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ON PARTIAL ORDERING OF CHIEF FACTORS IN SOLVABLE GROUPS

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The isomorphism classes of chief factors in a finite solvable group are partially ordered by taking one class higher than the other if a member of the first class appears as a chief factor of the action of the group on a member of the second class. Together with this partial ordering the characteristics of the chief factors are considered. It is shown that the two conditions found by G. Pazderski are not only necessary but also sufficient for a partially ordered set and a function to be representable as the poset of isomorphism classes of chief factors in a finite solvable group with the chief factors having the prescribed characteristics. In addition, the construction yields that every finite distributive lattice is the lattice of normal subgroups of some finite solvable group.

Introduction

We shall deal with finite solvable groups. The group G operates by conjugation on any chief factor. We say that two chief factors are isomorphic in G if they are isomorphic operator groups (cf. [1,p. 64]). For an isomorphism class α of chief factors of G let $C_G(\alpha)$ be the normal subgroup consisting of those elements of G which induce the identical automorphism on any (and hence on each) chief factor in the class α. G.Pazderski [5] introduced a par-

tial ordering on the set of isomorphism classes of chief factors in the following way: if α and β are isomorphism classes of chief factors in the finite solvable group G then we say that β is <u>higher</u> than α (and we write $\beta > \alpha$) if there is a chief factor U/V from the class β such that $V \geq C_G(\alpha)$. It is easy to see that $>$ is a transitive, irreflexive relation, hence $>$ makes the set Φ of all isomorphism classes of chief factors a partially ordered set. Further, any chief factor in a solvable group is an elementary abelian p-group for some prime p, so we have a function $p:\Phi \to P$ (P stands for the set of primes). We shall call p the <u>characteristics function</u>. G.Pazderski [5] pointed out two properties of the characteristics function:

(1) if α covers β in $(\Phi,>)$ then $p(\alpha) \neq p(\beta)$ and

(2) if α and β are distinct maximal elements in $(\Phi,>)$ then also $p(\alpha) \neq p(\beta)$.

We say that a finite partially ordered set $(\Phi,>)$ along with a function $p:\Phi \to P$ is representable by a finite solvable group G if the set of isomorphism classes of chief factors of G with respect to the order relation $>$ introduced above form a partially ordered set isomorphic to $(\Phi,>)$ so that p is the characteristics function. G.Pazderski conjectured that any finite partially ordered set $(\Phi,>)$ and function $p:\Phi \to P$ satisfying (1) and (2) is representable by a finite solvable group. The main goal of the present paper is to prove this conjecture.

First, we shall settle the case when p satisfies a condition stronger than (2), namely

(3) if the distinct elements $\alpha,\beta\in\Phi$ are such that $(\forall\phi\in\Phi)(\phi > \alpha$ iff $\phi > \beta)$ then $p(\alpha) \neq p(\beta)$.

If $p:\Phi \to P$ satisfies (1) and (3) then we shall construct a finite solvable group G representing $(\Phi,>)$ and p such that G has no multiple chief factors, i.e. no two chief factors in the same chief series are isomorphic in G. Such groups have distributive lattice of normal subgroups. As a corollary to our construction we obtain that every finite distributive lattice is isomorphic to the lattice of normal subgroups of a finite solvable group. The first attempt to prove this assertion was made by J.Kuntzmann [4] in 1947, but his construction was not correct (see E.T. Schmidt [6,p.101]). More recently H.L.Silcock [7] has proved that any finite distributive lattice is isomorphic to the lattice of normal subgroups of a finite group. His construction uses wreath products of nonabelian simple groups, but he also announced the solvable version [7,p.371], however it has not appeared yet.

Condition (3) is used to show that no "accidental" isomorphism occurs between the different chief factors. In the general case the construction will go so that distinct classes of chief factors will have distinct centralizers, except, of course, the maximal members of the poset, but for these the occurence of an "accidental" isomorphism is excluded by condition (2). To this end we shall modify the original construction by taking chief factors with suitable

multiplicities.

