Werk

Titel: Weil-Petersson Metric in the Moduli Space of Compact Polarized Kähler-Einstein Ma...

Autor: Nannicini, Antonella

Jahr: 1986

PURL: https://resolver.sub.uni-goettingen.de/purl?365956996_0054|log25

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

manuscripta math. 54, 405 - 438 (1986)

manuscripta
mathematica
© Springer-Verlag 1986

WEIL-PETERSSON METRIC IN THE MODULI SPACE OF COMPACT POLARIZED KÄHLER-EINSTEIN MANIFOLDS OF ZERO FIRST CHERN CLASS

Antonella Nannicini

We study the problem of computing the curvature of the Weil-Petersson metric of the moduli space of general compact polarized Kähler-Einstein manifolds of zero first Chern class. We use canonical lifting of vector fields from the moduli space to the total deformation space to obtain a formula for the curvature of the Weil-Petersson metric. From this formula we obtain negative bisectional curvature for certain directions. This formula also reprove and explain the recent result of Schumacher that the holomorphic sectional curvature of the Weil-Petersson metric for K3-surfaces and symplectic manifolds are negative.

Table of contents

0. Introduction

1. Kähler Condition

2. First Part of the Curvature Computation

3. Second Part of the Curvature Computation

4. K3-Surfaces Case

5. Generalisation of K3-Surfaces Case

References

NANNICINI

0. Introduction

In this paper we study the curvature tensor of the Weil-Petersson metric in the Moduli Space of polarized manifolds of zero first Chern class. Weil-Petersson metric was investigated first for Teichmüller spaces by Weil, Ahlfors, Royden [1], [2], [5], and recently by Wolpert, who proved the negativity of the Riemannian sectional curvature [8]. For higher dimensions, the problem of constructing a metric in the Moduli Space, was investigated after the solution of Yau-Calabi's conjecture about the existence of Einstein-Kähler metrics on compact Kähler manifolds of negative, or zero, first Chern class, [9]. Koizo [4] first proved the Kähler condition for this metric.
Recently Y.T. Siu [7] introduced the concept of canonical lifting of a tangent vector in the Moduli Space of compact Kähler-Einstein manifolds of negative first Chern class. He wrote the full curvature tensor in a quite simple form, proving the negativity of the holomorphic bisectional curvature in some special cases. In [6] Schumacher studied the Weil-Petersson metric in the Moduli space of polarized compact complex manifolds of zero first Chern class proving, indirectly, the negativity of the curvature for symplectic manifolds. Namely he remarks that this metric is nothing else than the Bergmann metric of a symmetric domain of III type.
Following Siu, [7], it seems to be natural to apply the tecnique of canonical liftings to the case

of polarized manifolds of zero first Chern class. The hypothesis of polarization with respect to a certain Kähler class is necessary, in this case, in order to define the Weil-Petersson metric. Namely we have to use the harmonic representative of the infinitesimal deformation with respect to the Einstein-Kähler metric of zero Ricci tensor, unique up to constants, for fixed Kähler class. Using canonical liftings it is much easier to prove the Kähler condition and write the full expression for the curvature tensor. In this paper, using this tecnique, we obtain a formula for the curvature of the Weil-Petersson metric that seems to be the best possible one for the general case. From this formula we obtain the negativity of the holomorphic sectional curvature in the case of symplectic manifolds. Namely the existence of a closed holomorphic 2-form nowhere zero makes easy to compute the L^2-norm of a certain element in $H^2(M,\Lambda^2 T_M)$, that seems to be like an obstruction to the negativity of the curvature. For the symplectic case this term can be compared to the others, however in the general case this comparison seems to be difficult, and, at the moment, we are able to state the negativity of the curvature only under the strong assumption that this term vanishes.

I would like to express my gratitude to Professor Yum-Tong Siu for having introduced me to this subject and for helping me during the preparation of this paper.

I. Kähler Condition

(1.1) Let M_o be a compact complex manifold and ω_o be a Kähler class on M_o, (M_o,ω_o) is called a polarized manifold.

Denote by R (C) the field of real (complex) numbers respectively. Let $I=(-\varepsilon,\varepsilon)$ the interval of radius ε in R, a smooth family of compact complex manifolds $\Pi:M\to I$ is called a family of polarized manifolds if: i) each $M_t=\Pi^{-1}(t)$ is a polarized manifold (M_t,ω_t) for some Kähler class ω_t on M_t, ii) the cohomology class $[\Phi_t^*\omega_t]\in H^2(M_o,C)$ is constant for any t, where $\Phi:M_o\times I\to M$ is a given smooth trivialisation of the family and $\Phi_t:M_o\to M_t$ is the restriction of Φ to $M_o\times\{t\}$.

When we are interested in to distinguish a manifold (M_o,ω_o) in the family we call (M_t,ω_t) a family of polarized manifolds with respect to ω_o.

Assume M_o is a compact complex manifold of zero first Chern class and ω_o is a fixed Kähler class on M_o. Let g_o be the unique Einstein-Kähler metric on M_o of zero Ricci tensor in the Kähler class ω_o, [9]. Assume (M_t,ω_t) is a family of polarized manifolds with respect to ω_o and g_t is the unique Einstein-Kähler metric on M_t of zero Ricci tensor in the fixed class ω_t. For notational simplicity in the rest of the paper we will denote by ω_t the Kähler class as well the representative of the class which is the Kähler form of the metric g_t. Let $(\tilde{U}_i,w_i^\alpha,t)$ be a covering of M and on $\tilde{U}_i\cap\tilde{U}_j$ let $w_i^\alpha=f_{ij}^\alpha(w_j^1,..,w_j^n,t)$, where n is the complex dimension of M_o. Let $U_i=\tilde{U}_i|_{M_o}$ and $z_i^\alpha=w_i^\alpha|_{U_i}$ the restriction of $\tilde{U}_i$ and w_i^α to M_o.

Here and in the following $\alpha,\beta,..\in\{1,2,..,n\}$.
Consider a smooth trivialisation $\Phi: M_o \times I \to \mathbf{M}$ such that $\Phi|_{M_o \times \{o\}}$ is the identity on M_o. In local coordinates we have: $w_i^\alpha = F_i^\alpha(z_i^1,..,z_i^n,t)$.
Assume that the lifting of the vector field $\frac{\partial}{\partial t}$ on I to the vector field

$$v = \frac{\partial}{\partial t} + \sum_\alpha \frac{\partial F_i^\alpha}{\partial t} \frac{\partial}{\partial w_i^\alpha} + \sum_\alpha \frac{\partial \overline{F_i^\alpha}}{\partial t} \frac{\partial}{\partial \overline{w_i^\alpha}}$$

is a canonical lifting in the sense of [7], that is $\phi = \sum_\alpha \phi_i^\alpha \frac{\partial}{\partial z_i^\alpha}$ on U_i, with $\phi_i^\alpha = \bar\partial(\frac{\partial F_i^\alpha}{\partial t})$, is the harmonic representative of the infinitesimal deformation.
As we are considering a family of polarized manifolds we have:

$$(1)\quad \Phi_t^* \omega_t = \omega_o + d\psi_t$$

where $\{\psi_t\}_{t\in I}$ is a family of 1-forms on M_o and d is the usual differential operator.
Compute the Lie derivative of ω along a canonical lifting v of the vector field $\frac{\partial}{\partial t}$ calculated at the point t=o, we have:

$$(L_v\omega)_{ab} = \frac{\partial}{\partial t}\left(\omega_{cd} \frac{\partial w^c}{\partial z^a} \frac{\partial w^d}{\partial z^b}\right)$$

$$= (\omega_{ab})^\cdot + \omega_{cb}(\partial_a w^c)^\cdot + \omega_{ad}(\partial_b w^d)^\cdot$$

$$= (\omega_{ab})^\cdot + \omega_{cb}\,\partial_a v^c + \omega_{ad}\,\partial_b v^d$$

where $(\ )^\cdot$ means derivative with respect to t, $\partial_a = \frac{\partial}{\partial z^a}$, $a,b,c,d\in\{1,2,..,n,\bar1,\bar2,..,\bar n\}$ and $\{z^a\}$,

$\{w^a\}$ mean $\{z^a_i\}$, $\{w^a_i\}$ for some i. The summation convention is used here as well in the rest of the paper. Let $\omega = \frac{1}{2}\sqrt{-1}g_{ab}dw^a\wedge dw^b$, we get:

$$(L_v\omega)_{\alpha\bar\beta} = \tfrac{1}{2}\sqrt{-1}\{(g_{\alpha\bar\beta})^{\cdot} + g_{\gamma\bar\beta}\,\partial_\alpha v^\gamma + g_{\alpha\bar\delta}\,\overline{\partial_\beta v^\delta}\},$$

$$(L_v\omega)_{\alpha\beta} = \tfrac{1}{2}\sqrt{-1}\{-g_{\beta\bar\gamma}\,\overline{\phi^\gamma_\alpha} + g_{\alpha\bar\delta}\,\overline{\phi^\delta_\beta}\}$$

$$= \tfrac{1}{2}\sqrt{-1}\{\phi_{\alpha\beta} - \phi_{\beta\alpha}\}$$

where we denote $\phi_{\alpha\beta} = g_{\alpha\bar\delta}\,\phi^{\bar\delta}_\beta = g_{\alpha\bar\delta}\,\overline{\phi^\delta_\beta}$,

$$(L_v\omega)_{\bar\alpha\bar\beta} = \tfrac{1}{2}\sqrt{-1}\{g_{\gamma\bar\beta}\phi^\gamma_{\bar\alpha} - g_{\delta\bar\alpha}\phi^\delta_{\bar\beta}\}$$

$$= \tfrac{1}{2}\sqrt{-1}\{\phi_{\bar\beta\bar\alpha} - \phi_{\bar\alpha\bar\beta}\}, \text{ where } \phi_{\bar\alpha\bar\beta} = \overline{\phi_{\alpha\beta}}$$

$$(L_v\omega)_{\bar\alpha\beta} = -(L_v\omega)_{\beta\bar\alpha}.$$

Or:

$$L_v\omega = \tfrac{1}{2}\sqrt{-1}(\phi_{\alpha\beta} - \phi_{\beta\alpha})\,dz^\alpha\wedge dz^\beta +$$

$$+\sqrt{-1}\{(g_{\alpha\bar\beta})^{\cdot} + g_{\gamma\bar\beta}\,\partial_\alpha v^\gamma + g_{\alpha\bar\delta}\,\overline{\partial_\beta v^\delta}\}\,dz^\alpha\wedge d\bar z^\beta +$$

$$+ \tfrac{1}{2}\sqrt{-1}(\phi_{\bar\beta\bar\alpha} - \phi_{\bar\alpha\bar\beta})\,d\bar z^\alpha\wedge d\bar z^\beta$$

$$= \sqrt{-1}\phi_{\alpha\beta}dz^\alpha\wedge dz^\beta + \sqrt{-1}\{(g_{\alpha\bar\beta})^{\cdot} + g_{\gamma\bar\beta}\,\partial_\alpha v^\gamma +$$

$$+ g_{\alpha\bar\delta}\,\overline{\partial_\beta v^\delta}\}\,dz^\alpha\wedge d\bar z^\beta + \sqrt{-1}\phi_{\bar\beta\bar\alpha}d\bar z^\alpha\wedge d\bar z^\beta.$$

From (1) there exists a 1-form $(E_\alpha dz^\alpha + E_{\bar\alpha}d\bar z^\alpha)$ on M_o such that:

$$(2)\quad L_v\omega = \nabla_\alpha E_\beta dz^\alpha\wedge dz^\beta + (\nabla_\alpha E_{\bar\beta} - \nabla_{\bar\beta}E_\alpha)dz^\alpha\wedge d\bar z^\beta +$$

$$+ \nabla_{\bar\alpha}E_{\bar\beta}d\bar z^\alpha\wedge d\bar z^\beta$$

where $\nabla_\alpha, \nabla_{\bar\alpha}$ denote covariant derivative with respect to $\frac{\partial}{\partial z^\alpha}$, $\frac{\partial}{\partial \bar z^\alpha}$ respectively.

