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ON RATIONALIZED H- AND CO-H-SPACES. WITH AN APPENDIX ON DECOMPOSABLE H- AND CO-H-SPACES

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Let R be a subring of the rationals with $1/2$, $1/3 \in R$; let S_R^n denote the R-local n-sphere and define $\Omega_R^n := S_R^n$ for n odd, $\Omega_R^n := \Omega\Sigma S_R^n$ for $n > 0$ even. An H-space (resp. a 1-conn. co-H-space) is "decomposable over R ", if it is homotopy equivalent to a weak product of spaces Ω_R^n (resp. to a wedge of R-local spheres). We prove that, if E is grouplike decomposable of finite type over R , the functor $[-,E]$ is determined on finite dim. complexes by the Hopf algebra $M^*(E;R)$; here M^* denotes the unstable cohomotopy functor of H.J. Baues. If C is cogrouplike decomposable over R , the functor $[C,-]$ is determined on 1-conn. R-local spaces by $\pi_*(\Omega C)$ as a cogroup in the category of M-Lie algebras. For $R = \mathbb{Q}$ the functor $[-,E]$ is also determined by the Lie algebra $\pi_*(E)$ and $[C,-]$ by the Berstein coalgebra associated to the comultiplication of C .

0. Introduction

The following investigation has been motivated by the remarks of [2] (chap. VI, (4.6)) on rationalized loop spaces and suspensions. We think it worthwhile to derive these results in a different, easier way for $\mathbb{Q}$-local H-spaces and co-H-spaces as well as to get rid of the finite type assumptions. In an appendix we will indicate that the view displayed here on the rational situation might also be obtained on the more general situation of decomposable H- and co-H-spaces.

We work in the homotopy category CW of well pointed spaces which are compactly generated and of the homotopy type of CW-complexes. For objects X and Y the set of morphisms from X to Y is denoted by [X,Y] .

Let R be a subring of $\mathbb{Q}$. If X is a nilpotent space, we denote by X_R the localization of X with respect to the set of primes which are not invertible in R . A nilpotent space X is called "R-local", if it is connected and if the localization $X \to X_R$ is a homotopy equivalence. Let ΩX denote the loop space of a space X and ΣX the suspension of X .

We begin with a résumé of results.

0.1. Rationalized H-spaces

Let E be an H-space in CW and $X \in CW$. The multiplication on E induces a multiplication on the set [X,E] .

Similarly, let C be a cocommutative coalgebra over R and let H be a quasi Hopf algebra over R (these notions being used in the sense of [15]), then the set of coalgebra morphisms $Mor_{co}(C,H)$ carries a multiplication defined as follows:

DEFINITION: *Let* Δ *be the comultiplication of* C , *let* m *be the multiplication of* H ; *for* $f,g \in Mor_{co}(C,H)$ *define*

$$f*g := m(f\otimes g)\Delta .$$

PROPOSITION 1: *Let* E *be a* $\mathbb{Q}$*-local H-space, let* $X \in CW$ *be connected.*

(1) *The canonical map* $[X,E] \to Mor_{co}(H_*(X;\mathbb{Q}),H_*(E;\mathbb{Q}))$ *is a multiplicative bijection.*

(2) *If E is grouplike, i.e. it is a group object in* CW *, then* $\mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(E;\mathbb{Q}))$ *is completely determined as a group by the coalgebra structure of* $H_*(X;\mathbb{Q})$ *and the Lie algebra structure of* $\pi_*(E)$ *(the Lie bracket being the Samelson product).*

This proposition will immediately be deduced from the following assertion.

LEMMA: *A* $\mathbb{Q}$*-local H-space is homotopy equivalent to a weak product of Eilenberg-MacLane spaces.*

Note: This fact is well known (see [21], [14], proposition 2 or [7], Satz 10.6.). We will indicate a short proof which seems to be "folklore" and which, in different terms, has already been given in [7], Satz 10.6. It also relates to [2], chap. V, (3.10).

The lemma also implies that for grouplike E the Hopf algebra $H_*(E;\mathbb{Q})$ is isomorphic to the universal enveloping algebra $U\pi_*(E)$ of $\pi_*(E)$ ([15], Appendix). Thus (2) follows from (1).

REMARK: In [19] the structure of the group $\mathrm{Mor}_{co}(H_*(X;\mathbb{Q}), H_*(E;\mathbb{Q}))$ has been studied more closely.

COROLLARY 1: *Two* $\mathbb{Q}$*-local grouplike spaces are H-equivalent, if and only if their Samelson Lie algebras are isomorphic.*

Proof: Let E_1, E_2 be two $\mathbb{Q}$-local grouplike spaces such that $\pi_*(E_1) \cong \pi_*(E_2)$ as Lie algebras. Then $H_*(E_1;\mathbb{Q}) \cong$ $\cong H_*(E_2;\mathbb{Q})$ as Hopf algebras, hence the functors $[-,E_1]$ and $[-,E_2]$ are isomorphic on the subcategory of connected spaces of CW . Therefore, E_1 and E_2 , being connected, are H-equivalent (see [6]). The other direction is obvious.

COROLLARY 2: <u>Any $\mathbb{Q}$-local grouplike space is H-equivalent to a loop space.</u>

<u>Proof</u>: Given such a space E there exists a 1-connected $\mathbb{Q}$-local space Y with $\pi_*(\Omega Y) \cong \pi_*(E)$ as Lie algebras by [18]. Apply corollary 1.

COROLLARY 3: <u>Let E_1, E_2 be $\mathbb{Q}$-local grouplike spaces. Then the set of homotopy classes of H-maps between E_1 and E_2 corresponds bijectively to</u> $\mathrm{Mor}_{\text{Hopf alg}}(H_*(E_1;\mathbb{Q}), H_*(E_2;\mathbb{Q}))$ <u>which corresponds bijectively to</u> $\mathrm{Mor}_{\text{Lie}}(\pi_*(E_1), \pi_*(E_2))$.