The groups will be constructed via consecutive split extensions (semidirect products). The definition and the basic properties of such "towers" will be dealt with in Section 1.

The key point of our construction is that certain groups have faithful irreducible representations. The exact necessary and sufficient condition was found by R.Kochendörffer [3], yet for our purposes the following handy sufficient condition will do:

Let G be a finite group and p a prime number. Then G has a faithful irreducible representation over the p-element field provided no two distinct minimal normal subgroups of G are isomorphic (as chief factors) in G and no minimal normal subgroup of G is a p-group.

1. Towers of split extensions

As it is customary, we say that the group $G = HN$ is a <u>split extension</u> if $H \leq G$, $N \lhd G$ and $H \cap N = 1$. Notice that it refers to the decomposition HN (in the given order!) rather than to the group G. Given two groups H, N and a homomorphism $\sigma: H \to \mathrm{Aut}\, N$ we can form the split extension (or semidirect product) of N by H with respect to σ (see [1,p.89]).

If $G_i = G_{i-1}A_i$ $(i=1,2,\dots,n;\ G_0=1)$ are split extensions then we say that $G = G_n = A_1A_2\dots A_n$ is a <u>tower of split extensions.</u> We establish some elementary properties of towers of split extensions.

<u>LEMMA 1.</u> <u>Let</u> $G = A_1A_2\dots A_n$ <u>be a tower of split extensions. Then</u>

(i) <u>for any</u> $I\subseteq\{1,2,\dots,n\}$, $A_I = \Pi\{A_i : i\in I\}$ <u>is a subgroup of</u> G (<u>if</u> $I=\emptyset$ <u>then</u> $A_I=1$);

(ii) <u>the mapping</u> $I \mapsto A_I$ <u>is a one-to-one</u> 0-1 <u>homomorphism of the boolean lattice</u> 2^n <u>into the subgroup lattice of</u> G.

<u>Proof.</u> Let $1 \le i < j \le n$, then $A_i \le N_G(A_j)$ hence $A_iA_j = A_jA_i$. Now it is obvious that $A_I \le G$, $A_IA_J = A_{I\cup J}$ and $A_I\cap A_J \supseteq A_{I\cap J}$. Moreover,

$$|A_I\cap A_J| = |A_I|\cdot|A_J|\cdot|A_IA_J|^{-1} = |A_I|\cdot|A_J|\cdot|A_{I\cup J}|^{-1} = $$
$$= |A_{I\cap J}|,$$

hence $A_I\cap A_J = A_{I\cap J}$.

<u>NOTATION.</u> If H and F are subgroups of $G = A_1A_2\dots A_n$ of the form $H = A_I$, $F = A_J$ for some $I,J\subseteq\{1,2,\dots,n\}$ then we shall write $H-F = A_{I-J}$.

<u>LEMMA 2.</u> <u>Let</u> $G = A_1A_2\dots A_n$ <u>be a tower of split extensions,</u> $1 \le j \le n$, $I\subseteq\{1,2,\dots,n\}$. <u>Then</u> $A_j \le N_G(A_I)$ <u>iff</u> $j\in I$ <u>or</u> $A_i \le C_G(A_j)$ <u>for all</u> $i < j$, $i\in I$.

Proof. If $j \in I$ then $A_j \leq A_I \leq N_G(A_I)$. If $A_i \leq C_G(A_j)$ for all $i < j$, $i \in I$ then A_j centralizes and hence normalizes each A_i where $i < j$, $i \in I$, but it also normalizes each A_i where $i \geq j$, $i \in I$, therefore $A_j \leq N_G(A_I)$. Conversely, if $A_j \leq N_G(A_I)$ but $j \notin I$ then A_j and $A_{I \cap \{1,2,\ldots,j\}}$ normalize each other and have a trivial intersection, hence they centralize each other.