First step will be to prove the symmetry condition $\phi_{\bar\alpha\bar\beta} = \phi_{\bar\beta\bar\alpha}$ (or equivalently $\phi_{\alpha\beta} = \phi_{\beta\alpha}$).

We use classical notations for the Ricci tensor $R_{\alpha\bar{\beta}}=\partial_\alpha\partial_{\bar{\beta}}\log(\det g_{\gamma\bar{\delta}})$, for the inverse matrix of the metric tensor $(g^{\alpha\bar{\beta}})=(g_{\gamma\bar{\delta}})^{-1}$ and for the adjoint operator $\bar{\partial}^*$ of $\bar{\partial}$.
Denote $\Psi=\phi_{\bar{\alpha}\bar{\beta}}d\bar{z}^\alpha\wedge d\bar{z}^\beta$, from (2) we get $\Psi=\sqrt{-1}\bar{\partial}(E_{\bar{\alpha}}d\bar{z}^\alpha)$. So Ψ is $\bar{\partial}$-exact and furthermore $\bar{\partial}$-closed. On the other hand Ψ is harmonic. In fact let $\Box$ be the complex laplacian, we have:

$$
\begin{aligned}
(\Box\Psi)_{\bar{\alpha}\bar{\beta}}&=(\bar{\partial}\bar{\partial}^*\Psi)_{\bar{\alpha}\bar{\beta}}=\nabla_{\bar{\alpha}}(\bar{\partial}^*\Psi)_{\bar{\beta}}-\nabla_{\bar{\beta}}(\bar{\partial}^*\Psi)_{\bar{\alpha}}\\
&=\nabla_{\bar{\alpha}}(g^{\gamma\bar{\mu}}\nabla_\gamma\Psi_{\bar{\mu}\bar{\beta}})-\nabla_{\bar{\beta}}(g^{\gamma\bar{\mu}}\nabla_\gamma\Psi_{\bar{\mu}\bar{\alpha}})\\
&=\tfrac{1}{2}\{\nabla_{\bar{\alpha}}g^{\gamma\bar{\mu}}\nabla_\gamma(\phi_{\bar{\mu}\bar{\beta}}-\phi_{\bar{\beta}\bar{\mu}})-\nabla_{\bar{\beta}}g^{\gamma\bar{\mu}}\nabla_\gamma(\phi_{\bar{\mu}\bar{\alpha}}-\phi_{\bar{\alpha}\bar{\mu}})\}\\
&=\nabla_{\bar{\alpha}}\nabla_\gamma\phi^\gamma_{\bar{\beta}}-\nabla_{\bar{\beta}}\nabla_\gamma\phi^\gamma_{\bar{\alpha}}\\
&=\nabla_\gamma\nabla_{\bar{\alpha}}\phi^\gamma_{\bar{\beta}}+R^\gamma{}_{\tau\bar{\alpha}\gamma}\phi^\tau_{\bar{\beta}}+R_{\bar{\beta}}{}^{\bar{\tau}}{}_{\bar{\alpha}\gamma}\phi^\gamma_{\bar{\tau}}+\\
&-\nabla_\gamma\nabla_{\bar{\beta}}\phi^\gamma_{\bar{\alpha}}-R^\gamma{}_{\tau\bar{\beta}\gamma}\phi^\tau_{\bar{\alpha}}-R_{\bar{\alpha}}{}^{\bar{\tau}}{}_{\bar{\beta}\gamma}\phi^\gamma_{\bar{\tau}}\\
&=R_{\tau\bar{\alpha}}\phi^\tau_{\bar{\beta}}-R_{\tau\bar{\beta}}\phi^\tau_{\bar{\alpha}}=0
\end{aligned}
$$

where the fifth equality follows from the $\bar{\partial}^*$-closedness of $\phi^\alpha_{\bar{\beta}}$, the seventh from the symmetry of the Riemann tensor and the $\bar{\partial}$-closedness of $\phi^\alpha_{\bar{\beta}}$ and the last expression is zero because of the vanishing of the Ricci tensor.
So Ψ is harmonic and $\bar{\partial}$-exact, by classical harmonic theory is $\Psi\equiv0$, or equivalently $\phi_{\bar{\alpha}\bar{\beta}}=\phi_{\bar{\beta}\bar{\alpha}}$.

(1.2) So far we have:

$(L_v\omega)_{\alpha\beta}=0$, $(L_v\omega)_{\bar{\alpha}\bar{\beta}}=0$ and
$(L_v\omega)=2(L_v\omega)_{\alpha\bar{\beta}}dz^\alpha\wedge dz^{\bar{\beta}}=d(E_\alpha dz^\alpha+E_{\bar{\alpha}}dz^{\bar{\alpha}})$.
Thus $(L_v\omega)$ is a 2-form on M_o of type (1.1), d-closed and d-exact.

We claim $(L_v\omega) = \partial\bar\partial F$, with F real function.

Let $\xi^{(1,0)} = E_\alpha dz^\alpha$, $\xi^{(0,1)} = E_{\bar\beta} d\bar z^\beta$. By (1.1) is

$\partial\xi^{(0,1)} = \bar\partial\xi^{(0,1)} = 0$, so we have $\xi^{(0,1)} = c_\nu \bar\chi_\nu + \bar\partial\zeta$

with χ_ν holomorphic 1-forms.

Thus:

$$(L_v\omega) = -\sqrt{-1}(\bar\partial\xi^{(1,0)} + \partial\xi^{(0,1)}) = -2\sqrt{-1}\,\mathrm{Re}\;\partial\xi^{(0,1)} =$$
$$= -2\sqrt{-1}\,\mathrm{Re}\;\partial\bar\partial\zeta = \partial\bar\partial(2\,\mathrm{Im}\;\zeta).$$

Or $(L_v\omega) = \partial\bar\partial F$ with $F = \mathrm{Im}\;\zeta$.

(1.3) Next step will be to prove that $L_v\omega = 0$.

From the Einstein condition $R_{\alpha\bar\beta} = 0$ we get

$\partial_\alpha\partial_{\bar\beta}\log(\det g_{\gamma\bar\mu}) = 0$ on M_t for all t.

Differentiating with respect to t we have:

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial z^\gamma}{\partial w^\alpha}\partial_\gamma + \frac{\partial\bar z^\gamma}{\partial w^\alpha}\partial_{\bar\gamma}\right)\left(\frac{\partial z^\mu}{\partial\bar w^\beta}\partial_\mu + \frac{\partial z^\mu}{\partial w^\beta}\partial_{\bar\mu}\right)\log(\det g_{\gamma\bar\delta})\right\} = 0$$

or:

$$-(\partial_\alpha w^\lambda)^\cdot\partial_\lambda\partial_{\bar\beta}\log(\det g_{\gamma\bar\delta}) - (\partial_\alpha\bar w^\lambda)^\cdot\partial_{\bar\lambda}\partial_{\bar\beta}\log(\det g_{\gamma\bar\delta}) +$$
$$- \partial_\alpha((\partial_{\bar\beta}w^\mu)^\cdot\partial_\mu\log(\det g_{\gamma\bar\delta}) - \partial_\alpha((\partial_\beta w^\mu)^\cdot\partial_{\bar\mu}\log(\det g_{\gamma\bar\delta}))$$
$$+ \partial_\alpha\partial_{\bar\beta}(g^{\lambda\bar\mu}(g_{\alpha\bar\beta})^\cdot) = 0$$

and:

$$-(\partial_\alpha w^\lambda)^\cdot R_{\lambda\bar\beta} - (\partial_\alpha\bar w^\lambda)^\cdot\partial_{\bar\lambda}\partial_{\bar\beta}\log(\det g_{\gamma\bar\delta}) - (\partial_\alpha(\partial_{\bar\beta}w^\mu)^\cdot)\cdot$$
$$\cdot\,\partial_\mu\log(\det g_{\gamma\bar\delta}) - (\partial_{\bar\beta}w^\mu)^\cdot\partial_\alpha\partial_\mu\log(\det g_{\gamma\bar\delta}) - (\partial_\alpha(\partial_\beta w^\mu)^\cdot)\cdot$$
$$\cdot\,\partial_{\bar\mu}\log(\det g_{\gamma\bar\delta}) - (\partial_\beta w^\mu)^\cdot R_{\alpha\bar\mu} + \partial_\alpha\partial_{\bar\beta}(g^{\gamma\bar\mu}(g_{\lambda\bar\mu})^\cdot) = 0.$$

At the point under consideration we use normal coordinates with $dg_{\alpha\bar\beta} = 0$ and $\partial_\alpha\partial_\beta g_{\lambda\bar\mu} = 0$, also we use the vanishing of the Ricci tensor and we get:

$\partial_\alpha\partial_{\bar\beta}(g^{\lambda\bar\mu}(g_{\lambda\bar\mu})^\cdot) = 0$.

On the other hand we have:

$$\frac{2}{\sqrt{-1}}\, g^{\alpha\bar{\beta}}(L_v\omega)_{\alpha\bar{\beta}} = g^{\alpha\bar{\beta}}(g_{\alpha\bar{\beta}})^{\cdot} + \partial_\alpha v^\alpha + \overline{\partial_\beta v^\beta}.$$

Thus:

$$\frac{2}{\sqrt{-1}}\, \partial_\lambda\partial_{\bar{\mu}}(g^{\alpha\bar{\beta}}(L_v\omega)_{\alpha\bar{\beta}}) = \partial_\lambda\partial_{\bar{\mu}}(g^{\alpha\bar{\beta}}(g_{\alpha\bar{\beta}})^{\cdot}) + \partial_\lambda\partial_{\bar{\mu}}(\partial_\alpha v^\alpha + \overline{\partial_\beta v^\beta})$$

$$= \partial_\lambda\partial_{\bar{\mu}}(g^{\alpha\bar{\beta}}(g_{\alpha\bar{\beta}})^{\cdot}) + \partial_\lambda\partial_\alpha\phi^{\alpha}_{\bar{\mu}} + \partial_{\bar{\beta}}\partial_{\bar{\mu}}\overline{\phi^{\beta}_{\lambda}}$$

$$= \partial_\lambda\partial_{\bar{\mu}}(g^{\alpha\bar{\beta}}(g_{\alpha\bar{\beta}})^{\cdot}) + \partial_\lambda(g^{\alpha\bar{\gamma}}\nabla_\alpha\phi_{\bar{\gamma}\bar{\mu}}) + \partial_{\bar{\mu}}(g^{\beta\bar{\delta}}\nabla_\beta\phi_{\bar{\delta}\lambda})$$

$$= \partial_\lambda\partial_{\bar{\mu}}(g^{\alpha\bar{\beta}}(g_{\alpha\bar{\beta}})^{\cdot})$$

because of the $\bar{\partial}^*$-closedness of $\phi^{\lambda}_{\bar{\mu}}$.