0.2. Rational co-H-spaces

In the dual situation we work only with 1-connected spaces, because any co-H-space has free fundamental group [4].

Let $1\text{-}CW_R$ denote the homotopy category of 1-connected R-local spaces.

For $X \in 1\text{-}CW_R$ denote the Lie algebra $\pi_*(\Omega X)$ by $L(X)$.

Let now C be a co-H-space in $1\text{-}CW_{\mathbb{Q}}$ with comultiplication $\sigma: C \to C \vee C$.

Note that $L(C \vee C) \cong L(C) \amalg L(C)$ where "$\amalg$" denotes the free product of Lie algebras (see section 2 for an explanation). Hence σ gives rise to a morphism of Lie algebras $L(C) \to L(C) \amalg L(C)$.

For any $X \in 1\text{-}CW_{\mathbb{Q}}$ the comultiplication σ defines a multiplication on $[C,X]$ in the usual way. The algebraic analogue is defined by using co-multiplicative objects in the category of Lie algebras (see [8] for a general study of comultiplicative objects in categories) as follows.

DEFINITION 2: Denote by Lie_R the category of connected graded Lie algebras over R. A comultiplicative object $L \in Lie_R$ is a Lie algebra together with a morphism $\sigma: L \to L \amalg L$ such that, if $r_1, r_2: L \amalg L \to L$ are the two retractions, then $r_i \circ \sigma = id_L$ for $i = 1,2$.

DEFINITION 3: Let L be a comultiplicative object in Lie_R, then for any $N \in Lie_R$ the set $Mor_{Lie}(L,N)$ carries a multiplication. Namely, for $f,g \in Mor_{Lie}(L,N)$ define

$$f*g := (f,g)\sigma ,$$

where (f,g) is the unique morphism of the coproduct extending f on the first and g on the second factor.

PROPOSITION 2: Let C be a $\mathbb{Q}$-local co-H-space and X a space in $1\text{-}CW_{\mathbb{Q}}$. Then the canonical map

$$[C,X] \to Mor_{Lie_{\mathbb{Q}}}(L(C),L(X))$$

is a multiplicative bijection.

This proposition will immediately be implied by the following result.

LEMMA: Let C be a co-H-space in $1\text{-}CW_{\mathbb{Q}}$. Then C is homotopy equivalent to a wedge of $\mathbb{Q}$-local spheres.

Note: This result has already been obtained in [10]. But the proof we will give here is simpler, though some of the arguments of [10] reappear in it.

For co-H-spaces satisfying some finiteness conditions the result has already been given in [3] and [22].

Let now L be a cogroup in Lie_R, let $U(L)$ be its universal enveloping algebra. Note that

$U(L \amalg L) \cong U(L) \amalg U(L)$. Therefore the map $U(\sigma): U(L) \to U(L) \amalg U(L)$ gives $U(L)$ the structure of a cogroup in the category of connected graded algebras. Let $B(L)$ be the coalgebra over R associated with this cogroup according to [4]. We call it the Berstein coalgebra of (L,σ) .

PROPOSITION 3: *Let* L *be a cogroup in* Lie_R *, let* $N \in Lie_R$. *Assume that the canonical map* $L \to U(L)$ *is injective and that* $N \to U(N)$ *maps* N *bijectively onto the subspace* $PU(N)$ *of primitive elements in* $U(N)$.

Then the group structure of $Mor_{Lie_R}(L,N)$ *depends only on the Lie algebra structure of* N *and the coalgebra structure of* $B(L)$. *In fact, there are isomorphisms of groups*

$$Mor_{Lie_R}(L,N) \cong Mor_{Hopf\ alg}(U(L),U(N)) \cong Mor_{co}(B(L),U(N)) .$$

Note that the group structure of the middle term can be defined through $U(\sigma): U(L) \to U(L) \amalg U(L)$.

Geometrically we thus obtain:

PROPOSITION 4: *Let* C *be a cogroup in* $1\text{-}CW_{\mathbb{Q}}$ *and* $X \in 1\text{-}CW_{\mathbb{Q}}$ *, then* $[C,X] \cong Mor_{Hopf\ alg}(H_*(\Omega C;\mathbb{Q}),H_*(\Omega X;\mathbb{Q})) \cong$ $\cong Mor_{co}(B(L(C)),U(L(X)))$. *I.e. the group structure of* $[C,X]$ *depends only on the Berstein coalgebra* $B(L(C))$ *and on the Lie algebra* $L(X)$.

COROLLARY 1: *Two cogrouplike spaces in* $1\text{-}CW_{\mathbb{Q}}$ *are co-H-equivalent, if and only if their Berstein coalgebras are isomorphic.*

COROLLARY 2: *Any cogrouplike space in* $2\text{-}CW_{\mathbb{Q}}$ *is co-H-equivalent to a suspension.*

Proof: Let C be a cogroup in $2\text{-}CW_{\mathbb{Q}}$ with Berstein coalgebra B. Choose a space $Y \in 1\text{-}CW_{\mathbb{Q}}$ such that $H_*(Y;\mathbb{Q})$ is isomorphic to B as a coalgebra (possible by [18]). Recall that $B(L(\Sigma Y)) \cong H_*(Y)$ ([4], section 3) and apply corollary 1.

COROLLARY 3: Let C_1, C_2 be cogroups in $1\text{-}CW_{\mathbb{Q}}$. Then there is a bijection between the set of homotopy classes of co-H-maps and $\mathrm{Mor}_{co}(B(L(C_1)),B(L(C_2)))$.