2. When p satisfies (1) and (3)

THEOREM 1. Let $(\Phi,>)$ be a finite partially ordered set and $p:\Phi \to P$ a function satisfying (1) and (3) (see the Introduction). Then $(\Phi,>)$ with p is representable by a finite solvable group G such that no two chief factors in the same chief series are isomorphic in G.

Proof. First fix an ordering of the elements of $\Phi = \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ so that higher elements have smaller indices, i.e. if $\alpha_i > \alpha_j$ in $(\Phi,>)$ then $i < j$. (It can be obtained by reversing the relation on Φ and then extending it to a linear order.)

Now starting with $G_0 = 1$ we construct split extensions $G_i = G_{i-1}A_i$, $i = 1,2,\ldots,n$, recurrently. It will be done so that

(a) $G_i = A_1A_2\ldots A_i > A_2\ldots A_i > \ldots > A_{i-1}A_i > A_i > 1$ be a chief series of G_i,

(b) A_i be an elementary abelian $p(\alpha_i)$-group and

(c) $C_{G_i}(A_jA_{j+1}\ldots A_i/A_{j+1}\ldots A_i) = G_i - D_{\alpha_j}$ (for notation see Section 1), where $D_{\alpha_j} = \Pi\{A_k : \alpha_k > \alpha_j\}$, $1 \leq j \leq i$.

For notational convenience, if $\alpha = \alpha_i$ then we shall use the alternative notations $A_i = A_\alpha$ and $D_\alpha = D_i$.

So suppose that G_{i-1} has already been constructed so that it satisfies (a),(b),(c). Let $\alpha = \alpha_i$. If $\beta > \alpha$ then β precedes α in the fixed ordering of Φ hence $D_\alpha = \Pi\{A_\beta : \beta > \alpha\}$ is a subgroup of G_{i-1} by Lemma 1. Now we show that $G_{i-1} - D_\alpha$ is normal in G_{i-1}. In virtue of Lemma 2 we have to show that A_j centralizes A_k whenever $j < k < i$, $A_j \leq G_{i-1} - D_\alpha$ (i.e. $\alpha_j \not> \alpha$) and $A_k \leq D_\alpha$ (i.e. $\alpha_k > \alpha$). It follows that $\alpha_j \not> \alpha_k$, hence by (c), $A_j \leq$ $\leq G_k - D_k = C_{G_k}(A_k)$. Let ν_i denote the retraction of G_{i-1} onto D_α. Now we seek a faithful irreducible representation of D_α over the $p(\alpha)$-element field. In order to make use of the sufficient condition cited in the Introduction we have to determine the socle of D_α. If α_m covers α in $(\Phi,>)$ then, again by Lemma 2 and (c), $A_m \triangleleft D_\alpha$. Now (a) asserts that $A_m A_{m+1} \ldots A_{i-1}/A_{m+1} \ldots A_{i-1}$ is a chief factor of G_{i-1}, i.e. G_{i-1} acts irreducibly on it. Notice that $G_{i-1} = (G_{i-1} - D_\alpha)D_\alpha$ and $G_{i-1} - D_\alpha \leq$ $\leq G_{i-1} - D_{\alpha_m} = C_{G_{i-1}}(A_m A_{m+1} \ldots A_{i-1}/A_{m+1} \ldots A_{i-1})$ hence D_α also acts irreducibly on this factor. Since $A_m \triangleleft D_\alpha$, D_α must act irreducibly on A_m, hence A_m is a minimal normal subgroup of D_α. Moreover, making use of (c) we obtain $C_{D_\alpha}(A_m) = D_\alpha \cap C_{G_{i-1}}(A_m A_{m+1} \ldots A_{i-1}/A_{m+1} \ldots A_{i-1}) = D_\alpha \cap (G_{i-1} - D_m) =$ $= D_\alpha - D_m$.

Conversely, let M be an arbitrary minimal normal subgroup of D_α. For any cover γ of α in $(\Phi,>)$ either $M = A_\gamma$ or $M \cap A_\gamma = 1$, in both cases $M \leq C_{D_\alpha}(A_\gamma) = D_\alpha - D_\gamma$.