Thus in normal coordinates is:

$$\partial_\lambda\partial_{\bar{\mu}}(g^{\alpha\bar{\beta}}(L_v\omega)_{\alpha\bar{\beta}}) = 0$$

or: $g^{\alpha\bar{\beta}}(L_v\omega)_{\alpha\bar{\beta}}$ is harmonic on M_o and hence is a costant function on M_o.

(1.4) From (1.2) and (1.3) we get

$$\Delta F = g^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\, F = \text{constant}.$$

By the compactness of M_o and the fact that F is real this implies $\Delta F=0$. In particular F is harmonic on M_o and that means F is constant on M_o, or $L_v\omega=0$.

(1.5) Now replace the real interval I by an open neighbourhood Ω of C^N, and suppose $\Pi:\mathcal{M}\to\Omega$ is a holomorphic family of compact polarized Kähler-Einstein manifolds of zero Ricci curvature. Assume $t=(t_1,..,t_N)$ are holomorphic coordinates on Ω, and let t'_k, t''_k be respectively the real and the imaginary part of t_k. Let v'_k and v''_k be

canonical liftings of $\frac{\partial}{\partial t'_k}$, $\frac{\partial}{\partial t''_k}$ respectively.
We define:

$$L_{v_k} = \tfrac{1}{2}(L_{v'_k} - \sqrt{-1}\, L_{v''_k}),$$

$$L_{\bar{v}_k} = \tfrac{1}{2}(L_{v'_k} + \sqrt{-1}\, L_{v''_k}).$$

Denote $\phi(\frac{\partial}{\partial t'_k})$, $\phi(\frac{\partial}{\partial t''_k})$ the harmonic representative of the infinitesimal deformation in the direction $\frac{\partial}{\partial t'_k}$, $\frac{\partial}{\partial t''_k}$ respectively. Then let

$$\phi(\frac{\partial}{\partial t_k}) = \tfrac{1}{2}(\phi(\frac{\partial}{\partial t'_k}) - \sqrt{-1}\phi(\frac{\partial}{\partial t''_k})),$$

$$\phi(\frac{\partial}{\partial \bar{t}_k}) = \tfrac{1}{2}(\overline{\phi(\frac{\partial}{\partial t'_k})} + \sqrt{-1}\overline{\phi(\frac{\partial}{\partial t''_k})}).$$

We have:

$$\phi(\frac{\partial}{\partial t'_k}) = -\sqrt{-1}\ \phi(\frac{\partial}{\partial t''_k}).$$

Let $J=J_t$ denote the (1.1)-tensor representing the complex structure on M_t, from $J_\alpha^\beta = \sqrt{-1}\delta_\alpha^\beta$, $J_{\bar{\alpha}}^{\beta} = 0$, $J_{\bar{\alpha}}^{\bar{\beta}} = -\sqrt{-1}\delta_\alpha^\beta$, $J_\alpha^{\bar{\beta}} = 0$ (where δ_α^β is the Kronecker delta), it follows:

$$(L_{v_k} J)_{\bar{\alpha}}^{\beta} = 2\sqrt{-1}\ \phi(\frac{\partial}{\partial t_k})_{\bar{\alpha}}^{\beta},$$

$$(L_{\bar{v}_k} J)_{\alpha}^{\bar{\beta}} = (L_{v_k} J)_{\alpha}^{\beta} = (L_{v_k} J)_{\bar{\alpha}}^{\bar{\beta}} = 0.$$

Also we have:

$$(L_{\bar{v}_k} J) = \overline{(L_{v_k} J)}, \quad \phi(\frac{\partial}{\partial \bar{t}_k}) = \overline{\phi(\frac{\partial}{\partial t_k})}.$$

(1.6) We define the <u>Weil-Petersson metric</u> on Ω by:

$$h_{i\bar{j}}(t) = \int_{M_t} (L_{v_i} J)_a^p \overline{(L_{v_j} J)_b^q}\ g_{p\bar{q}}\ g^{a\bar{b}}\ dV_t$$

where dV_t denotes the volume form of the Einstein-Kähler metric on M_t, and denote $dV_0 = dV$.
In order to prove the Kähler condition for the

Weil-Petersson metric we have to show that $\frac{\partial}{\partial \bar{t}_k} h_{i\bar{j}}(o) = \frac{\partial}{\partial \bar{t}_j} h_{i\bar{k}}(o)$. As $dV_t = \omega_t^{\wedge n}/n!$, from (1.4) we get $L_v(dV_t) = 0$. Moreover from $(L_v g)_{\alpha\bar{\beta}} = \frac{2}{\sqrt{-1}}(L_v g)_{\alpha\bar{\beta}}$ we get $(L_v g)_{\alpha\bar{\beta}} = 0$.

The proof now is the same as in [7], we report it for sake of completness.

As the only non zero component of $L_v J$ is of type $(L_v J)^{\beta}_{\bar{\alpha}}$ and $(L_v g)^{\alpha\bar{\beta}} = -g^{\alpha\bar{\mu}}(L_v g)_{\lambda\bar{\mu}} g^{\lambda\bar{\beta}} = 0$, we have:

$$\frac{\partial h_{i\bar{j}}}{\partial \bar{t}_k}(o) = \int_{M_o} (L_{\bar{v}_k} L_{v_i} J)^p_a \overline{(L_{v_j} J)^q_b}\, g_{p\bar{q}}\, g^{a\bar{b}}\, dV + \int_{M_o} (L_{v_i} J)^p_a \overline{(L_{v_k} L_{v_j} J)^q_b}\, g_{p\bar{q}}\, g^{a\bar{b}} dV.$$

We claim the first integral vanishes. As the only non zero component of $(L_{\bar{v}_k} J)$ is of type $(L_{\bar{v}_k} J)^{\bar{\alpha}}_{\beta}$ it follows that $(L_{v_i} L_{\bar{v}_k} J)^{\alpha}_{\bar{\beta}} = 0$. Thus we get:

$$\int_{M_o} (L_{\bar{v}_k} L_{v_i} J)^p_a \overline{(L_{v_j} J)^q_b}\, g_{p\bar{q}}\, g^{a\bar{b}} dV =$$

$$= \int_{M_o} (L_{[\bar{v}_k, v_i]} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_j} J)^{\gamma}_{\delta}}\, g_{\alpha\bar{\gamma}}\, g^{\beta\bar{\delta}}\, dV.$$

Last integral is zero because $(L_{[\bar{v}_k, v_i]} J)^{\alpha}_{\bar{\beta}} = 2\sqrt{-1}\,\partial_{\bar{\beta}}[\bar{v}_k, v_i]^{\alpha}$ is $\bar{\partial}$-exact and $(L_{v_j} J)^{\alpha}_{\bar{\beta}}$ is $\bar{\partial}^*$-closed, where [,] means Lie product of vectors fields.

Thus:

$$\frac{\partial h_{i\bar{j}}}{\partial \bar{t}_k}(o) - \frac{\partial h_{i\bar{k}}}{\partial \bar{t}_j}(o) =$$

$$= \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_k} L_{v_j} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\beta\bar{\delta}}\, dV +$$

$$- \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_j} L_{v_k} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\beta\bar{\delta}}\, dV$$

$$= \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{[v_k, v_j]} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\beta\bar{\delta}}\, dV \equiv 0,$$

this concludes the proof of the Kähler property of the Weil-Petersson metric.

2. First Part of the Curvature Computation

(2.1) Consider normal coordinates $(t_1, .., t_N)$ at the point o in Ω, such that the Riemann tensor at o is given by:

$$(3) \quad R_{i\bar{j}l\bar{k}}^{(W.P.)}(o) = \frac{\partial^2 h_{i\bar{j}}}{\partial t_l \partial \bar{t}_k}(o)$$

Direct computation yields:

$(L_{v_k} L_{v_j} J)^{\alpha}_{\beta} = (L_{v_k} L_{v_j} J)^{\bar{\alpha}}_{\bar{\beta}} = 0,$

moreover use $(L_v g)_{\alpha\bar{\beta}} = 0$, we get:

$$(4) \quad R_{i\bar{j}l\bar{k}}^{(W.P.)}(o) = \int_{M_o} (L_{v_l} L_{v_i} J)^{p}_{a} \overline{(L_{v_k} L_{v_j} J)^{q}_{b}}\, g_{p\bar{q}}\, g^{a\bar{b}}\, dV +$$

$$+ \int_{M_o} (L_{v_i} J)^{p}_{a} \overline{(L_{\bar{v}_l} L_{v_k} L_{v_j} J)^{q}_{b}}\, g_{p\bar{q}}\, g^{a\bar{b}}\, dV.$$

(2.2) We look at the last integral in the expression (4), call it term II, we have:

$$L_{\bar{v}_l} L_{v_k} L_{v_j} = L_{[[\bar{v}_l, v_k], v_j]} + L_{v_j} L_{[\bar{v}_l, v_k]} +$$

$$+ L_{v_k} L_{[\bar{v}_l, v_j]} + L_{v_k} L_{v_j} L_{\bar{v}_l}.$$

Thus is $II = A + B + C + D$, where:

$$A = \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{[[\bar{v}_1, v_k], v_j]} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\bar{\beta}\delta}\, dV$$

$$B = \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_j} L_{[\bar{v}_1, v_k]} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\bar{\beta}\bar{\delta}}\, dV$$

$$C = \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_k} L_{[\bar{v}_1, v_j]} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\bar{\beta}\bar{\delta}}\, dV$$

$$D = \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_k} L_{v_j} L_{\bar{v}_1} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\bar{\beta}\bar{\delta}}\, dV.$$

The term A vanishes because $(L_{[[\bar{v}_1, v_k], v_j]} J)^{\gamma}_{\bar{\delta}}$ is $\bar{\partial}$-exact and $(L_{v_i} J)^{\alpha}_{\bar{\beta}}$ is $\bar{\partial}^*$-closed.

