Werk

Titel: Approximation of Quasiconvex Functions, and Lower Semicontinuity of Multiple Inte...

Autor: Marcellini, Paolo

Jahr: 1985

PURL: https://resolver.sub.uni-goettingen.de/purl?365956996_0051|log4

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

manuscripta math. 51, 1 - 28 (1985)

manuscripta mathematica
© Springer-Verlag 1985

APPROXIMATION OF QUASICONVEX FUNCTIONS, AND LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS

Paolo Marcellini

We study semicontinuity of multiple integrals $\int_\Omega f(x,u,Du)dx$, where the vector-valued function u is defined for $x \in \Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}^N$. The function $f(x,s,\xi)$ is assumed to be Carathéodory and quasiconvex in Morrey's sense. We give conditions on the growth of f that guarantee the sequential lower semicontinuity of the given integral in the weak topology of the Sobolev space $H^{1,p}(\Omega;\mathbb{R}^N)$. The proofs are based on some approximation results for f. In particular we can approximate f by a nondecreasing sequence of quasiconvex functions, each of them being *convex* and *independent* of (x,s) for large values of ξ. In the special polyconvex case, for example if $n=N$ and $f(Du)$ is equal to a convex function of the Jacobian detDu, then we obtain semicontinuity in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^n)$ for small p, in particular for some p smaller than n.

1. Introduction

Let us consider a function $f(x,s,\xi)$ defined for x in a bounded open set Ω of $\mathbb{R}^n$, $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$. We assume that f is a Carathéodory function, i.e., it is measurable with respect to x and continuous with respect to (s,ξ), and satisfies the growth conditions

$$(1.1) \qquad -C_1|\xi|^r - C_2|s|^t - C_3(x) \leq f(x,s,\xi) \leq g(x,s)(1+|\xi|^p).$$

Here $C_1, C_2 \geq 0$; $C_3 \in L^1(\Omega)$; $g \geq 0$ is a Carathéodory function (no growth conditions are required for g). For the exponents we assume: $p \geq 1$; $1 \leq r < p$ ($r = 1$ if $p = 1$) and $1 \leq t < np/(n-p)$ ($t \geq 1$ if $p \geq n$).

Moreover, if $C_2 \neq 0$, we assume also that the boundary $\partial\Omega$ is Lipschitz continuous.

Finally we assume that f is *quasiconvex* with respect to ξ in Morrey's sense ([25]; [26], section 4.4):

$$(1.2) \qquad \int_\Omega f(x,s,\xi + D\phi(y))dy \geq |\Omega| f(x,s,\xi) \quad \forall\phi\in H_0^{1,\infty}(\Omega;\mathbb{R}^N),$$

and also for every $\phi \in H_0^{1,p}(\Omega;\mathbb{R}^N)$, by mean of (1.1).

In section 4 we prove the following result:

THEOREM 1.1 - *Let* $f(x,s,\xi)$ *be e Carathéodory function, quasiconvex with respect to* ξ, *and satisfying the growth condition* (1.1). *Then the integral*

$$(1.3) \qquad \int_\Omega f(x,u(x),Du(x))\, dx$$

is sequentially lower semicontinuous in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$.

Theorem 1.1 improves the analogous result by Morrey [25], [26] and by Meyers [24], who assume a type of uniform continuity of f with respect to its arguments, and a recent result by Acerbi and Fusco [2], obtained assuming slightly more restrictive growth conditions.

We recall that either if N = 1 or n = 1, then f is quasiconvex if and only if f is convex; while, if both n and N are greater than one, then quasiconvexity is a more general condition than convexity (for properties of quasiconvex functions we refer to [26], [5], [8], [20]; see also our section 5). Therefore it seems not possible to reduce theorem 1.1 to the semicontinuity results known in convex case (see, i.e., Serrin [31], De Giorgi [10], and more recently Ekeland and Temam [15], chap. 8, theorem 2.1, and Eisen

[13]).

