L(q)I(p,v)$$

for $(p,v,q,w) \in J \times J$. Then (2) goes over into (5) and with

(10) $\qquad \alpha = I(0,0) \ , \ \beta = I(0,1) - I(0,0)$

Lemma 1 yields

$$(11) \qquad I(pq,vw) - G(p)I(q,w) - L(q)I(p,v) =$$

$$= (v-p)(I(0,w)-\alpha-\beta L(q)) + (w-q)(I(0,v)-\alpha-\beta G(p)) +$$

$$+ \alpha(1 - np)(1 - L(q)) + \alpha(1 - mq)(1 - G(p)) -$$

$$- \alpha(1 - np)(1 - mq) - \beta(v - p)(w - q).$$

Defining

$$(12) \quad \begin{cases} F(p,v) = I(p,v) - \alpha(1 - nmp) + \beta(p - v) \\ H(p,v) = I(p,v) - \alpha(1 - mp) + \beta(p - v) \\ K(p,v) = I(p,v) - \alpha(1 - np) + \beta(p - v) \end{cases}$$

for all $(p,v) \in J$ equation (11) can be rewritten into

$$(13) \quad F(pq,vw) = G(p)H(q,w)+L(q)K(p,v)+(v-p)F(0,w)+(w-q)F(0,v)$$

for all $(p,v,q,w) \in J \times J$ (We remark that

$$F(p,v) = H(p,v) = K(p,v) = I(p,v) + \beta(p-v) \quad , \quad (p,v) \in J$$

if $\alpha = 0$. This case was treated in [8]).
Using the fact that

$$(14) \quad F(0,v) = H(0,v) = K(0,v) = I(0,v) - \alpha - \beta v \quad , \quad v \in (0,1]$$

and setting $p = 0$, $w = 1$ or $q = 0$, $v = 1$ or $p = 0$, $q = 0$ into (13) we get

$$(15) \qquad (L(q)-q)F(0,v) = 0 \; , \qquad q,v \in (0,1]$$

and

$$(16) \qquad (G(p)-p)F(0,w) = 0 \; , \qquad p,w \in (0,1]$$

and

$$(17) \qquad F(0,vw) = vF(0,w) + wF(0,v) , \qquad v,w \in (0,1],$$

respectively. Like in [8] we distinguish two cases which are more complicated now.

Case (I): $F(0,v)$ is not identically zero.

Case (II): $F(0,v)$ is identically zero.

In the first case (15) and (16) imply

$$(18) \qquad G(p) = L(p) = p , \qquad p \in (0,1]$$

which holds also for $p = 0$ (see (4)). Since (17) is a Cauchy type equation where F is measurable in the second variable and $F(0,v) \not\equiv 0$ we obtain (cf. [1])

$$(19) \qquad F(0,v) = \gamma v \log v , \qquad \gamma \in \mathbb{R},\ v \in (0,1].$$

Note that (19) is also valid for $v = 0$ (see (14) and our conventions). Substituting (18) and (19) into (13) we get

$$(20) \quad f(pq,vw) = ph(q,w) + qk(p,v) , \quad (p,v,q,w) \in J \times J$$

where

$$(21) \quad \begin{cases} f(p,v) = F(p,v) - \gamma v \log v \\ h(p,v) = H(p,v) - \gamma v \log v \\ k(p,v) = K(p,v) - \gamma v \log v \end{cases} , \quad (p,v) \in J.$$

Since equation (20) is - for fixed $v,w \in (0,1]$ - a Pexider equation we have the solutions (cf. [1])

$$(22) \quad \begin{cases} f(p,v) = p(a(v)\log p + b(v) + c(v)) \\ h(p,v) = p(a(v)\log p + b(v)) \\ k(p,v) = p(a(v)\log p + c(v)) \end{cases} , \quad p,v \in (0,1].$$

Setting $q = 1$ into (20) we get using (22)

$$(23) \quad a(vw)\log p + b(vw) + c(vw) = b(w) + a(v)\log p + c(v)$$

for all $p,v,w \in (0,1]$. Thus (23) implies

$$(24) \quad a(vw) = a(v) = \delta \text{ (say)}$$

and

$$b(vw) + c(vw) = b(w) + c(v)$$

which is again a Pexider equation with the solutions

$$(25) \quad \begin{cases} b(v) = B\log v + C \\ c(v) = B\log v + D & , v \in (0,1] \\ b(v) + c(v) = B\log v + C + D \end{cases}$$

where B,C,D are constants. Putting $p = q = 1$ into (12) and using (21), (22), (24) and (25) we get

$$\begin{cases} I(1,1) = \alpha(1 - nm) + C + D \\ \qquad\quad\; = \alpha(1 - m) + C \\ \qquad\quad\; = \alpha(1 - n) + D \end{cases}$$

which implies

$$(26) \quad C = n(m - 1)\alpha \quad \text{and} \quad D = m(n - 1)\alpha.$$

Finally we obtain from (12), (21), (22), (24), (25) and (26)

$$(27) \quad I(p,v) = \alpha(1 + (nm-n-m)p) - \beta(p - v) +$$

$$+ \gamma v\log v + \delta p\log p + Bp\log v \quad , \quad p,v \in (0,1].$$

But (27) is also valid for $p = 0$, $v \in (0,1]$ and for $p = v = 0$ (see (14), (19) and (10)). Thus we have solution (7).