Now we compute term D. Direct computation yields:

$(L_{v_j} L_{\bar{v}_1} J)^{\alpha}_{\beta} = \frac{1}{2}\sqrt{-1}(L_{v_j} J)^{\alpha}_{\bar{\lambda}}(L_{\bar{v}_1} J)^{\bar{\lambda}}_{\beta}$, $(L_{v_j} L_{\bar{v}_1} J)^{\alpha}_{\bar{\beta}} = 0$,

$(L_{v_j} L_{\bar{v}_1} J)^{\bar{\alpha}}_{\bar{\beta}} = -\frac{1}{2}\sqrt{-1}(L_{\bar{v}_1} J)^{\bar{\alpha}}_{\mu}(L_{v_j} J)^{\mu}_{\bar{\beta}}$.

Thus:

$$D = \frac{1}{2} \int_{M_o} (L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{v_j} J)^{\gamma}_{\alpha}}\, (L_{v_1} J)^{\lambda}_{\gamma} \overline{(L_{v_k} J)^{\beta}_{\lambda}}\, dV.$$

(2.3) Term C can be obtained by term B after changing v_k and v_j, so consider term B.

From the harmonicity of $(L_{v_i} J)^{\alpha}_{\bar{\beta}}$, the $\bar{\partial}$-exactness of $(L_{[\bar{v}_1, \bar{v}_k]} J)^{\gamma}_{\bar{\delta}}$ and the previous computations we get:

$$B = -\int_{M_o} (L_{\bar{v}_j} L_{v_i} J)^{\alpha}_{\bar{\beta}} \overline{(L_{\bar{v}_1} L_{v_k} J)^{\gamma}_{\bar{\delta}}}\, g_{\alpha\bar{\gamma}}\, g^{\bar{\beta}\delta}\, dV.$$

We have to compute $(L_{\bar{v}_j} L_{v_i} J)^{\alpha}_{\bar{\beta}}$.

From the harmonicity of the $T_{M_t}^{(1,0)}$-valued $(0,1)$-form $(L_{v_i} J)^{\alpha}_{\bar{\beta}}$ for all t, where $T_{M_t}^{(1,0)} = T_{M_t}$ denote the holomorphic tangent space to M_t, we have:

(5) $g^{\alpha\bar{\beta}} \frac{\partial}{\partial w^{\alpha}} (g_{\bar{\delta}\gamma}(L_{v_i} J)^{\gamma}_{\bar{\beta}}) \equiv 0$ on M_t for all t.

Denote $(L_{v_i} J)^{\alpha}_{\bar{\beta}} = T^{\alpha}_{\bar{\beta}}$. Differentiating identity (5) with respect to t_j and denoting, as usual, by $(\ )^{\cdot}$ such differentiation we get:

$$(6)\quad (g^{\alpha\bar{\beta}})^{\cdot}\,\partial_{\alpha}(g_{\bar{\delta}\gamma}T^{\gamma}_{\bar{\beta}}) - g^{\alpha\bar{\beta}}(\partial_{\alpha}w^{\lambda})^{\cdot}\partial_{\lambda}(g_{\bar{\delta}\gamma}T^{\gamma}_{\bar{\beta}}) - g^{\alpha\bar{\beta}}(\partial_{\alpha}\bar{w}^{\lambda})^{\cdot}\cdot$$
$$\cdot\,\partial_{\bar{\lambda}}(g_{\bar{\delta}\gamma}T^{\gamma}_{\bar{\beta}}) + g^{\alpha\bar{\beta}}\partial_{\alpha}((g_{\bar{\delta}\gamma})^{\cdot}T^{\gamma}_{\bar{\beta}}) + g^{\alpha\bar{\beta}}\partial_{\alpha}(g_{\bar{\delta}\gamma}(T^{\gamma}_{\bar{\beta}})^{\cdot}) = 0$$

on M_o. On the other hand:

$$(T^{\gamma}_{\bar{\beta}})^{\cdot} = (L_{\bar{v}_j} T)^{\gamma}_{\bar{\beta}} + T^{\lambda}_{\bar{\beta}}(\partial_{\lambda}w^{\gamma})^{\cdot} - T^{\gamma}_{\bar{\mu}}(\partial_{\bar{\beta}}\bar{w}^{\mu})^{\cdot}$$
$$(g_{\bar{\delta}\gamma})^{\cdot} = -g_{\bar{\lambda}\gamma}(\partial_{\bar{\delta}}\bar{w}^{\lambda})^{\cdot} - g_{\bar{\delta}\lambda}(\partial_{\gamma}w^{\lambda})^{\cdot}$$
$$(g^{\alpha\bar{\beta}})^{\cdot} = g^{\lambda\bar{\beta}}(\partial_{\lambda}w^{\alpha})^{\cdot} + g^{\alpha\bar{\lambda}}(\partial_{\bar{\lambda}}\bar{w}^{\beta})^{\cdot}.$$

Substituting last expressions in (6) and using the harmonicity of $T^{\alpha}_{\bar{\beta}}$ we get finally:

$$(7)\quad g^{\alpha\bar{\beta}}\partial_{\alpha}(g_{\bar{\delta}\gamma}(L_{\bar{v}_j} T)^{\gamma}_{\bar{\beta}}) = g^{\alpha\bar{\beta}}g_{\bar{\lambda}\gamma}\partial_{\bar{\delta}}(\partial_{\alpha}\bar{w}^{\lambda})^{\cdot}T^{\gamma}_{\bar{\beta}} +$$
$$+ g^{\alpha\bar{\beta}}g_{\bar{\delta}\gamma}T^{\gamma}_{\bar{\mu}}\partial_{\bar{\beta}}(\partial_{\alpha}\bar{w}^{\mu})^{\cdot} +$$
$$+ g^{\alpha\bar{\beta}}(\partial_{\alpha}\bar{w}^{\lambda})^{\cdot}\partial_{\bar{\lambda}}(g_{\bar{\delta}\gamma}T^{\gamma}_{\bar{\beta}}).$$

Denote $(\partial_{\alpha}\bar{w}^{\lambda})^{\cdot} = S^{\bar{\lambda}}_{\alpha} = \overline{S^{\lambda}_{\alpha}}$ $(= \tfrac{1}{2}\sqrt{-1}\,\overline{(L_{v_j} J)^{\lambda}_{\alpha}})$.

Using the symmetry condition proved in (1.1) ($T_{\bar{\delta}\bar{\beta}} = T_{\bar{\beta}\bar{\delta}}$), a direct computation of (7) yields:

$$(8)\quad g^{\alpha\bar{\beta}}\partial_{\alpha}g_{\bar{\delta}\gamma}(L_{\bar{v}_j} L_{v_i} J)^{\gamma}_{\bar{\beta}} = \partial_{\bar{\delta}}(T^{\alpha}_{\lambda}S^{\lambda}_{\alpha}).$$

Using the fact that the Lie derivative of a symmetric tensor is a symmetric tensor and the vanishing of the (1,1)-component of $(L_v g)$ we have

$$g_{\bar{\delta}\gamma}(L_{\bar{v}_j} L_{v_i} J)^{\gamma}_{\bar{\beta}} = g_{\bar{\beta}\gamma}(L_{\bar{v}_j} L_{v_i} J)^{\gamma}_{\bar{\delta}}.$$

Thus expression (8) becomes:

$$(9)\quad \nabla_{\alpha}(L_{\bar{v}_j} L_{v_i} J)^{\alpha}_{\bar{\delta}} = \partial_{\bar{\delta}}(T^{\alpha}_{\lambda}S^{\bar{\lambda}}_{\alpha}).$$

From the $\bar{\partial}$-exactness is $(L_{\bar{v}_j} L_{v_i} J)^{\alpha}_{\bar{\delta}} = \nabla_{\bar{\delta}} H^{\alpha}$, for some vector field H on M_o; from the vanishing of the Ricci tensor is $\nabla_{\alpha}\nabla_{\bar{\delta}} H^{\alpha} = \nabla_{\bar{\delta}}\nabla_{\alpha} H^{\alpha}$. Thus (9) becomes:

(10) $\nabla_{\bar{\delta}}(\nabla_{\alpha} H^{\alpha} - T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha}) = 0$ for any δ.

Identity (10) means that $\nabla_{\alpha} H^{\alpha} - T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha}$ is a holomorphic function on M_o, hence a constant function. In particular if we denote V=volume of M_o we have:

$$\nabla_{\alpha} H^{\alpha} - T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha} = \frac{1}{V} \int_{M_o} (\nabla_{\alpha} H^{\alpha} - T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha}) dV = - \frac{1}{V} \int_{M_o} (T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha}) dV,$$

or:

$$\nabla_{\alpha} H^{\alpha} = T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha} - \frac{1}{V} \int_{M_o} (T^{\alpha}_{\bar{\lambda}} S^{\bar{\lambda}}_{\alpha}) dV.$$

Now go back to integral B:

$$- \int_{M_o} (L_{\bar{v}_j} L_{v_i} J)^{\alpha}_{\beta} \overline{(L_{\bar{v}_1} L_{v_k} J)^{\gamma}_{\delta}}\, g_{\alpha\bar{\gamma}} g^{\bar{\beta}\delta} dV$$

$$= - \int_{M_o} \nabla_{\bar{\beta}} H^{\alpha} \overline{(L_{\bar{v}_1} L_{v_k} J)^{\gamma}_{\delta}}\, g_{\alpha\bar{\gamma}} g^{\bar{\beta}\delta} dV$$

$$= \int_{M_o} H^{\alpha} \overline{g^{\beta\delta} \nabla_{\beta} (g_{\gamma\bar{\alpha}} (L_{\bar{v}_1} L_{v_k} J)^{\gamma}_{\delta})}\, dV$$

$$= \int_{M_o} H^{\alpha} \overline{\nabla_{\bar{\alpha}} (\tfrac{1}{2}\sqrt{-1} (L_{v_k} J)^{\nu}_{\lambda}\, (L_{\bar{v}_1} J)^{\lambda}_{\nu})}\, dV$$

$$= \tfrac{1}{2}\sqrt{-1} \int_{M_o} (\nabla_{\alpha} H^{\alpha})\, (L_{\bar{v}_k} J)^{\bar{\nu}}_{\lambda}\, (L_{v_1} J)^{\lambda}_{\bar{\nu}}\, dV$$

$$= -(\tfrac{1}{2})^2 \int_{M_o} (L_{v_i} J)^{\alpha}_{\mu}\, (L_{\bar{v}_j} J)^{\bar{\mu}}_{\alpha}\, (L_{\bar{v}_k} J)^{\bar{\nu}}_{\lambda}\, (L_{v_1} J)^{\lambda}_{\bar{\nu}}\, dV +$$

$$+ (\tfrac{1}{2})^2 \frac{1}{V} \Big(\int_{M_o} (L_{v_1} J)^{\lambda}_{\bar{\nu}} (L_{\bar{v}_k} J)^{\bar{\nu}}_{\lambda} dV \Big) \Big(\int_{M_o} (L_{\bar{v}_j} J)^{\bar{\lambda}}_{\alpha} (L_{v_i} J)^{\alpha}_{\bar{\lambda}} dV \Big).$$