So

$$M \leq \cap\{D_\alpha - D_\gamma : \gamma \succ \alpha\} = D_\alpha - \Pi\{D_\gamma : \gamma \succ \alpha\} = \Pi\{A_\gamma : \gamma \succ \alpha\}.$$

Thus the socle of D_α is $\Pi\{A_\gamma : \gamma \succ \alpha\}$. No two of the minimal normal subgroups A_γ where γ covers α are isomorphic in D_α since either their centralizers $D_\alpha - D_\gamma$ differ or their characteristics are different by condition (3) and (b). Moreover, none of them is a $p(\alpha)$-group by condition (1) and (b). Hence both assumptions are satisfied to guarantee the existence of an elementary abelian $p(\alpha)$-group A_α and a faithful irreducible representation $\rho_\alpha : D_\alpha \to \mathrm{Aut}\, A_\alpha$.

Now if we form the split extension $G_i = G_{i-1}A_\alpha$ with respect to $\nu_i\rho_\alpha : G_{i-1} \to \mathrm{Aut}\, A_\alpha$ then A_α is a minimal normal subgroup of G_i since ρ_α is irreducible. By construction, A_α is an elementary abelian $p(\alpha)$-group and $C_{G_i}(A_\alpha) = G_i - D_\alpha$. So by induction (a),(b),(c) are satisfied for G_i as well.

Finally, let $G = G_n$. Then (a) yields a chief series $G = A_1A_2\ldots A_n > A_2\ldots A_n > \ldots > A_{n-1}A_n > A_n > 1$. No two chief factors in this series are isomorphic since either they have different centralizers or by (c), condition (3) and (b) they have different characteristics. Now in this chief series we have a single representative for each isomorphism class of chief factors. By (c), the ordering of the classes of chief factors is the prescribed one, and by (b), p is the characteristics function.

3. Distributive lattices of normal subgroups

The lattice of normal subgroups of a group without

multiple chief factors is distributive (see [4]). For the groups constructed in the previous section the normal subgroups can be determined easily.

LEMMA 3. *Let G be the group constructed in the proof of Theorem 1. Then every normal subgroup of G has the form A_I for some $I \subseteq \{1,2,\ldots,n\}$. The subgroup A_I is normal in G if and only if $\{\alpha_i : i \in I\}$ is an order-ideal in $(\Phi,>)$.*

Proof. Let $I \subseteq \{1,2,\ldots,n\}$ be such that $\{\alpha_i : i \in I\}$ is an order-ideal in $(\Phi,>)$. Then Lemma 2 can be applied to show that A_I is normal in G. Indeed, if $1 \leq i < j \leq n$ are such that $i \in I$, $j \notin I$ then $\alpha_i \not> \alpha_j$ since $\{\alpha_i : i \in I\}$ is an order-ideal, and hence A_i centralizes A_j by property (c) of the construction. Conversely, if I is such that there are $1 \leq i, j \leq n$ for which $i \in I$, $j \notin I$ and $\alpha_i > \alpha_j$ then $i < j$ and A_i does not centralize A_j again by property (c), and so A_I is not normal in G.

Now take any normal subgroup N of G. Since G has no multiple chief factors, it follows that for any class α_i either the interval below N or the interval above N contains a chief factor from the class α_i but not both. Let I be the set of those indices i for which there is a chief factor from the isomorphism class α_i in the interval below N. Now if $j \notin I$ and $\alpha_k > \alpha_j$ then there is a chief factor in class α_j in the interval above N, so $C_G(\alpha_j) \geq N$; moreover, because of $\alpha_k > \alpha_j$ the interval above $C_G(\alpha_j)$ contains a chief factor from class

α_k hence no chief factor from class α_k can occur in the interval below N, therefore we have $k \notin I$ as well. This proves that $\{\alpha_i : i \in I\}$ is an order-ideal in $(\Phi,>)$. We have already shown that in this case $A_I \triangleleft G$. Now $A_I N$ is also normal in G and any chief factor in the interval below $A_I N$ belongs to a class α_i where $i \in I$. Since G has no multiple chief factors, it follows that $A_I N = A_I = N$.