1. Rationalized H-spaces, proofs

Let E be a $\mathbb{Q}$-local H-space in CW. Write $\pi_*(E) = \bigoplus_{i\in J} L_i$ with $L_i \subset \pi_{n_i}(E)$ for some n_i and $\dim(L_i) = 1$ (where J is some indexing set).

LEMMA: The space E is homotopy equivalent to the weak product ${}^{w}\prod_{i\in J} K(L_i,n_i)$.

Proof: Without loss of generality we may assume that E has a strict unit element e which is the base point.

Let α_i be a generator of L_i with representative $f_i\colon S^{n_i}_{\mathbb{Q}} \to E$. Note that we may identify $K(\mathbb{Q},n)$ with $S^n_{\mathbb{Q}}$ for n odd and with $\Omega\Sigma S^n_{\mathbb{Q}}$ for n even, $n > 0$. For $i \in J$ we construct a map $g_i\colon K(L_i,n_i) \to E$ as follows:

If n_i is odd, take $g_i := f_i$.

Let n_i be even and consider the diagram

$$\begin{array}{ccc} \Omega\Sigma S^{n_i}_{\mathbb{Q}} & \xrightarrow{\Omega\Sigma f_i} & \Omega\Sigma E \\ \uparrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \rho} \\ S^{n_i}_{\mathbb{Q}} & \xrightarrow[f_i]{} & E \end{array}$$

where ι is the inclusion and ρ is a retraction. Then define $g_i := \rho\circ\Omega\Sigma f_i$.

We order J totally. For finite $F = \{i_1 < \ldots < i_r\} \subset J$ let $K_F := \prod_{i\in F} K(L_i,n_i)$. Define $\alpha_F \colon K_F \to E$,

$(x_{i_1},\ldots,x_{i_r}) \to (\ldots((g_{i_1}(x_{i_1})\cdot g_{i_2}(x_{i_2}))\cdot g_{i_3}(x_{i_3}))\ldots)$.

For $F' \subset F$ we consider $K_{F'} \subset K_F$ in the obvious way. Since e is neutral, we have $\alpha_F|K_{F'} = \alpha_{F'}$. Hence the α_F induce a map $\alpha\colon \varinjlim_{F\subset J} K_F \to E$.

By construction α induces isomorphisms of homotopy groups, hence is a homotopy equivalence.

REMARK 1: The same argument shows that there is also a homotopy equivalence $E \sim {}^{w}\prod_{n>0} K(\pi_n(E),n)$.

COROLLARY (W. Meier): *Let E be a $\mathbb{Q}$-local H-space. Let $X = \varinjlim X_\alpha$ be a connected complex where the X_α are the finite connected subcomplexes. Then* $[X,E] \cong \varprojlim [X_\alpha,E]$.

Proof: Since ${}^{w}\prod K(\pi_n(E),n)$ is a CW-approximation of $\prod K(\pi_n(E),n)$, we deduce that $[X,E] \sim \prod_{n>0} [X,K(\pi_n(E),n)]$. The result now follows from [11]. (See also [14], proposition 2).

Proof of proposition 1: By the lemma we may assume $E = \varinjlim K_F$ where each K_F is a finite product of spaces $K(\mathbb{Q},n_i)$.

Let X be a finite complex.

Since the continuous image of any finite complex (e.g. X or $X\times I$) in E is contained in some K_F , it follows that $[X,E] \sim \varinjlim [E,K_F]$. But $[X,K_F] \sim$ $\sim \mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(K_F;\mathbb{Q}))$ (see [19]). Hence $[X,E] \sim$

$\sim \varinjlim \mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(K_F;\mathbb{Q})) \sim \mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(\varinjlim K_F;\mathbb{Q})) \sim$
$\sim \mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(E;\mathbb{Q}))$.

If X is connected infinite, let $X = \varinjlim X_\alpha$ where the X_α are the finite connected subcomplexes of X . By the corollary we have $[X,E] \sim \varprojlim [X_\alpha,E]$ and $\varprojlim [X_\alpha,E] \sim$
$\sim \lim \mathrm{Mor}_{co}(H_*(X_\alpha;\mathbb{Q}),H_*(E;\mathbb{Q})) \sim \mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(E;\mathbb{Q}))$.

It is immediate from the definitions that the map $[X,E] \to \mathrm{Mor}_{co}(H_*(X;\mathbb{Q}),H_*(E;\mathbb{Q}))$ preserves the multiplications.

Thus part (1) is proved; part (2) is then clear.

REMARK: Let C be a coalgebra and H a Hopf algebra over $\mathbb{Q}$; assume that C, H are connected and cocommutative. Then $\mathrm{Mor}_{co}(C,H)$ is a group (comp. [15], section 8), in fact, it is a special kind of N-group in the sense of [12]. Its Lie algebra was determined in [19] as the vector space of module homomorphisms $\mathrm{Mor}_{mo}(C,PH)$ with Lie bracket given as follows:

If $f,g \in \mathrm{Mor}_{mo}(C,PH)$ then define

$$[f,g]:= m_H(f\otimes g)\Delta_C - m_H(g\otimes f)\Delta_C \ .$$

In particular (see [19]) $\exp\colon \mathrm{Mor}_{mo}(C,PH) \to \mathrm{Mor}_{co}(C,H)$ and its inverse log are defined, relating the group structure of $\mathrm{Mor}_{co}(C,H)$ with the Lie algebra structure of $\mathrm{Mor}_{mo}(C,PH)$ via the Baker-Campbell-Hausdorff formula.