In the case that $F(0,v) \equiv 0$ equation (13) reduces to

$$(28) \qquad F(pq,vw) = G(p)H(q,w) + L(q)K(p,v) \quad , \quad (p,v,q,w) \in J \times J.$$

Moreover we can assume (see (15) and (16))

$$(29) \qquad G(p) \not\equiv p \quad \text{or} \quad L(p) \not\equiv p \quad , \quad p \in [0,1].$$

Putting $v = w = 1$ into (28) we get the functional equation

$$(30) \qquad F'(pq) = G(p)H'(q) + L(q)K'(p) \quad , \quad p,q \in [0,1]$$

where

$$(31) \quad F'(p) = F(p,1), \ H'(p) = H(p,1), \ K'(p) = K(p,1), \ p \in [0,1].$$

Moreover we define

$$(32) \qquad M(p,v) = I(p,v) - I(p,1) + \beta(1 - v) \quad , \quad (p,v) \in J$$

so that (12) can be rewritten into

$$(33) \qquad \begin{cases} F(p,v) = F'(p) + M(p,v) \\ H(p,v) = H'(p) + M(p,v) \, , \\ K(p,v) = K'(p) + M(p,v) \end{cases} \qquad (p,v) \in J.$$

Note that by definition

$$(34) \qquad M(p,1) = 0 \quad , \qquad p \in [0,1].$$

Substituting (33) into (28) we get by means of (30) the following functional equation

$$(35) \quad M(pq,vw) = G(p)M(q,w) + L(q)M(p,v) \quad , \quad (p,v,q,w) \in J \times J.$$

Setting $p = q = 1$ into (35) we get the Pexider equation

$$(36) \qquad M(1,vw) = G(1)M(1,w) + L(1)M(1,v) \quad , \quad v,w \in (0,1]$$

with the solution

$$(37) \qquad M(1,v) = e + d\log v \ , \quad d,e \in \mathbb{R},\ v \in (0,1]$$

(We note that $e = 0$ in (37) if $G(1) = L(1) = 1$ in (36)). Substituting $q = v = 1$ or $p = w = 1$ into (36) and using (34) we obtain

$$(38) \qquad M(p,w) = G(p)(e + d\log w) \quad , \quad p,w \in (0,1]$$

and

$$(39) \qquad M(q,v) = L(q)(e + d\log v) \quad , \quad q,v \in (0,1],$$

respectively. We now show that (38) and (39) imply that there are only two possibilities for the function $M(p,v)$: Either

$$(40) \qquad M(p,v) = dp^{\rho}\log v \quad , \ \rho \neq 1 \ , \ p,v \in (0,1]$$

or

$$(41) \qquad M(p,v) = 0 \quad , \quad p,v \in (0,1].$$

From (38) and (39) we get

$$(42) \qquad (G(p) - L(p))(e + d\log v) = 0 \quad , \ p,v \in (0,1].$$

If $e + d\log v = 0$ for all $v \in (0,1]$ then (41) is valid. If there exists a $v_o \in (0,1]$ with $e + d\log v_o \neq 0$ then we conclude from (42) that

$$(43) \qquad G(p) = L(p) \ , \qquad p \in (0,1].$$

Substituting (43) and (38) into (35) (with $w = 1$ and $v = v_o$) we obtain (use (34))

$$G(pq)(e + d\log v_o) = G(p)G(q)(e + d\log v_o) \quad , \ p,q \in (0,1]$$

that is

$$(44)\qquad G(p) = L(p) = p^{\rho} \quad \text{or} \quad G(p) = L(p) = 0 \ , \ p \in (0,1]$$

which yields (40) and (41), respectively (see the remark after (37)); because of (29) we can assume that $\rho \neq 1$.
Let us first consider the case that $M(p,v)$ is given by (40) where $G(p) = L(p) = p^{\rho}$, $\rho \neq 1$. Then (30) goes over into a Pexider equation with the solutions

$$(45)\qquad \begin{cases} F'(p) = Rp^{\rho}\log p + (S+T)p^{\rho} \\ H'(p) = Rp^{\rho}\log p + Sp^{\rho} & , \qquad p \in (0,1] \\ K'(p) = Rp^{\rho}\log p + Tp^{\rho} \end{cases}$$