THEOREM 2 (cf. [7,p. 371]). *For any finite distributive lattice* L *there exists a finite solvable group* G *such that the lattice of normal subgroups of* G *is isomorphic to* L.

Proof. Let L be a finite distributive lattice and Φ the set of nonzero join-irreducible elements of L. Then, as it is well-known (see, e.g. [2,p.72]), the set of order-ideals of $(\Phi,>)$ with respect to inclusion form a lattice isomorphic to L. Now we choose a $p:\Phi \to P$ which satisfies conditions (1) and (3), for example we can assign different primes to different elements of Φ. Then the group G constructed in the proof of Theorem 1 is such that its lattice of normal subgroups is isomorphic to L by Lemma 3.

4. Proof of Pazderski's conjecture

THEOREM 3. *Let* $(\Phi,>)$ *be a finite partially ordered set and* $p:\Phi \to P$ *a function satisfying* (1) *and* (2) (*see the Introduction*). *Then* $(\Phi,>)$ *with* p *is representable by a finite solvable group* G.

Proof. For preparation, first take a new symbol $\omega \notin \Phi$. Let Φ_2 be the set of those pairs (α,β) where $\alpha,\beta \in \Phi$ and α covers β or α is a minimal element of $(\Phi,>)$ and $\beta = \omega$. List these pairs in such an order

$$\Phi_2 = \{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_m,\beta_m)\} \quad \text{that if} \quad \beta_i \geq \alpha_j$$

then $i < j$. The construction will closely follow the one in the proof of Theorem 1.

Starting with $G_0 = 1$ we construct split extensions $G_i = G_{i-1}A_i$, $i=1,2,\ldots,m$, recurrently so that

(a) $G_i = A_1A_2\ldots A_i > A_2\ldots A_i > \ldots > A_{i-1}A_i > A_i > 1$ is a chief series of G_i,

(b) A_i is an elementary abelian $p(\alpha_i)$-group,

(c) $C_{G_i}(A_jA_{j+1}\ldots A_i/A_{j+1}\ldots A_i) = G_i - D_{\alpha_j}$ where $D_{\alpha_j} = \Pi\{A_k : \beta_k \geq \alpha_j\}$, $1 \leq j \leq i$, and

(d) if $\alpha_j = \alpha_k$ $(1 \leq j,\ k \leq i)$ then the chief factors $A_jA_{j+1}\ldots A_i/A_{j+1}\ldots A_i$ and $A_kA_{k+1}\ldots A_i/A_{k+1}\ldots A_i$ are isomorphic in G_i.

For notational convenience, if $(\alpha,\beta) = (\alpha_i,\beta_i)$ then we shall use the alternative notation $A_i = A_{\alpha\beta}$.

So suppose that G_{i-1} has already been constructed so that it satisfies (a),(b),(c), and (d). Let $\alpha = \alpha_i$, $\beta = \beta_i$. If $\beta_k \geq \alpha$ then $k < i$ hence $D_\alpha = \Pi\{A_k : \beta_k \geq \alpha\}$ is a subgroup of G_{i-1}. It clearly does not depend on β hence we obtain the same D_α for different occurences of α as a first component of a pair in Φ_2. Application of Lemma 2 shows in this case as well that $G_{i-1} - D_\alpha \triangleleft G_{i-1}$. Let ν_i denote the retraction of G_{i-1} onto D_α. As