Instead of transfering the group structure of $\mathrm{Mor}_{co}(C,H)$ to the Lie algebra $\mathrm{Mor}_{mo}(C,PH)$ by exp , one can also transfer it by a "coordinate system of the second kind", e.g. the map

$$\widetilde{\exp}\colon \mathrm{Mor}_{mo}(C,PH) \to \mathrm{Mor}_{co}(C,H),\ \{x_n\}_{n\geq 0} \to \prod_{n\geq 0} \exp(x_n)\ ,$$

where each x_n of the family $\{x_n\}_{n\geq 0}$ is a homomorphism $C \to PH$ concentrated in degree n. The group $\mathrm{Mor}_{co}(C,H)$ can now be described as an inverse limit using the elements $\exp(x_n)$ as generators and the Zassenhaus formulas as relations. The formulas given in [2] describe the group using only some of the elements $\exp(x_n)$ as generators together with the corresponding relations.

It would be interesting to have an explicit Baker-Campbell-Hausdorff formula-like formula with respect to $\widetilde{\exp}$.

2. Rational co-H-spaces, proofs

LEMMA: Let C be a co-H-space in $1\text{-}CW_{\mathbb{Q}}$. Then C is homotopy equivalent to a wedge of $\mathbb{Q}$-local spheres.

Proof: The comultiplication $\sigma: C \to C \vee C$ defines a coretraction $r: C \to \Sigma\Omega C$, i.e. $v \circ r \sim id_C$ where $v: \Sigma\Omega C \to C$ is the evaluation map (see [9]). Now, $\Omega C \sim \varinjlim K_F$ as in the proof of 1. lemma, hence $\Sigma\Omega C \sim \varinjlim \Sigma K_F$.

Let $F = \{j_1 < \ldots < j_r\}$ and $K_F = \prod_{i\in F} K(\mathbb{Q},n_i)$, then $\Sigma K_F \sim \vee\Sigma(K(\mathbb{Q},n_{a_1}) \wedge \ldots \wedge K(\mathbb{Q},n_{a_s}))$, the wedge extending over all s-tupels $a_1 < \ldots < a_s$ with $\{a_1,\ldots,a_s\} \subset F$ and $s = 1,\ldots,r$ (see [17]). Now, $\Sigma K(\mathbb{Q},n_i)$ is itself homotopy equivalent to a wedge of rationalized spheres; this is trivial for n_i odd; for n_i even $\Sigma K(\mathbb{Q},n_i) \sim \Sigma\Omega\Sigma S^{n_i}_{\mathbb{Q}} \sim \bigvee_{k\geq 1} \Sigma S^{kn_i}_{\mathbb{Q}}$ (see e.g. [23], chap. VII, section 2). Hence ΣK_F is homotopy equivalent to a wedge of rationalized spheres.

Now, note that a 1-connected $\mathbb{Q}$-local space X is homotopy equivalent to a wedge of $\mathbb{Q}$-local spheres, if and only if the Hurewicz homomorphism $h: \pi_*(X) \to \widetilde{H}_*(X;\mathbb{Z})$ is surjective (see [10], or the lemma below in A1).

The surjectivity of h for $X = \Sigma K_F$ implies the surjectivity of h for $\varinjlim \Sigma K_F$ and hence for any retract up to homotopy of $\Sigma\Omega C \sim \varinjlim \Sigma K_F$. Therefore C is homotopy equivalent to a wedge of $\mathbb{Q}$-local spheres.

Note that $\Sigma\Omega C$ is itself homotopy equivalent to a wedge of $\mathbb{Q}$-local spheres.

COROLLARY: The Lie algebra L(C) *is free* (*by* [5]).

REMARK 1: It also follows that $L(C \vee C) \cong L(C) \amalg L(C)$.

Proof of proposition 2: Let $C \sim \vee S_{\mathbb{Q}}^{n_k}$. Then any element of [C,X] is determined by the restrictions to the $S_{\mathbb{Q}}^{n_k}$. But the adjoints of the inclusions $S_{\mathbb{Q}}^{n_k} \to C$ are the generators of the free Lie algebra L(C) . Hence for any Lie algebra map $L(C) \to L(X)$ there is a unique geometric realization.

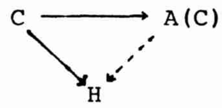
It is immediate from the definitions that the map $[C,X] \to \mathrm{Mor}_{\mathrm{Lie}}(L(C),L(X))$ is multiplicative.

Proof of proposition 3: Any Hopf algebra morphism $U(L) \to U(N)$ is determined by its effect on $\mathrm{image}(L \to U(L))$. Since $L \to U(L)$ is injective, it follows that the canonical map $\mathrm{Mor}_{\mathrm{Lie}_R}(L,N) \to \mathrm{Mor}_{\mathrm{Hopf\ alg}}(U(L),U(N))$ is injective. But it is also surjective, because any Hopf algebra map $U(L) \to U(N)$ restricts to a Lie algebra map $L \subset PU(L) \to PU(N) = N$.

We have to recall a definition from [16].