where R, S, T are constants; because of (10), (12) and (31) equation (45) is also valid for $p = 0$. Substituting (33), (40) and (45) into (12) we get (note that $\rho \neq 1$)
$S = T = \alpha = 0$ and thus solution (8).
Finally, if $M(p,v) = 0$ (see (41)) then (12) together with (33) yields (we denote the term $I(p,v) + \beta(p-v)$, which is not dependent upon v, by $I'(p)$)

$$(46)\qquad \begin{cases} I'(p) = I(p,v) + \beta(p-v) = F'(p) + \alpha(1-nmp) \\ \qquad\qquad\qquad\qquad\qquad\quad\; = H'(p) + \alpha(1-mp) \quad , \ p \in [0,1] \\ \qquad\qquad\qquad\qquad\qquad\quad\; = K'(p) + \alpha(1-np) \end{cases}$$

where F', H', K', G, L is a measurable solution of (30) and

$$G(p) \equiv p^{\rho} \quad \text{or} \quad L(p) \equiv p^{\rho} \ , \quad \rho \in \mathbb{R}, \ p \in [0,1].$$

To finish the proof we show that for fixed n,m the measurable triples (I',G,L) satisfying (1) and (4) are exactly given by all measurable solutions F', H', K', G, L fulfilling (46) and (30) (cf. [9]).

Let first (I',G,L) satisfy (1) and (4). Setting

$$(47) \qquad B(p,q) = I'(pq) - G(p)I'(q) - L(q)I'(p) \quad , \; p,q \in [0,1]$$

we get from (1)

$$\sum_{i=1}^{n}\sum_{j=1}^{m} B(p_i,q_j) = 0.$$

Now, Lemma 1 (see also Lemma 1 in [3]) leads to

$$(48) \quad B(p,q) = B(0,q)(1-np) + B(p,0)(1-mq) - B(0,0)(1-np)(1-mq)$$

for all $p,q \in [0,1]$. Thus substitution of (47) into (48) yields

$$(49) \qquad I'(pq) - (1-mnpq)I'(0) = G(p)(I'(q) - (1-mq)I'(0)) + L(q)(I'(p) - (1-np)I'(0))$$

for all $p,q \in [0,1]$. Defining

$$(50) \quad \begin{cases} F'(p) = I'(p) - (1-mnp)I'(0) \\ H'(p) = I'(p) - (1-mp)I'(0) \qquad , \quad p \in [0,1] \\ K'(p) = I'(p) - (1-np)I'(0) \end{cases}$$

equation (49) goes over into (30); moreover (50) is exactly (46) with $\alpha = I'(0)$.

Conversely, if F', H', K', G, L satisfy (46) and (30) then we get immediately equation (1) since

$$\sum_{i=1}^{n}\sum_{j=1}^{m} (1-nmp_iq_j) = \sum_{j=1}^{m} (1-mq_j) = \sum_{i=1}^{n} (1-np_i) = 0$$

for $P \in \Gamma_n$ and $Q \in \Gamma_m$. Thus Theorem 2 is proven.

REMARK 3. The general complex-valued solutions $F', G, H', K', L: (0,1] \to \mathbb{C}$ of (30) were given in [10] so that also all real-valued measurable solutions of (30) can be derived from this result; this was done in [9]. Thus all measurable triples (I,G,L) satisfying (2) and (4) are explicitely known and it turns out that the function I is independent upon n and m if, and only if $I(0,0) = 0$.

Literatur

[1] ACZÉL,J.: Lectures on functional equations and their applications, New York-London: Academic Press 1966

[2] ACZÉL,J.,DARÓCZY,Z.: On measures of information and their characterizations, New York-San Francisco-London: Academic Press 1975

[3] KANNAPPAN,Pl.: On some functional equations from additive and non-additive measures-I.Proc.Edin.Math.Soc. 23,145-150 (1980)

[4] KANNAPPAN,Pl.: On a generalization of sum form functional equation-III.Demonstratio Math.13, 749-754(1980)

[5] KANNAPPAN,Pl.: On some functional equation from additive and non-additive measures-V.Utilitas Math. 22, 141-147 (1982)

[6] KANNAPPAN,Pl.: On a generalization of sum form functional equation-V.Aequationes Math.28,255-261 (1985).

[7] SANDER,W.: On a sum form functional equation. Aequationes Math.28, 321-326 (1985)

[8] SANDER,W.: On a functional equation arising in information theory. Appears in Manuscripta Math

[9] SANDER,W.: Weighted additive entropies with the sum property. Submitted

[10] VINCZE,E.: Eine allgemeinere Methode in der Theorie der Funktionalgleichungen II.Publ.Math.Debrecen 9, 414-323 (1962)

Wolfgang Sander

Institut für Analysis

Technische Universität Braunschweig

Pockelsstr. 14

D 3300 Braunschweig

(Received September 19, 1985;
in revised form December 11, 1985)