(2.4) So far we have:

$$(11)\quad R_{i\bar{j}l\bar{k}}^{(W.P.)}(o) =$$

$$= \int_{M_o} (L_{v_l} L_{v_i} J)_a^p \overline{(L_{v_k} L_{v_j} J)_b^q}\, g_{p\bar{q}}\, g^{a\bar{b}}\, dV +$$

$$+ \tfrac{1}{2} \int_{M_o} (L_{v_i} J)_{\bar{\beta}}^{\alpha} (L_{\bar{v}_j} J)_{\alpha}^{\bar{\gamma}} (L_{v_l} J)_{\bar{\gamma}}^{\lambda} (L_{\bar{v}_k} J)_{\lambda}^{\bar{\beta}}\, dV +$$

$$+ (\tfrac{1}{2})^2 \frac{1}{V} \Big(\int_{M_o} (L_{v_l} J)_{\bar{\nu}}^{\lambda} (L_{\bar{v}_k} J)_{\lambda}^{\bar{\nu}} dV \Big) \Big(\int_{M_o} (L_{\bar{v}_j} J)_{\alpha}^{\bar{\lambda}} (L_{v_i} J)_{\bar{\lambda}}^{\alpha} dV$$

$$+ (\tfrac{1}{2})^2 \frac{1}{V} \Big(\int_{M_o} (L_{v_l} J)_{\bar{\nu}}^{\lambda} (L_{\bar{v}_j} J)_{\lambda}^{\bar{\nu}} dV \Big) \Big(\int_{M_o} (L_{\bar{v}_k} J)_{\alpha}^{\bar{\lambda}} (L_{v_i} J)_{\bar{\lambda}}^{\alpha} dV$$

$$- (\tfrac{1}{2})^2 \int_{M_o} (L_{v_i} J)_{\bar{\mu}}^{\alpha} (L_{\bar{v}_j} J)_{\alpha}^{\bar{\mu}} (L_{\bar{v}_k} J)_{\lambda}^{\bar{\nu}} (L_{v_l} J)_{\bar{\nu}}^{\lambda}\, dV +$$

$$- (\tfrac{1}{2})^2 \int_{M_o} (L_{v_i} J)_{\bar{\mu}}^{\alpha} (L_{\bar{v}_k} J)_{\alpha}^{\bar{\mu}} (L_{\bar{v}_j} J)_{\lambda}^{\bar{\nu}} (L_{v_l} J)_{\bar{\nu}}^{\lambda}\, dV .$$

We are interested in to prove the negativity of the curvature of the Weil-Petersson metric at least in some special case, but expression [11] gives us the result only in the known case n=1. So,as in the case of deformations of manifolds with negative first Chern class [7], we need a second part of the curvature computation.

3. Second Part of the Curvature Computation

(3.1) In this part we will use the argument and the notations of [7] (§5), we will recall the notations for sake of completness.

Denote $T_{\bar{\beta}}^{\alpha} = (L_{v_i} J)_{\bar{\beta}}^{\alpha}$, where, as usual, v is a canonical lifting of a tangent vector $\frac{\partial}{\partial t_i}$ in Ω at o.

From the $\bar{\partial}$-closedness of $T^{\alpha}_{\bar{\beta}}$ we have:

$$(12)\quad \frac{\partial}{\partial \bar{w}^{\alpha}} T^{\gamma}_{\bar{\beta}} - \frac{\partial}{\partial \bar{w}^{\beta}} T^{\gamma}_{\bar{\alpha}} = 0 \quad \text{on } M_t \text{ for all } t.$$

Taking the differentiation with respect to t_i of the previous identity at the point o we have:

$$(13)\quad \partial_{\bar{\alpha}}(L_{v_i} T)^{\gamma}_{\bar{\beta}} - \partial_{\bar{\beta}}(L_{v_i} T)^{\gamma}_{\bar{\alpha}} = \sqrt{-1}(T^{\sigma}_{\bar{\beta}} \partial_{\sigma} T^{\gamma}_{\bar{\alpha}} - T^{\sigma}_{\bar{\alpha}} \partial_{\sigma} T^{\gamma}_{\bar{\beta}}).$$

Using normal coordinates at the point under consideration and the $\bar{\partial}^*$-closedness of $T^{\alpha}_{\bar{\beta}}$, (13) becomes:

$$(14)\quad \partial_{\bar{\alpha}}(L_{v_i} T)^{\gamma}_{\bar{\beta}} - \partial_{\bar{\beta}}(L_{v_i} T)^{\gamma}_{\bar{\alpha}} = \sqrt{-1}\,\partial_{\sigma}(T^{\sigma}_{\bar{\beta}} T^{\gamma}_{\bar{\alpha}} - T^{\sigma}_{\bar{\alpha}} T^{\gamma}_{\bar{\beta}}).$$

Denote:

$$\Theta_{\bar{\beta}\bar{\alpha},\bar{\tau}\bar{\gamma}} = T_{\bar{\beta}\bar{\tau}} T_{\bar{\alpha}\bar{\gamma}} - T_{\bar{\alpha}\bar{\tau}} T_{\bar{\beta}\bar{\gamma}}, \text{ and}$$

$$\Theta^{\beta\alpha}_{\bar{\tau}\bar{\gamma}} = g^{\beta\bar{\lambda}}\, g^{\alpha\bar{\mu}}\, \Theta_{\bar{\lambda}\bar{\mu},\bar{\tau}\bar{\gamma}}.$$

Let S be a tensor of type $S^{\alpha}_{\bar{\beta}}$, then denote $\bar{D}_1 S$ the $\bar{\partial}$-operator applied to the first index of the tensor $S_{\bar{\alpha}\bar{\beta}} = g_{\bar{\alpha}\gamma} S^{\gamma}_{\bar{\beta}}$.

Let Θ be a tensor of type $\Theta^{\alpha\beta}_{\bar{\lambda}\bar{\delta}}$, then denote $\bar{D}_2^* \Theta$ the $\bar{\partial}^*$-operator applied to the second couple of indices of the tensor $\Theta_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}} = g_{\bar{\alpha}\lambda} g_{\bar{\beta}\nu} \Theta^{\lambda\nu}_{\bar{\gamma}\bar{\delta}}$, then we have:

$$(15)\quad \bar{D}_1(L_{v_i} T) = \sqrt{-1}\ \bar{D}_2^* \Theta\ .$$

Denote X the space of tensor of type $\Xi_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}}$ such that:

1) $\Xi_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}} = -\Xi_{\bar{\beta}\bar{\alpha},\bar{\gamma}\bar{\delta}}$

2) $\Xi_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}} = \Xi_{\bar{\gamma}\bar{\delta},\bar{\alpha}\bar{\beta}}$

3) $\Xi_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}} + \Xi_{\bar{\alpha}\bar{\gamma},\bar{\delta}\bar{\beta}} + \Xi_{\bar{\alpha}\bar{\delta},\bar{\beta}\bar{\gamma}} = 0.$

We have $\Theta_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}} \in X$.

Also denote $\bar{D}_\delta$ ($\delta=1,2$) the $\bar{\partial}$-operator applied to the sth set of indices of elements of X, and $\bar{D}_\delta^*$ the adjoint operator of $\bar{D}_\delta$.
Let $\Box_\delta=\bar{D}_\delta^*\bar{D}_\delta+\bar{D}_\delta\bar{D}_\delta^*$ the Laplace operator, H_δ the harmonic projector and G_δ the Green's operator, so that $\Xi=H_\delta\Xi+\Box_\delta G_\delta\Xi$.
For $\Xi\epsilon X$ we have:

a) $\bar{D}_1\bar{D}_2\Xi=\bar{D}_2\bar{D}_1\Xi$,

b) $\bar{D}_2\bar{D}_1^*\Xi=\bar{D}_1^*\bar{D}_2\Xi$,

c) $\bar{D}_1^*\bar{D}_2^*\Xi=\bar{D}_2^*\bar{D}_1^*\Xi$,

d) $\bar{D}_1\bar{D}_2^*\Xi=\bar{D}_2^*\bar{D}_1\Xi$,

e) $\Box_1\Xi\epsilon X$,

f) $\Box_1\Xi=\Box_2\Xi$,

g) $[\bar{D}_2^*,\Box_1]\Xi=(\bar{D}_2^*\Box_1-\Box_1\bar{D}_2^*)\Xi=0$ if $\bar{D}_1\Xi=0$.

The proof consists in a direct computation, remark that condition g) is the same as in [7] when we force the scalar curvature to be zero.
Denote $X_o=\{\Xi\epsilon X : \bar{D}_1\Xi=0\}$.
By property g) on X_o is $\bar{D}_2^*\Box_1=\Box_1\bar{D}_2^*$.
We have $\Theta\epsilon X_o$ because, from the harmonicity of $T^{\alpha}_{\bar{\beta}}$, is $\bar{D}_1(T_{\bar{\alpha}\bar{\beta}}T_{\bar{\gamma}\bar{\delta}}-T_{\bar{\gamma}\bar{\beta}}T_{\bar{\alpha}\bar{\delta}})=0$.
Also $H_1\bar{D}_2^*\Theta=0$, in fact:

$$\begin{aligned}H_1\bar{D}_2^*\Theta &= H_1(-\sqrt{-1}\,\bar{D}_1(L_{v_i}T))=-\sqrt{-1}H_1\bar{D}_1(L_{v_i}T)\\ &= -\sqrt{-1}H_1\bar{D}_1((L_{v_i}T)-H_1(L_{v_i}T))\\ &= -\sqrt{-1}H_1\bar{D}_1\Box_1G_1(L_{v_i}T)\\ &= -\sqrt{-1}H_1\Box_1G_1\bar{D}_1(L_{v_i}T)\end{aligned}$$

$$= -\sqrt{-1}\ H_1(\bar{D}_1(L_{v_i}T) - H_1\bar{D}_1(L_{v_i}T))$$
$$= -\sqrt{-1}\ (H_1\bar{D}_1(L_{v_i}T) - H_1\bar{D}_1(L_{v_i}T))$$
$$= 0.$$

Hence:

$$\bar{D}_2^*\theta = \Box_1 G_1 \bar{D}_2^*\theta.$$

On the other hand:

$$\bar{D}_2^*\theta = \bar{D}_2^*(H_2\theta + \Box_2 G_2\theta) = \bar{D}_2^*\Box_2 G_2\theta\ ,$$

as consequence: $\Box_1 G_1 \bar{D}_2^*\theta = \bar{D}_2^*\Box_2 G_2\theta$.