DEFINITION: *Let* C *be a connected coalgebra over* R . *Let* A(C) *be a Hopf algebra together with a coalgebra morphism* $C \to A(C)$. *Then* A(C) *is called the "universal Hopf algebra on* C ", *if for any Hopf algebra* H *and coalgebra map*

$C \to H$ *there exists a unique Hopf algebra morphism* $A(C) \to H$ *such that the following diagram commutes:*



Note that, if $A(C)$ *exists and* C *is cocommutative, then* $A(C)$ *is cocommutative.*

Given a cogroup A in the category of connected graded algebras over R with comultiplication $\sigma: A \to A \amalg A$ it can be made into a Hopf algebra by introducing the diagonal $A \to A \amalg A \to A \otimes A$. By [4] there is a subcoalgebra B of this Hopf algebra depending functorially on the cogroup A such that as a Hopf algebra $A \cong A(B)$. I.e. if $\overline{B}$ is the kernel of the counit of B, the algebra A is the free algebra $T\overline{B}$ on $\overline{B}$. The comultiplication $\sigma: A \to A \amalg A$ is related to B as follows: Let $i_1, i_2: A \to A \amalg A$ be the inclusions of A as first resp. as second factor; for $b \in \overline{B}$ – it suffices to know $\sigma(b)$ for $b \in \overline{B}$ – let $\Delta_B(b) = \Sigma\, b'_j \otimes b''_j \in B \otimes B$, then $\sigma(b) = \Sigma\mu(i_1(b'_j) \otimes i_2(b''_j))$, where $\mu: (A \amalg A) \otimes (A \amalg A) \to A \amalg A$ is the multiplication.

In particular, in the case of the cogroup $U(L)$ the Hopf algebra structure derived from the cogroup structure is the original Hopf algebra structure on $U(L)$. Hence $U(L)$ is the universal Hopf algebra on $B(L)$. This is the bijection

$$\mathrm{Mor}_{\text{Hopf alg}}(U(L), U(N)) \approx \mathrm{Mor}_{co}(B(L), U(N)) .$$

The discussion above implies that the bijection is multiplicative. Corollary 3 also follows.

Proof of proposition 4: From section 1 we know that $H_*(\Omega C;\mathbb{Q}) \cong U(L(C))$, $H_*(\Omega X;\mathbb{Q}) \cong U(L(X))$ and $PU(L(C)) = L(C)$, $PU(L(X)) = L(X)$. Thus proposition 3 can be applied.

REMARK 2: The bijection $Mor_{Lie}(L,N) \approx Mor_{Hopf\ alg}(U(L),U(N))$ has a general geometric analogue:

Let C be a co-H-space, $X \in CW$. Then there is a bijection between $[C,X]$ and the set $[\Omega C,\Omega X]_H$ of homotopy classes of H-maps $\Omega C \to \Omega X$.

Proof: Define α: $[C,X] \to [\Omega C,\Omega X]_H$ by $[f] \to [\Omega f]$.

Let r: $C \to \Sigma\Omega C$ be the coretraction corresponding to the comultiplication σ: $C \to C \vee C$. Define β: $[\Omega C,\Omega X]_H \to [C,X]$, $[g] \to [v\circ\Sigma g\circ r]$.

One has $\beta\circ\alpha([f]) = [v\circ\Sigma\Omega f\circ r] = [f\circ v\circ r] = [f]$ (because $v\circ r \sim id_C$). On the other hand $\alpha\circ\beta([g]) = [\Omega v\circ\Omega\Sigma g\circ\Omega r] = [g\circ\Omega v\circ\Omega r] = [g]$ (because $\Omega v\circ\Omega\Sigma g \sim g\circ\Omega v$).

REMARK 3: The dual of the statement of remark 2 is true as well.

REMARK 4: Let $L \in Lie_{\mathbb{Q}}$ be a cogroup in $Lie_{\mathbb{Q}}$, then L is a free Lie algebra. In fact, a set of free generators can be obtained as follows:

Let i: $B(L) \to U(L)$ be the inclusion. Then $\log(i)$ (in the sense of [19]) maps $B(L)$ into $PU(L) = L$ and $image(\log(i))$ still generates $U(L)$ freely. Hence L is a free Lie algebra.

APPENDIX: DECOMPOSABLE H- AND CO-H-SPACES.

A1. Geometric part.

DEFINITION 1: For any natural number $n > 0$ define

$$\Omega_R^n := \begin{cases} S_R^n & n \text{ odd,} \\ \Omega\Sigma S_R^n & n \text{ even.} \end{cases}$$

DEFINITION 2: (1) An H-space E is said to be "decomposable over R", if it is homotopy equivalent to a weak product of spaces Ω_R^n.

(2) A co-H-space C is said to be decomposable over R, if it is homotopy equivalent to a wedge of R-local spheres S_R^n with $n > 1$.

We now look for decomposability criteria.

Let E be a connected R-local H-space with $H_*(E;R)$ free as R-module. Assume that E has a strict unit.

Let $h: \pi_*(E) \to H_*(E)$ be the Hurewicz homomorphism. Choose a basis $\{\alpha_i\}_{i\in I}$ of homogeneous elements of $h(\pi_*(E))$. For each $i \in I$ choose $f_i: S_R^{n_i} \to E$ with $h([f_i]) = \alpha_i$ and define a map $g_i: \Omega_R^{n_i} \to E$ as follows: If n_i is odd, define $g_i := f_i$; if n_i is even, define g_i as the composition $\Omega\Sigma S_R^{n_i} \xrightarrow{\Omega\Sigma f_i} \Omega\Sigma E \xrightarrow{\rho} E$ where ρ is a retraction.

The g_i combine to define a map

$$\alpha: {}^{w}\prod_{i\in I} \Omega_R^{n_i} \to E$$

just as in the proof of 1. lemma.

PROPOSITION 1 (Compare [2], chap. V, (3.10)): *Let E be a connected R-local grouplike space with $H_*(E;R)$ free as R-module. Assume 1/2 , 1/3 $\in$ R . Then the following are equivalent:*

(a) *The H-space E is decomposable over R .*

(b) *Any map α constructed as above is a homotopy equivalence.*

(c) *The map $U(\pi_*(E)) \to H_*(E;R)$ from the universal enveloping algebra of $\pi_*(E)$ into the homology of E induced by the Hurewicz homomorphism $h\colon \pi_*(E) \to H_*(E;R)$ is surjective.*

Proof: This is essentially a reformulation of [2], chap. V, (3.10). Compare also [20].