The proof of theorem 1.1, different from that of [25], [24], [2], but similarly to other classical semicontinuity results, is based on the possibility to approximate f by a nondecreasing sequence of functions f_k, each of them being easier to handle. To quote the main approximation theorem, proved in section 3, we assume also that $p > 1$ and that f satisfies the coercivity condition ($C_0 > 0$):

$$C_0 |\xi|^p \leq f(x,s,\xi) \leq g(x,s)(1 + |\xi|^p) \quad . \tag{1.4}$$

THEOREM 1.2 - *Let* $f(x,s,\xi)$ *be a Carathéodory function, quasiconvex with respect to* ξ, *and satisfying the growth condition* (1.4) *with* $p > 1$. *Then there exists a sequence* $f_k(x,s,\xi)$ *of Carathéodory functions, quasiconvex with respect to* ξ, *and such that:*

$$C_0 |\xi|^p \leq f_k(x,s,\xi) \leq k(1 + |\xi|^p) \quad ; \tag{1.5}$$

$$f_k(x,s,\xi) = C_0 |\xi|^p, \quad \text{either for } |s| \geq k \text{ or } |\xi| \geq k \quad ; \tag{1.6}$$

$$f_k \leq f_{k+1} \quad , \quad \sup_k f_k = f \quad . \tag{1.7}$$

It is clear that this approximation result is useful to prove theorem 1.1; in fact, in the domain $|Du| \geq k$, that is critical of the integral (1.3), f_k reduces to a convex function which is independent of x and s. One of the difficulties in the proof of theorem 1.2 is that the definition of quasiconvexity involves an integral inequality instead that a pointwise inequality such as convexity does. In our proof we follow a procedure introduced in a similar context by Marcellini and Sbordone [22], and we use a representation formula by Dacorogna [7], a variational principle by Ekeland [14], and a regularity result by Giaquinta and Giusti [17].

In section 2 we prove a semicontinuity result for $f = f(\xi)$, independent of x and s. The proof is particularly simple and selfcontained. Although this is a special case, it is a crucial step to obtian theorem 1.1.

In sections 3 and 4 we prove theorems 1.2 and 1.1 respectively.

In section 5 we specialize (1.2). By assuming that f is *polyconvex* In Ball's sense [5] (an example is given by (6.8)), we can prove a semicontinuity result that, with respect to theorem 1.1, roughly speaking allows us to consider semicontinuity in the weak topology of $H^{1,p}$, for p strictly smaller than in theorem 1.1. Let us mention that, in the same context of polyconvex integrals, Acerbi, Buttazzo and Fusco [1] proved a semicontinuity theorem in the strong topology of L^{∞}, while they have shown a counterexample to semicontinuity in the strong topology of L^p, if p is finite.

In section 6 we give some counterexamples to the semicontinuity theorems 1.1 and 5.5, when some of the assumptions are not satisfied.

We thank J.M. Ball and F. Murat for interesting discussions.

2. The case $f = f(\xi)$

THOEREM 2.1 - *Let* $f = f(\xi)$ *be a quasiconvex function such that*

(2.1) $$0 \le f(\xi) \le C_4(1 + |\xi|^p) ,$$

for $C_4 > 0$ *and* $p > 1$. *Then the integral*

(2.2) $$\int_\Omega f(Du(x))\, dx$$

is sequentially lower semicontinuous in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$.

We divide the proof of this theorem into 3 steps:

Step 1 - We assume first that u is affine, i.e., $Du = \xi$ in Ω for some $\xi \in \mathbb{R}^{nN}$.

Let u_h be a sequence in $H^{1,p}(\Omega;\mathbb{R}^N)$ that converges to u in the weak topology. If u_h had the same boundary values as u, then the semicontinuity result would trivially follow from quasiconvexity of f. To change the boundary data of u_h, we use a method introduced by De Giorgi [11], and well known in the context of Γ-convergence theory (see, i.e. Sbordone [30] and Dal Maso - Modica [9], theorem 6.1).

Let Ω_o be a fixed open set compactly contained in Ω, let $R = 1/2 \operatorname{dist}(\bar{\Omega}_o, \partial\Omega)$, let ν be a positive integer, and for i = = 1,2,...,ν let us define

$$\Omega_i = \{x \in \Omega : \operatorname{dist}(x,\Omega_o) < \frac{i}{\nu} R\}. \tag{2.3}$$

Let us choose smooth functions $\phi_i \in C^1_o(\Omega_i)$ such that

$$\begin{aligned} &0 \le \phi_i \le 1; \ \phi_i = 1 \text{ on } \Omega_{i-1}; \ \phi_i = 0 \quad \text{in } \Omega \setminus \Omega_i ; \\ &|D\phi_i| \le (\nu+1)/R. \end{aligned} \tag{2.4}$$