Since on X is $\Box_1 = \Box_2$ and $G_1 = G_2$ we have:

$$\bar{D}_1 G_2\theta = \bar{D}_1 G_1\theta = G_1\bar{D}_1\theta = 0.$$

That means $G_2\theta$ belongs to X_o, and as consequence:

$$\bar{D}_2^*\Box_1 G_2\theta = \Box_1\bar{D}_2^* G_2\theta.$$

We get:

$$\Box_1 G_1\bar{D}_2^*\theta = \bar{D}_2^*\Box_2 G_2\theta$$
$$= \bar{D}_2^*\Box_1 G_2\theta$$
$$= \Box_1\bar{D}_2^* G_2\theta$$
$$= \Box_1 G_2\bar{D}_2^*\theta.$$

Thus:

$$\bar{D}_2^*\theta = \Box_1 G_2\bar{D}_2^*\theta.$$

(3.2) Denote $J = (L_{v_i}T) - H_1(L_{v_i}T)$, or equivalently $J = \Box_1 G_1(L_{v_i}T)$.

From

$$\bar{D}_1^* J = \bar{D}_1^*(L_{v_i}T) = 0\ ,$$

it follows:

$$J = H_1 J + \Box_1 G_1 J = \Box_1 G_1 J = \bar{D}_1^* \bar{D}_1 G_1 J.$$

Hence:

$$J = \bar{D}_1^* G_1 (\bar{D}_1 (L_{v_i} T))$$
$$= \bar{D}_1^* G_1 (\sqrt{-1} \bar{D}_2^* \Theta)$$

where last equality follows from [15].

Denote by $(\ , \ )_{M_o}$ the global inner product on M_o.

We have:

$$\begin{aligned}
(J,J)_{M_o} &= (\bar{D}_1^* G_1 (\sqrt{-1} \bar{D}_2^* \Theta), J)_{M_o} \\
&= (G_1 (\sqrt{-1} \bar{D}_2^* \Theta), \bar{D}_1 J)_{M_o} \\
&= \sqrt{-1} (G_1 \bar{D}_2^* \Theta, \sqrt{-1} \bar{D}_2^* \Theta)_{M_o} \\
&= (G_1 \bar{D}_2^* \Theta, \Box_1 G_2 \bar{D}_2^* \Theta)_{M_o} \\
&= (\Box_1 G_1 \bar{D}_2^* \Theta, G_2 \bar{D}_2^* \Theta)_{M_o} \\
&= (\bar{D}_2^* \Theta, \bar{D}_2^* G_2 \Theta)_{M_o} \\
&= (\Theta, \Box_2 G_2 \Theta)_{M_o} - (\Theta, \bar{D}_2^* \bar{D}_2 G_2 \Theta)_{M_o} \\
&= (\Theta, \Theta - H_2 \Theta)_{M_o} \\
&= (\Theta, \Theta)_{M_o} - (H_2 \Theta, H_2 \Theta)_{M_o}
\end{aligned}$$

where we used:

$$\bar{D}_2 \Theta = \bar{D}_1 \Theta = 0.$$

Moreover we have:

$$(L_{v_i} L_{v_i} J, L_{v_i} L_{v_i} J)_{M_o} = (J,J)_{M_o} + (H_1 L_{v_i} L_{v_i} J, H_1 L_{v_i} L_{v_i} J)_{M_o}$$

and:

$$(\Theta, \Theta)_{M_o} = \frac{1}{2!2!} \int_{M_o} \Theta^{\alpha\beta}{}_{\bar{\gamma}\bar{\delta}} \, \overline{\Theta^{\gamma\delta}{}_{\bar{\alpha}\bar{\beta}}} \, dV$$

$$= \tfrac{1}{2} \int_{M_o} (L_{v_i} J)^{\alpha}_{\beta} \overline{(L_{v_i} J)^{\gamma}_{\alpha}} (L_{v_i} J)^{\lambda}_{\gamma} \overline{(L_{v_i} J)^{\beta}_{\lambda}} \, dV$$

$$- \tfrac{1}{2} \int_{M_o} (L_{v_i} J)^{\alpha}_{\mu} \overline{(L_{v_i} J)^{\mu}_{\alpha}} (L_{v_i} J)^{\lambda}_{\nu} \overline{(L_{v_i} J)^{\nu}_{\lambda}} \, dV \, .$$

Thus:

$$R_{i\bar{i}i\bar{i}}^{(W.P.)}(o) = \tfrac{1}{2} \frac{1}{V} (h_{i\bar{i}}(o))^2 +$$

$$+ (HL_{v_i} L_{v_i} J, HL_{v_i} L_{v_i} J)_{M_o} - (H_2\Theta, H_2\Theta)_{M_o}$$

where we don't need to distinguish between $H_1(L_{v_i} L_{v_i} J)$ and $H_2(L_{v_i} L_{v_i} J)$, as we are considering $(L_{v_i} L_{v_i} J)$ as a $T_{M_o}^{(1,0)}$-valued (0,1)-form.

The complete expression for the Riemann tensor can be obtained by polarization of $R_{i\bar{i}i\bar{i}}$:

$$R_{i\bar{j}l\bar{k}}^{(W.P.)}(o) = \frac{1}{4V} (h_{i\bar{j}}(o) h_{l\bar{k}}(o) + h_{i\bar{k}}(o) h_{l\bar{j}}(o)) +$$

$$+ \int_{M_o} (HL_{v_i} L_{v_l} J)^{p}_{a} \overline{(HL_{v_j} L_{v_k} J)^{q}_{b}} \, g_{p\bar{q}} \, g^{a\bar{b}} \, dV +$$

$$- (H_2(\phi(\frac{\partial}{\partial t_i}) \wedge \phi(\frac{\partial}{\partial t_l})), H_2(\phi(\frac{\partial}{\partial t_j}) \wedge \phi(\frac{\partial}{\partial t_k})))_{M_o}$$

where

$$(\phi(\frac{\partial}{\partial t_i}) \wedge \phi(\frac{\partial}{\partial t_l}))^{\lambda\mu}_{\alpha\beta} = - \frac{1}{4} ((L_{v_i} J) \wedge (L_{v_l} J))^{\lambda\mu}_{\alpha\beta} =$$

$$= - \frac{1}{8} \{ (L_{v_i} J)^{\lambda}_{\alpha} (L_{v_l} J)^{\mu}_{\beta} - (L_{v_i} J)^{\lambda}_{\beta} (L_{v_l} J)^{\mu}_{\alpha} +$$

$$+ (L_{v_l} J)^{\lambda}_{\alpha} (L_{v_i} J)^{\mu}_{\beta} - (L_{v_l} J)^{\lambda}_{\beta} (L_{v_i} J)^{\mu}_{\alpha} \} .$$

Let $H^1(M_o,T_{M_o})_{\omega_o}$ be the subspace of $H^1(M_o,T_{M_o})$ defined by the infinitesimal deformations polarized with respect to the Kähler class ω_o. Then to associate to any couple of elements in $H^1(M_o,T_{M_o})_{\omega_o}$ $(\phi(\frac{\partial}{\partial t_i}),\phi(\frac{\partial}{\partial t_l}))$ the element $\phi(\frac{\partial}{\partial t_i})\wedge\phi(\frac{\partial}{\partial t_l})$, defines a map

$$\Psi : H^1(M_o,T_{M_o})_{\omega_o} \times H^1(M_o,T_{M_o})_{\omega_o} \longrightarrow H^2(M_o,\Lambda^2 T_{M_o}).$$

Now we are going to state the generalization of the negativity of the holomorphic bisectional curvature of the Weil-Petersson metric in [7] to the case of compact polarized Kähler-Einstein manifolds of zero first Chern class:

THEOREM 1. Let Ω be an open neighbourhood of o in C^N and let $\Pi:M\to\Omega$ be a holomorphic family of polarized manifolds with respect to a Kähler class ω_o on $M_o=\Pi^{-1}(o)$. Let X and Y be two tangent vectors on Ω at the point o, and let $\phi(X)$, $\phi(Y)$ be the elements of $H^1(M_o,T_{M_o})_{\omega_o}$ defined by X and Y.
If $\Psi(\phi(X),\phi(Y))=\phi(X)\wedge\phi(Y)$ is zero in $H^2(M_o,\Lambda^2 T_{M_o})$ then the holomorphic bisectional curvature of the Weil-Petersson metric in the direction X,Y is negative.

Proof: It follows directly from the expression of the Riemann tensor. In fact let $X=\partial/\partial t_i$ and $Y=\partial/\partial t_j$, then

$$\text{Hol.bis. } (X,Y)=-R_{i\bar{i}j\bar{j}}=-\tfrac{1}{2}\frac{1}{V}\, h_{i\bar{i}}h_{j\bar{j}}-(HL_{v_i}L_{v_i}J,HL_{v_i}L_{v_i}J)<0$$

4. K3-Surfaces Case

(4.1) The negativity of the Weil-Petersson metric for polarized K3-surfaces has been proved in [6] just computing explicitely the metric and remarking that in this case it turns out to be the Bergmann metric of a symmetric space of type III. In this part we will compute directly the term $(H_2\Theta, H_2\Theta)$ of (3.2) and we will prove that this term is smaller than the remaining terms of the Riemann tensor.

Suppose M_o is a K3-surface and let ψ_o be a nowhere zero holomorphic 2-form on M_o. In local coordinates is:

$$\psi_o = \tfrac{1}{2} \sum_{\alpha,\beta=1}^{2} \psi_{\alpha\beta} dz^\alpha \wedge dz^\beta, \quad \psi_{\alpha\beta} = -\psi_{\beta\alpha}.$$

Let $\omega_o \in H^{1,1}(M_o)$ be a Kähler class. Let $\partial/\partial t$ be a tangent vector at o in the moduli space of polarized K3-surfaces with respect to ω_o, and let v be a canonical lifting of $\partial/\partial t$. Consider the deformation $\psi = \psi_t$ of ψ_o along the family defined by $\partial/\partial t$ and compute the Lie derivative of ψ along v.

Denote $(L_v\psi)_{(r,s)}$ the component of type (r,s) of the 2-form $(L_v\psi)$. From $\psi\wedge\psi = 0$ on M_t for all t it follows:

$(L_v\psi)\wedge\psi = 0$, or $(L_v\psi)_{(0,2)} = 0$ on M_t for all t.