PROPOSITION 2: *Assume* 1/2 $\in$ R .

(a) An H-space E is decomposable over R , if and only if the co-H-space ΣE is decomposable over R .

(b) A co-H-space C is decomposable over R , if and only if ΩC is decomposable over R .

Proof: (a) Let E be decomposable. Each $\Sigma\Omega_R^n$ is homotopy equivalent to a wedge of R-local spheres; this is trivial for n odd. If n is even, $\Sigma\Omega_R^n = \Sigma\Omega\Sigma S_R^n \sim \bigvee_{i\geq 1} \Sigma S_R^{ni}$ (see [23], chap. VII, section 2). The assertion now follows by the methods of the proof of 2. lemma using the lemma below.

Let ΣE be decomposable. It follows from the Hilton-Milnor theorem (see [23], chap. XI, section 6) that $\Omega\Sigma E$ is homotopy equivalent to a weak product of spaces $\Omega\Sigma S_R^n$ with $n \geq 1$. Since 1/2 $\in$ R the space $\Omega\Sigma S_R^n$ for n odd is homotopy equivalent to $S_R^n \times \Omega\Sigma S_R^{2n}$. Hence $\Omega\Sigma E$ is decomposable over R . One now observes that a retract (up to homotopy) of a decomposable H-space is again decomposable (see [20]).

(b) Let C be decomposable, then by the Hilton-Milnor theorem and the assumption $1/2 \in R$ the H-space ΩC is decomposable.

If ΩC is decomposable, we know by (a) that $\Sigma\Omega C$ is decomposable. Since C is a retract up to homotopy of $\Sigma\Omega C$ by [9], the assertion is implied by the following lemma.

LEMMA: *Let X be 1-connected with $\tilde{H}_*(X;\mathbb{Z})$ a free R-module. Then X is homotopy equivalent to a wedge of R-local spheres, if and only if the Hurewicz homomorhpism $h: \pi_*(X) \to \tilde{H}_*(X;\mathbb{Z})$ is surjective.*

Proof: Let h be surjective; choose a basis of homogeneous elements $\{a_i\}_{i\in I}$ of $\tilde{H}_*(X;\mathbb{Z}) \cong \tilde{H}_*(X;R)$. For each i choose $\alpha_i: S_R^{n_i} \to E$ with $h(\alpha_i) = a_i$. Then the maps α_i define a map $\bigvee_{i\in I} S_R^{n_i} \to X$ inducing an isomorphism of homology. Both spaces being 1-connected this map is a homotopy equivalence.

The other direction does not require a proof.

In fact, the lemma is a special case of the dual of a theorem of J.C. Moore (see [23], chap. IX, (1.9) theorem).

A2. The functor M^* and decomposable H-spaces

From now on let $1/2$, $1/3 \in R$. We recall the definition of $M^*(X;R)$ from [2], chap. V.

DEFINITION 1: *Let X be a connected space. Then define $M^\circ(X;R) := R$. For finite dimensional X let $M^i(X;R) := [X,\Omega_R^i]$ for $i > 0$. For infinite dimensional X let*

$M^i(X;R) := \varprojlim_N M^i(X^N;R)$ *(where* X^N *denotes the N-skeleton of* X *).*

Then $M^*(X;R)$ is an object in the category div alg_R of connected commutative graded algebras over R with divided powers. Moreover, the module of homotopy coefficients $M^{*,*}$ (over R) operates on $M^*(X;R)$, so $M^*(X;R)$ is also an object in the category div alg_M of M-algebras (loc. cit.).

DEFINITION 2: *We say that* X *is phantom free with respect to* M^* *, if* $M^i(X;R) \cong [X,\Omega_R^i]$ *for all* $i > 0$.

Examples: Each Ω_R^r is phantom free w.r.t. M^* , because $\Sigma\Omega_R^r$ is decomposable.

But the infinite complex projective space is not phantom free w.r.t. M^* for some rings R (see [13]).

DEFINITION 3: *Let* A *be an algebra in* div alg_M .

(a) *We say that* A *is of geometric type if there exist algebras* $A_i \in$ div alg_M *with* $A_i^r = 0$ *for large* r *such that* $A \cong \varprojlim A_i$.

(b) *The full subcategory of* div alg_M *of algebras of geometric type is denoted by* g-div alg_M .

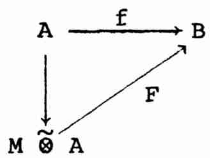
Examples: For any connected space X the algebra $M^*(X)$ is of geometric type by definition.

Let $A \in$ div alg_R , then the M -extension $M \tilde{\otimes} A$ in the sense of [2], chap. VII, is of geometric type.

REMARK 1: The forgetful functor g-div $alg_M \to$ div alg_R has a left adjoint, the M -extension of [2], chap. VII.

It is claimed in [2] that the M -extension is left adjoint to the forgetful functor div $alg_M \to$ div alg_R .

But we cannot see this in general. The M -extension $M \tilde{\otimes} A$ of $A \in \text{div alg}_R$ should be an algebra in div alg_M together with a morphism $A \to M \tilde{\otimes} A$ in div alg_R such that for any algebra B in div alg_M and morphism $f: A \to B$ in div alg_R there exists a unique morphism $F: M \tilde{\otimes} A \to B$ in div alg_M such that the diagram



commutes. Using the definitions of [2] this universal property can be verified, if B has the property that $B^r = 0$ for large r; hence it also holds for B of geometric type.

REMARK 2: In div alg_R finite coproducts exist (i.e. the tensor product). Hence certain coproducts exist in g-div alg_M by remark 1.

It can be shown that finite coproducts exist in g-div alg_M; but we want to avoid this verification here.