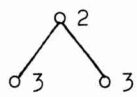
before, we need a faithful irreducible representation ρ_α of D_α over the $p(\alpha)$-element field. The same considerations yield that the socle of D_α is $\Pi\{A_{\gamma\alpha}:\gamma \succ \alpha\}$. No two minimal normal subgroups $A_{\gamma\alpha}(\gamma \succ \alpha)$ are isomorphic in D_α, since all of them have different centralizers $D_\alpha - D_\gamma$ except for the central ones, i.e. those for which γ is maximal in $(\Phi,>)$, but in case of these condition (2) and (b) applies. By condition (1) and (b) none of the subgroups $A_{\gamma\alpha}$ $(\gamma \succ \alpha)$ is a $p(\alpha)$-group. So there exists a faithful irreducible representation $\rho_\alpha : D_\alpha \to \mathrm{Aut}\, A_\alpha$ where A_α is an elementary abelian $p(\alpha)$-group. Let $A_{\alpha\beta}$ be an isomorphic copy of A_α with a fixed isomorphism $\mu_i : A_\alpha \to A_{\alpha\beta}$ and denote by $\mu_i^* : \mathrm{Aut}\, A_\alpha \to \mathrm{Aut}\, A_{\alpha\beta}$ the induced isomorphism.

Now we form the split extension $G_i = G_{i-1}A_{\alpha\beta}$ with respect to $\nu_i \rho_\alpha \mu_i^* : G_{i-1} \to \mathrm{Aut}\, A_{\alpha\beta}$. Then $A_{\alpha\beta}$ is a minimal normal subgroup of G_i since ρ_α is an irreducible representation. By construction, $A_{\alpha\beta}$ is an elementary abelian $p(\alpha)$-group and $C_{G_i}(A_{\alpha\beta}) = G_i - D_\alpha$. If $\alpha_j = \alpha(=\alpha_i)$ for some $j < i$ then $A_jA_{j+1}\ldots A_i/A_{j+1}\ldots A_i$ is isomorphic to A_i in G_i since G_i operates on these chief factors as given by $G_i \to G_i/A_j\ldots A_i \cong G_{j-1} \to G_{j-1}/(G_{j-1} - D_\alpha) \cong D_\alpha \xrightarrow{\rho_\alpha} \mathrm{Aut}\, A_\alpha \cong \mathrm{Aut}\, A_j = \mathrm{Aut}\, A_jA_{j+1}\ldots A_i/A_{j+1}\ldots A_i$ and $G_i \to G_i/A_i \cong G_{i-1} \to G_{i-1}/(G_{i-1} - D_\alpha) \cong D_\alpha \xrightarrow{\rho_\alpha} \mathrm{Aut}_\alpha \cong \mathrm{Aut}\, A_i$. So by induction (a),(b),(c), and (d) are satisfied in G_i as well.

Finally, let $G = G_m$. Then (a) yields a chief series $G = A_1A_2\ldots A_m > A_2\ldots A_m > \ldots > A_{m-1}A_m > A_m > 1$. Two chief factors, $A_jA_{j+1}\ldots A_m/A_{j+1}\ldots A_m$ and $A_kA_{k+1}\ldots A_m/A_{k+1}\ldots A_m$

are isomorphic in G iff $\alpha_j = \alpha_k$ as it follows easily from (d) and (c),(2),(b). By (c), the ordering of the isomorphism classes of chief factors is the prescribed one and by (b) p is the characteristics function of G.

REMARK. It is necessary to consider groups with multiple chief factors since for example there is no finite solvable group without multiple chief factors representing



(Here the characteristics function satisfies (1) and (2) but not (3).) Indeed, suppose by way of contradiction that there were a finite solvable group G without multiple chief factors representing this poset and characteristics function. Then G has a chief series $G > N > M > 1$ such that $|G/N| = 2$, $|N/M| = 3^e$, $|M| = 3^f$, moreover $|G/C_G(N/M)| = 2$ hence $C_G(N/M) = N$ and similarly $C_G(M) = N$. Since G/N acts irreducibly on N/M and on M, it follows that $|N/M| = |M| = 3$. Now N is abelian of order 9 and any element $g \in G-N$ operates as inversion on N. Hence N/M and M are G-isomorphic chief factors, a contradiction.

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