Thus $(L_v\psi)$ has only the (2,0) and the (1,1) component and direct computation gives at the point o:

$$(L_v\psi)_{(1,1)}\big|_o = \frac{1}{2\sqrt{-1}} \psi_{\alpha\beta} (L_v J)^{\beta}_{\bar\gamma} \, dz^\alpha \wedge d\bar z^\gamma.$$

Since v is a canonical lifting and ψ is holomorphic $(L_v\psi)_{(1,1)}$ is a harmonic $(1,1)$-form, thus $(L_v\psi)_{(1,1)}$ is the harmonic representative of the infinitesimal deformation in the space $H^1(M_t,\Omega^1_t)\simeq$ $\simeq H^1(M_t,T_{M_t})$, where Ω^1_t denotes the sheaf of holomorphic 1-forms on M_t and the above isomorphism is the canonical one given by the form ψ.
In particular the Weil-Petersson metric can be computed by:

$$h_{i\bar{j}}(t) = -\int_{M_t} (L_{v_i}\psi)_{(1,1)} \wedge \overline{(L_{v_j}\psi)_{(1,1)}}$$

$$= \frac{1}{4}\int_{M_t} (L_{v_i}J)^{\alpha}_{\bar{\beta}}\,\overline{(L_{v_j}J)^{\beta}_{\alpha}}\;\psi\wedge\bar{\psi}$$

after normalizing the volum form $\psi\wedge\bar{\psi}$ by some constant, for example such that $(1/4)\psi_t\wedge\bar{\psi}_t = dV_t$, where dV_t is the volum form in the given Eistein-Kähler metric of zero Ricci tensor on M_t.
Remark that the volum form $\psi\wedge\bar{\psi}$ is necessarily a constant multiple of dV_t because the holomorphicity of ψ on M_t implies that $\psi\wedge\bar{\psi}$ gives zero Ricci tensor.
Moreover we have $(L_v\psi)_{(2,0)}$ is a holomorphic form, because $(L_v\psi)$ is d-closed and $(L_v\psi)_{(1,1)}$ is harmonic. Thus $(L_v\psi)_{(2,0)}=c\psi$, where c is a constant on M_t.
Without loos of generality we can assume that $(L_v\psi)_{(2,0)}|_o = 0$.
In fact let ψ_o be the holomorphic 2-form on M_o and $\psi(w,t)=\frac{1}{2}\psi_{\alpha\beta}(w,t)dw^{\alpha}\wedge dw^{\beta}$ the deformation of ψ_o along a family trivialized by a given diffeo-

morphism $\Phi: M_o \times \Omega \to M$ such that $w(z,t)=\Phi_t(z)$. Then $\psi(w,t)=\psi_{12}\det(\partial w/\partial z)dz^1\wedge dz^2+\eta(w,t)$, and $\eta(w,t)$ has no components of type $(2,0)$ on M_o.

Define $f(t)=(\psi_{12}(w,t)\det(\partial w/\partial z))/\psi_{12}(z,o)$. Then $f(t)\neq 0$ for t in a neighbourhood of o in C^N and $\tilde{\psi}(w,t)=\psi(w,t)/f(t)=\psi_o+\tilde{\eta}(w,t)$ satisfies $(L_v\tilde{\psi})_{(2,0)}\big|_o=0$.

In order to compute the Riemann tensor we need to compute $L_v(L_v\psi)_{(1,1)}$.

By direct computation and using the assumption $(L_v\psi)_{(2,0)}\big|_o=0$, at the point $o\in\Omega$ we have:

$$L_v(L_v\psi)_{(1,1)}\big|_o = \frac{1}{4}\psi_{\lambda\gamma}(L_vJ)^{\gamma}_{\beta}(L_vJ)^{\lambda}_{\alpha}d\bar{z}^{\beta}\wedge d\bar{z}^{\alpha}+$$

$$+\frac{1}{2\sqrt{-1}}\psi_{\alpha\gamma}(L_vL_vJ)^{\gamma}_{\beta}dz^{\alpha}\wedge d\bar{z}^{\beta}.$$

In particular:

$$\int_{M_o}(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\overline{(L_v(L_v\psi)_{(1,1)})_{(0,2)}} =$$

$$=\tfrac{1}{2}\int_{M_o}(L_vJ)^{\alpha}_{\beta}\overline{(L_vJ)^{\beta}_{\alpha}}(L_vJ)^{\gamma}_{\delta}\overline{(L_vJ)^{\delta}_{\gamma}}dV+$$

$$-\tfrac{1}{2}\int_{M_o}(L_vJ)^{\alpha}_{\beta}\overline{(L_vJ)^{\gamma}_{\alpha}}(L_vJ)^{\lambda}_{\gamma}\overline{(L_vJ)^{\beta}_{\lambda}}dV$$

$$=(\Theta,\Theta)_{M_o}$$

where Θ is defined as in (3.1).

Moreover:

$$(H\Theta,H\Theta)_{M_o}=\int_{M_o}H(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\overline{H(L_v(L_v\psi)_{(1,1)})_{(0,2)}}$$

where we don't need to distinguish between H_1 and H_2. Also:

$$\int_{M_o}(L_v(L_v\psi)_{(1,1)})_{(1,1)}\wedge\overline{(L_v(L_v\psi)_{(1,1)})_{(1,1)}} =$$

$$= -\int_{M_o} (L_v L_v J)\frac{\alpha}{\beta}\ \overline{(L_v L_v J)\frac{\beta}{\alpha}}\ dV \ , \text{ and:}$$

$$(H(L_v L_v J), H(L_v L_v J))_{M_o} =$$

$$= -\int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(1,1)} \wedge \overline{H(L_v(L_v\psi)_{(1,1)})_{(1,1)}}.$$

This gives a very simple expression of the Riemann tensor in terms of ψ. Denote $v_i = v$, we have:

$$R_{i\bar{i}i\bar{i}}(o) = \frac{1}{2V}\cdot(h_{i\bar{i}}(o))^2 - \int_{M_o} L_v(L_v\psi)_{(1,1)} \wedge \overline{L_v(L_v\psi)_{(1,1)}}$$

or:

$$R_{i\bar{i}i\bar{i}}(o) = \frac{1}{2V}\ (h_{i\bar{i}}(o))^2 +$$

$$- \int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(1,1)} \wedge \overline{H(L_v(L_v\psi)_{(1,1)})_{(1,1)}} +$$

$$- \int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(0,2)} \wedge \overline{H(L_v(L_v\psi)_{(1,1)})_{(0,2)}}.$$

(4.2) In order to prove the negativity, at least, of the holomorphic sectional curvature we need to express the last term of the previous expression, which is clearly negative, in a more computable way. Actually we will prove the following:

LEMMA 1. With the above notations we have:

$$\int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(0,2)} \wedge \overline{H(L_v(L_v\psi)_{(1,1)})_{(0,2)}}$$

$$= \frac{1}{4V}\ \Big(\int_{M_o} (L_v\psi)_{(1,1)} \wedge \overline{(L_v\psi)_{(1,1)}}\Big)\Big(\int_{M_o} (L_v\psi)_{(1,1)} \wedge \overline{(L_v\psi)_{(1,1)}}\Big).$$

Proof. From the harmonicity of $H(L_v(L_v\psi)_{(1,1)})_{(0,2)}$ and the antiholomorphicity, and furthermore harmonicity, of $\bar{\psi}$, we have: $H(L_v(L_v\psi)_{(1,1)})_{(0,2)}=c_t\bar{\psi}$, where c_t is a constant on M_t. On the other hand, from $(L_v\psi)_{(1,1)}\wedge\psi=0$ for all t, we have:

$$(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)} + (L_v(L_v\psi)_{(1,1)})\wedge\psi=0$$

or:

$$(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\psi = -(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}$$

and:

$$\overline{(L_v(L_v\psi)_{(1,1)})_{(0,2)}}\wedge\bar{\psi} = -\overline{(L_v\psi)_{(1,1)}}\wedge\overline{(L_v\psi)_{(1,1)}}.$$

In particular, denote $c=c_o$, we have:

$$\int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\psi = \int_{M_o} c\psi\wedge\bar{\psi} = 4cV$$

and, from the harmonicity of ψ,

$$\int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\psi$$
$$= \int_{M_o} (L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\psi$$
$$= -\int_{M_o} (L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}.$$

Thus:

$$c = -\frac{1}{4V}\int_{M_o} (L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}.$$

On the other hand:

$$\int_{M_o} H(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\overline{H(L_v(L_v\psi)_{(1,1)})_{(0,2)}}$$
$$= \int_{M_o} c\bar{\psi}\wedge\overline{(L_v(L_v\psi)_{(1,1)})_{(0,2)}}$$
$$= \frac{1}{4V}\left(\int_{M_o} (L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\right)\left(\int_{M_o}\overline{(L_v\psi)_{(1,1)}}\wedge\overline{(L_v\psi)_{(1,1)}}\right).$$

As consequence of Lemma 1 we have the following expression for the Riemann tensor:

$$(16)\quad R_{i\bar{i}i\bar{i}}(o)=\frac{1}{4V}\Big(\int_{M_o}(L_v\psi)_{(1,1)}\wedge\overline{(L_v\psi)_{(1,1)}}\Big)^2+$$

$$-\frac{1}{4V}\Big(\int_{M_o}(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\Big)\Big(\int_{M_o}\overline{(L_v\psi)_{(1,1)}}\wedge\overline{(L_v\psi)_{(1,1)}}\Big)+$$

$$+\,(H(L_vL_vJ),H(L_vL_vJ))_{M_o}.$$

(4.3) Last step will be to prove that the first term in the (16) expression is bigger than the second. Actually, with completely elementary argument we will prove the following:

LEMMA 2. Let η be a (1,1)-form on M_o, then

$$\Big(\int_{M_o}\eta\wedge\bar{\eta}\Big)^2\geq\Big(\int_{M_o}\eta\wedge\eta\Big)\Big(\int_{M_o}\bar{\eta}\wedge\bar{\eta}\Big)$$

and the equality holds if and only if η is real or $\mathrm{Re}\eta=\pm\mathrm{Im}\eta$.

Proof. It suffies to decompose η by $\eta=\eta'+\sqrt{-1}\eta''$ where $\eta'=\mathrm{Re}\eta$ and $\eta''=\mathrm{Im}\eta$. In fact:

$$\Big(\int_{M_o}\eta\wedge\bar{\eta}\Big)^2-\Big(\int_{M_o}\eta\wedge\eta\Big)\Big(\int_{M_o}\bar{\eta}\wedge\bar{\eta}\Big)$$

$$=2\Big(\int_{M_o}\eta''\wedge\eta''\Big)\Big(\int_{M_o}\eta'\wedge\eta'\Big)-2\Big(\int_{M_o}\eta'\wedge\eta''\Big)^2\geq 0$$

because of the Schwartz inequality.

Finally we have:

$$(17)\quad R_{i\bar{i}i\bar{i}}(o)\geq\frac{1}{4}\Big(\int_{M_o}(L_v\psi)_{(1,1)}\wedge\overline{(L_v\psi)_{(1,1)}}\Big)^2+$$

$$+\,(H(L_vL_vJ),H(L_vL_vJ))_{M_o}.$$

Expression (17) gives the following statement:

THEOREM 2. The Weil-Petersson metric in the Moduli Space of polarized K3-Surfaces has negative holomorphic sectional curvature.