LEMMA 1: *Let E be a decomposable H-space over R of finite type over R, then the coproduct in g-div alg_M of $M^*(E;R)$ with itself, henceforth denoted by $M^*(E;R) \otimes_M M^*(E;R)$, exists and $M^*(E \times E;R) \cong$ $\cong M^*(E;R) \otimes_M M^*(E;R)$.*

Proof: This follows from the facts that the forgetful functor $\text{g-div alg}_M \to \text{div alg}_R$ has a left adjoint and that (see [2], chap. VII) for any decomposable H-space F over R the M-algebra $M^*(F;R)$ is the M-extension of $H^*(F;R)$. One has $H^*(E \times E;R) \cong H^*(E;R) \otimes_R H^*(E;R)$, hence $M^*(E \times E;R)$ is the M-extension of the coproduct $H^*(E;R) \otimes_R H^*(E;R)$, i.e. the coproduct $M^*(E;R) \otimes_M M^*(E;R)$.

The universal property of the coproduct implies that the algebra multiplication $M^*(E;R) \otimes_R M^*(E;R) \to M^*(E;R)$ factors over $M^*(E;R) \otimes_M M^*(E;R)$, hence $M^*(E;R)$ is equipped with the structure of a comultiplicative object in g-div alg_M with diagonal $\Delta: M^*(E;R) \to M^*(E;R) \otimes_M M^*(E;R)$ induced by the multiplication of E .

It follows that for any $A \in$ g-div alg_M there is a multiplication on $Mor_{div\ alg_M}(M^*(E;R),A)$ defined via the comultiplication Δ .

If E is grouplike, then $M^*(E;R)$ is a Hopf algebra in g-div alg_M .

REMARK 3: Let H be a connected commutative Hopf algebra and A a connected commutative algebra over $\mathbb{Q}$. Then the group structure on $Mor_{alg}(H,A)$ is determined by the Lie coalgebra $Q(H)$ (as a vector space $Q(H)$ is the space of "indecomposables" of H) and the algebra A , because H is the universal coenveloping algebra of $Q(H)$ (see [1], section 6). Moreover, $Mor_{alg}(H,A)$ is again a special kind of N-group. Its Lie algebra is $Mor_{mo}(Q(H),A)$, the Lie bracket being defined by $[f,g] := m_A(f\otimes g)\lambda$; here $\lambda: Q(H) \to Q(H) \otimes Q(H)$ is the Lie "cobracket" and m_A the algebra multiplication of A .

PROPOSITION: *Let X be connected and phantom free w.r.t. M^* , let E be a decomposable H-space of finite type over R . Then there is a multiplicative bijection*

$$[X,E] \cong Mor_{div\ alg_M}(M^*(E;R),M^*(X;E)) .$$

Proof: Let $E \sim {}^{w}\prod \Omega_R^{n_i}$ and let $\pi_i: {}^{w}\prod \Omega_R^{n_i} \to \Omega_R^{n_i}$ be the projections.

For each $n \geq 1$ the identity map of Ω_R^n defines an element $\iota_n \in M^n(\Omega_R^n;R)$; it is then a tautology, that

$[X,\Omega^n_R] \to M^n(X;R), [f] \to f^*(\iota_n)$, is an isomorphism.

Given a map $f: X \to {}^w\prod \Omega^{n_i}_R$ with components $f_i: X \to \Omega^{n_i}_R$ we have $f^*(\pi^*_i(\iota_{n_i})) = f^*_i(\iota_{n_i})$; hence f is completely determined by the elements $f^*(\pi^*_i(\iota_{n_i})) \in M^*(X;R)$, i.e. $[X,E] \to \mathrm{Mor}_{\text{div alg}_M}(M^*(E;R), M^*(X;R))$ is injective.

Recall that $M^*(\Omega^n_R;R)$ is a free algebra in g-div alg_M generated by ι_n ; this follows from [2], chap. VII, by computing $\deg(\iota_n)$ as a generator of the free algebra $H^*(\Omega^n_R;R)$ in div alg_R . Hence $M^*(E;R)$ is the free M-algebra of geometric type in the generators $\pi^*_i(\iota_{n_i})$ and any M-algebra morphism $M^*(E;R) \to M^*(X;R)$ can be realized geometrically.

<u>REMARK 4</u>: As a set $\mathrm{Mor}_{\text{div alg}_M}(M^*(E;R), M^*(X;R)) \approx$ $\approx \mathrm{Mor}_{\text{div alg}_R}(H^*(E;R), M^*(X;R))$ because $M^*(E;R)$ is the M-extension of $H^*(E;R)$.

<u>DEFINITION 4</u>: *The H-space E is said to have "property (s)", if E is decomposable of finite type over R and if $\deg: M^*(E;R) \to H^*(E;R)$ has a right inverse $s: H^*(E;R) \to M^*(E;R)$ in div alg_R which is also a morphism of quasi Hopf algebras, i.e. the diagram*

$$\begin{array}{ccc}
H^*(E;R) & \xrightarrow{\Delta} & H^*(E;R) \otimes_R H^*(E;R) \\
 & & \downarrow{\scriptstyle s \otimes s} \\
\downarrow{\scriptstyle s} & & M^*(E;R) \otimes_R M^*(E;R) \\
 & & \downarrow \\
M^*(E;R) & \xrightarrow{\Delta} & M^*(E;R) \otimes_M M^*(E;R)
\end{array}$$

commutes.

LEMMA 2: *Let the H-space E have property (s) and let X be connected and phantom free w.r.t. M^*. Then there is a multiplicative bijection*

$$[X,E] \cong \mathrm{Mor}_{\mathrm{div\ alg}_R}(H^*(E;R), M^*(X;R)) .$$

Proof: The diagonal $M^*(E;R) \xrightarrow{\Delta} M^*(E;R) \otimes_M M^*(E;R)$ is the M-extension of the diagonal $H^*(E;R) \to H^*(E;R) \otimes H^*(E;R)$.

COROLLARY: *Two grouplike spaces having property (s) are H-equivalent, if and only if their Hopf algebras $H^*(-;R)$ (resp. $H_*(-;R)$ resp. their Lie algebras $PH_*(-;R)$) are isomorphic.*

Proof: For a decomposable grouplike space E of finite type over R, the three objects $H^*(E;R)$, $H_*(E;R)$ and $PH_*(E;R)$ determine each other (by A1 $H_*(E;R) \cong U(PH_*(E;R))$ and $H^*(E;R)$ is the dual Hopf algebra to $H_*(E;R)$). Let now the grouplike spaces E_1, E_2 have property (s) with $H^*(E_1;R) \cong H^*(E_2;R)$ as Hopf algebras. Then by the lemma the functors $[-,E_1]$ and $[-,E_2]$ are isomorphic on the category of connected spaces having no phantom maps w.r.t. M^*. Since E_1, E_2 belong to that category, they are H-equivalent.

A3. The functor M^* and decomposable co-H-spaces

Let C be a decomposable co-H-space over R; i.e. $C \sim \bigvee_{i\in I} S_R^{n_i}$ with $n_i > 1$. Recall our definition $L(C) := \pi_*(\Omega C)$.

Let $\alpha_i \in \pi_{n_i-1}(\Omega C)$ be the adjoint of the inclusion $S_R^{n_i} \to C$. Then by [5] the Lie algebra $PH_*(\Omega C;R)$ is freely generated by $\{h(\alpha_i) \mid i\in I\}$ (h being the Hurewicz homomorphism).

Note that L(C) is an object of Lie_R and of Lie_M (see [2], chap. V, for a definition of Lie_M).

As a consequence the H-space ΩC is splittable in the sense of [2], chap. VI, and L(C) is the M-extension of $PH_*(\Omega C)$ (see [2], chap. VII); in particular L(C) is a free object in Lie_M . Thus we obtain:

LEMMA 1: The coproduct $L(C) \amalg_M L(C)$ of L(C) with itself exists in Lie_M and $L(C \vee C) \cong L(C) \amalg_M L(C)$ is the M-extension of $PH_*(\Omega(C \vee C);R)$.

In particular L(C) together with $L(C) \xrightarrow{L(\sigma)} L(C) \amalg_M L(C)$ ($\sigma: C \to C \vee C$ being the comultiplication of C) is a comultiplicative object in Lie_M .

REMARK 1: Finite coproducts exist in Lie_M .

PROPOSITION: Let C be a decomposable co-H-space over R , let $X \in 1\text{-}CW_R$. Then there is a multiplicative bijection $[C,X] \cong Mor_{Lie_M}(L(C),L(X))$ and a bijection $Mor_{Lie_M}(L(C),L(X)) \approx Mor_{Lie_R}(PH_*(\Omega C;R);L(X))$.

Proof: The isomorphism is established similarly to the proof of proposition 2 in section 2. The bijection follows from the fact that L(C) is the M-extension of $PH_*(\Omega C;R)$.

DEFINITION: Let C be a co-H-space in $1\text{-}CW_R$. We say that it has property (s), if it is decomposable over R and if $h: L(C) \to PH_*(\Omega C;R)$ has a right inverse $s: PH_*(\Omega C;R) \to L(C)$ in Lie_R compatible with the structures of comultiplicative objects; i.e. the diagram

$$\begin{array}{ccccc} L(C) & \longrightarrow & L(C) \amalg_M L(C) & \longleftarrow & L(C) \amalg_R L(C) \\ \uparrow{\scriptstyle s} & & & \nearrow{\scriptstyle s \amalg s} & \\ PH_*(\Omega C;R) & \longrightarrow & PH_*(\Omega C;R) \amalg_R PH_*(\Omega C;R) & & \end{array}$$

commutes.

LEMMA 2: *Let* C *be a co-H-space in* $1\text{-}CW_R$ *which has property* (s), *let* $X \in 1\text{-}CW_R$. *Then there is a multiplicative bijection*

$$[C,X] \cong \mathrm{Mor}_{\mathrm{Lie}_R}(PH_*(\Omega C;R),L(X)) .$$

Proof: The structure of $L(C)$ as comultiplicative object is just the M-extension of the structure of $PH_*(\Omega C;R)$ as comultiplicative object.

COROLLARY: *Two cogroups* C_1, C_2 *having property* (s) *are co-H-equivalent, if and only if the Berstein coalgebras associated to* $H_*(\Omega C_i;R)$ *are isomorphic* $(i = 1,2)$.

Proof: If C is decomposable over R the Berstein coalgebra $B(H_*(\Omega C;R))$ and the cogroup $PH_*(\Omega C;R)$ in Lie_R determine each other. Now follow the line of proof of A2, corollary.

REMARK 2: In [20] it is shown that the Samelson Lie algebra of an R-local grouplike space E admits the structure of an M-Lie algebra and that, if E is decomposable of finite type over R , the functor $[-,E]$ is determined by this M-Lie algebra on finite dimensional complexes. Dually, a Berstein algebra (with respect to M^*) is defined for an R-local cogrouplike space C , such that, if C is decomposable with $H_*(C;R)$ of finite rank, the functor $[C,-]$

on $1\text{-}CW_R$ is completely determined by this Berstein M-algebra.

Thus the problem arises to relate the propositions in A2 and A3 to these results.

We also conjecture that "having property (s)" is equivalent to "splittable" in the sense of [2], chap. VI.

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