5. Generalization of K3-Surfaces Case

(5.1) In the attempt to generalize the argument of previous paragraph to higher dimensional case it seems to be natural to consider manifolds endowed with a holomorphic n-form nowhere zero. Actually in this case we have an analogous expression for the Riemann tensor, namely by direct computation we can prove that the Weil-Petersson metric can be expressed by:

$$h_{i\bar{j}}(t) = -\int_{M_t} (L_{v_i}\psi)_{(n-1,1)} \wedge \overline{(L_{v_j}\psi)_{(n-1,1)}}$$

and the Riemann tensor at $o \in \Omega$ by:

$$R_{i\bar{i}i\bar{i}}^{(W.P.)}(o) = \frac{1}{2V}(h_{i\bar{i}}(o))^2 +$$

$$- \int_{M_o} (L_{v_i}(L_{v_i}\psi)_{(n-1,1)})_{(n-1,1)} \wedge \overline{(L_{v_i}(L_{v_i}\psi)_{(n-1,1)})_{(n-1,1)}} +$$

$$- \int_{M_o} (L_{v_i}(L_{v_i}\psi)_{(n-1,1)})_{(n-2,2)} \wedge \overline{(L_{v_i}(L_{v_i}\psi)_{(n-1,1)})_{(n-2,2)}}$$

where $n = \dim_C M_o$, ψ is the holomorphic n-form and v_i, v_j are canonical liftings.
However it seems to be difficult to compare last integral in the previous expression to the first one, except in the case n is even and the n-form

decomposes in a wedge product $\frac{n}{2}$ times of a closed holomorphic 2-form.
This is exactly the case of symplectic manifolds, considered also in [6].
In the following we will study the Riemann tensor of the Weil-Petersson metric in the Moduli Space of polarized symplectic manifolds. This will turn out to be quite a simple manipulation of K3-surfaces case. The theory of moduli space of symplectic manifolds also may be found in [3].

(5.2) Let M_o be a compact complex manifold endowed with a closed holomorphic 2-form nowhere zero, call it ψ_o. Let $n=\dim_C M_o=2m$. Let $\omega_o \in H^{1,1}(M_o)$ be a Kähler class, and Ω_{ω_o} the moduli space of polarized deformations of M_o with respect to ω_o, near M_o.
Let $\{\frac{\partial}{\partial t_1},..,\frac{\partial}{\partial t_N}\}$ be a basis for the tangent space of Ω_{ω_o} at the point t and $\{v_1,..,v_N\}$ be canonical liftings, then, in this case, the Weil-Petersson metric is given by:

$$h_{i\bar{j}}(t)=-\int_{M_t}(L_{v_i}\psi)_{(1,1)}\wedge\overline{(L_{v_j}\psi)_{(1,1)}}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)}.$$

In fact:

$$\int_{M_t}(L_v\psi)_{(1,1)}\wedge\overline{(L_v\psi)_{(1,1)}}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)}$$

$$=\frac{1}{4}\int_{M_t}\psi_{\alpha\beta}(L_vJ)^{\beta}_{\gamma}\overline{\psi_{\lambda\mu}(L_vJ)^{\mu}_{\nu}}\,dz^{\alpha}\wedge d\bar{z}^{\gamma}\wedge d\bar{z}^{\lambda}\wedge dz^{\nu}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)}$$

$$=-\frac{1}{4m^2}\int_{M_t}(L_vJ)^{\alpha}_{\beta}\overline{(L_vJ)^{\beta}_{\alpha}}\;(\psi\wedge\bar{\psi})^{\wedge m},$$

just take a normalization such that :

$\frac{1}{4m^2}(\psi\wedge\bar{\psi})^{\wedge m}=dV_t$, where dV_t is the volum form of the given Einstein-Kähler metric on M_t.

Since $\psi^{\wedge(m+1)}=0$ on M_t for all t we have:

$(m+1)(L_v\psi)_{(0,2)}\wedge\psi^{\wedge m}=0$, or $(L_v\psi)_{(0,2)}=0$ for all t.

So $(L_v\psi)$ has only the (2,0) and the (1,1)-component.

Since $(L_v\psi)_{(1,1)}$ is harmonic and $(L_v\psi)$ is d-closed, $(L_v\psi)_{(2,0)}$ is holomorphic, hence $(L_v\psi)_{(2,0)}=$ $=c_t\psi$, with c_t constant on M_t. Namely it is a general fact that for a symplectic manifold M is $H^0(M,\Omega_M^{2p})=C\cdot\psi^{\wedge p}$ for $0\leq p\leq m$, [3], where we denote Ω_M^{2p} the sheaf of holomorphic 2p-forms on M.

As in (4.1) we can assume without loosing generality that $(L_v\psi)_{(2,0)}|_o=0$.

In fact consider the holomorphic n-form on M_t nowhere zero $\tilde{\psi}=\psi^{\wedge m}$. We can normalize $\tilde{\psi}$ such that $\tilde{\psi}_t=\psi_o^{\wedge m}+\eta_t$, where η_t has no component of type (n,0) on M_o.

Then $(L_v\tilde{\psi})_{(n,0)}|_o=0$. On the other hand:

$(L_v\tilde{\psi})_{(n,0)}|_o=m(L_v\psi)_{(2,0)}|_o\wedge\psi_o^{\wedge(m-1)}=mc\psi_o^{\wedge m}$,

where we denote $c=c_o$; for this normalization c must be zero.

Direct computation gives:

$$L_v(L_v\psi)_{(1,1)}|_o=\frac{1}{4}\psi_{\alpha\beta}(L_vJ)^{\beta}_{\gamma}(L_vJ)^{\alpha}_{\delta}d\bar{z}^{\gamma}\wedge d\bar{z}^{\delta}+$$

$$+\frac{1}{2\sqrt{-1}}\psi_{\alpha\gamma}(L_vL_vJ)^{\gamma}_{\beta}dz^{\alpha}\wedge d\bar{z}^{\beta}$$

where we used the assumption $(L_v\psi)_{(2,0)}|_o=0$.
Moreover we have:

$$\int_{M_o}(L_v(L_v\psi)_{(1,1)})_{(1,1)}\wedge\overline{(L_v(L_v\psi)_{(1,1)})_{(1,1)}}\wedge(\psi\wedge\bar\psi)^{\wedge(m-1)}$$

$$=-\frac{1}{4m^2}\int_{M_o}(L_vL_vJ)^{\alpha}_{\beta}\,\overline{(L_vL_vJ)^{\beta}_{\alpha}}\,(\psi\wedge\bar\psi)^{\wedge m}$$

and:

$$\int_{M_o}(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\overline{(L_v(L_v\psi)_{(1,1)})_{(0,2)}}\wedge(\psi\wedge\bar\psi)^{\wedge(m-1)}$$

$$=\frac{1}{16}\int_{M_o}\psi_{\alpha\beta}(L_vJ)^{\beta}_{\gamma}(L_vJ)^{\alpha}_{\delta}\overline{\psi_{\lambda\mu}}\,\overline{(L_vJ)^{\mu}_{\nu}(L_vJ)^{\lambda}_{\sigma}}\,d\bar z^{\gamma}\wedge d\bar z^{\delta}\wedge dz^{\nu}\wedge dz^{\sigma}\wedge$$
$$\wedge(\psi\wedge\bar\psi)^{\wedge(m-1)}$$

$$=\frac{1}{4}\int_{M_o}\psi_{\alpha\beta}\Theta^{\alpha\beta}_{\gamma\delta}\overline{\psi_{\lambda\mu}}\,\overline{\Theta^{\lambda\mu}_{\nu\sigma}}\,d\bar z^{\gamma}\wedge d\bar z^{\delta}\wedge dz^{\nu}\wedge dz^{\sigma}\wedge(\psi\wedge\bar\psi)^{\wedge(m-1)}$$

$$=\frac{1}{4}\int_{M_o}\Theta^{\alpha\beta}_{\gamma\delta}\,\overline{\Theta^{\gamma\delta}_{\alpha\beta}}\,(\psi\wedge\bar\psi)^{\wedge m}$$

$$=(\Theta,\Theta)_{M_o}.$$

We are interested in computing $(H\Theta,H\Theta)_{M_o}$ =

$$=(H(L_v(L_v\psi)_{(1,1)})_{(0,2)},H(L_v(L_v\psi)_{(1,1)})_{(0,2)})_{M_o}.$$

From $(L_v\psi)_{(1,1)}\wedge\psi^{\wedge m}=0$ on M_t for all t, we get:

$$(L_v(L_v\psi)_{(1,1)})_{(0,2)}\wedge\psi^{\wedge m}+$$

$$+\,m(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge\psi^{\wedge(m-1)}=0.$$

On the other hand:

$$H(L_v(L_v\psi)_{(1,1)})_{(0,2)}=c'\bar\psi$$

and:

$$c'(\bar\psi\wedge\psi^{\wedge m})=-m(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge\psi^{\wedge(m-1)}$$

or

$$c'(\psi\wedge\bar\psi)^{\wedge m}=-m(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge(\psi\wedge\bar\psi)^{\wedge(m-1)}.$$

We get:

$$c' = -\frac{1}{4mV}\int_{M_o}(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)}.$$

From

$$(H\Theta,H\Theta)_{M_o} = (c'\bar{\psi},c'\bar{\psi})_{M_o} = c'\bar{c}'(\bar{\psi},\bar{\psi})_{M_o}$$

we get:

$$(H\Theta,H\Theta)_{M_o} =$$

$$= -\frac{1}{4mV}\int_{M_o}(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)} \cdot$$

$$\cdot \frac{-1}{4mV}\int_{M_o}\overline{(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)}}\cdot\int_{M_o}(\psi\wedge\bar{\psi})^{\wedge m}$$

$$= \frac{1}{4V}\left|\int_{M_o}(L_v\psi)_{(1,1)}\wedge(L_v\psi)_{(1,1)}\wedge(\psi\wedge\bar{\psi})^{\wedge(m-1)}\right|^2$$

where we denote by $|\cdot|$ the module of a complex number.

The negativity of the holomorphic sectional curvature follows now, as in (4.3), from Schwartz inequality. So far we proved the following:

THEOREM 3. The Weil-Petersson metric in the Moduli Space of polarized Symplectic manifolds has negative holomorphic sectional curvature.

References

[1] AHLFORS,L.: Some remarks on Teichmüller space of Riemann surfaces. Ann. of Math. 72, 171-191 (1961)

[2] AHLFORS,L.: Curvature properties of Teichmüller space. J. Analyse Math. 9, 161-176 (1961)

[3] BEAUVILLE,A.: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Diff. Geom. I8, 753-782 (I983)

[4] KOISO,N.: Einstein metrics and complex structures. Invent. Math. 73, 7I-I06 (I983)

[5] ROYDEN,H.: Intrinsic metric on Teichmüller spaces. Proceedings International Congress Math. 2, 2I7-22I (I974)

[6] SCHUMACHER,G.: On the geometry of moduli spaces. Preprint (I984) (To appear in Manuscripta Mathematica)

[7] SIU,Y.T.: Curvature of the Weil-Petersson metric in the moduli space of compact Kähler -Einstein manifolds of negative first Chern class. Preprint (I985)

[8] WOLPERT,S.: Chern forms and the Riemann tensor for the moduli space of curves. Preprint (I984)

[9] YAU,S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. Comm. Pure and Appl. Math. 3I, 339-44I (I978)

The paper has been prepared during the stay of the author to the Mathematics Department of Harvard University, Cambridge, Massachusetts 02I38 U.S.A., partially supported by a C.N.R. grant.

Antonella Nannicini
Istituto di Matematica Applicata
"Giovanni Sansone"
Via di S. Marta, 3
50I39 Firenze Italy

(Received August 2, 1985)