Let us define $v_{hi} = u + \phi_i(u_h - u)$. The support of v_{hi} is contained in Ω; thus by quasiconvexity of f we have

$$\begin{aligned} &\int_\Omega f(Du)dx = f(\xi)|\Omega| \le \int_\Omega f(Dv_{hi})dx \\ &= \int_{\Omega\setminus\Omega_i} f(Du)dx + \int_{\Omega_i\setminus\Omega_{i-1}} f(Dv_{hi})dx + \int_{\Omega_{i-1}} f(Du_h)dx . \end{aligned} \tag{2.5}$$

We sum up with respect to $i = 1,2,\dots,\nu$, and we divide by ν. We obtain

$$(2.6)\quad \int_\Omega f(Du)dx \le \int_{\Omega\setminus\Omega_0} f(Du)dx + \frac{1}{\nu}\int_{\Omega_\nu} f(Dv_{hi})dx + \int_\Omega f(Du_h)dx\ .$$

Since $Dv_{hi} = (1-\phi_i)Du + \phi_i Du_h + (u_h-u)D\phi_i$, we have

$$(2.7)\quad \begin{aligned} \int_{\Omega_\nu} f(Dv_{hi})dx &\le C_4|\Omega| + C_5\{\int_\Omega |Du|^p dx + \\ &+ \int_\Omega |Du_h|^p dx + (\frac{\nu+1}{R})^p \int_{\Omega_\nu} |u_h-u|^p dx\}. \end{aligned}$$

Let us go to the limit as $h \to +\infty$ in (2.6), (2.7). The sequence Du_h is bounded in $L^p(\Omega;\mathbb{R}^{nN})$, and u_h converges strongly to u in $L^p(\Omega_\nu,\mathbb{R}^N)$. Thus we have

$$(2.8)\qquad \int_\Omega f(Du)dx \le \int_{\Omega\setminus\Omega_0} f(Du)dx + \frac{C_6}{\nu} + \liminf_h \int_\Omega f(Du_h)dx.$$

As $\nu \to +\infty$ and $\Omega_0 \nearrow \Omega$ we obtain our result.

<u>Step 2</u> - f is continuous in the following way:

$$(2.9)\qquad |f(\xi) - f(\eta)| \le C_7(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi-\eta|$$

(this step is similar to theorem 4.4.1 of Morrey [26]). The function ϕ: $\mathbb{R} \to \mathbb{R}^+$, defined by $f(\xi)$ when only one component (say ξ_i) of ξ varies, is convex. Thus the derivative $\phi'(\xi_i)$ is definite almost everywhere, and we have

$$(2.10)\qquad \phi'(\xi_i) \gtrless (\phi(\xi_i+h) - \phi(\xi_i))/h \quad \text{if } h \gtrless 0.$$

For $h = \pm(|\xi|+1)$ we obtain

$$|f_{\xi_i}| = |\phi'| \leq \frac{\phi(\xi_i \pm |\xi| \pm 1) + \phi(\xi_i)}{|\xi| + 1}$$

(2.11)

$$\leq C_8 (1 + |\xi|^{p-1}).$$

Of course this inequality implies (2.9).

<u>Step 3</u> - We prove the semicontinuity result for general $u \in H^{1,p}(\Omega;\mathbb{R}^N)$. Let us consider a partition of Ω into open cubes Ω_i ($\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, $\bar{\Omega} = \cup \bar{\Omega}_i$) and let us define vectors $\xi_i \in \mathbb{R}^{nN}$ by

$$\text{(2.12)} \qquad \xi_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} Du \, dx .$$

Let us define $\xi \in L^p(\Omega;\mathbb{R}^{nN})$ by $\xi(x) = \xi_i$ for $x \in \Omega_i$. As the diameter of the partition goes to zero, ξ converges to Du in $L^p(\Omega;\mathbb{R}^{nN})$. Therefore for every $\epsilon > 0$ we can choose the partition of Ω in such a way that

$$\text{(2.13)} \qquad \sum_i \int_{\Omega_i} |Du - \xi_i|^p \, dx < \epsilon .$$

Let u_h be a sequence that converges to u in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$. For every i, let us define in Ω_i the sequence $v_h(x) = u_h(x) - u(x) + \langle \xi_i, x \rangle$. As $h \to +\infty$, v_h converges to $v(x) = \langle \xi_i, x \rangle$ in the weak topology of $H^{1,p}(\Omega_i;\mathbb{R}^N)$. Thus, by step 1, we have

$$\text{(2.14)} \qquad \liminf_h \sum_i \int_{\Omega_i} f(Dv_h)dx \geq \sum_i \int_{\Omega_i} f(\xi_i)dx .$$

By step 2, and by Hölder's inequality with exponents $p/(p-1)$ and p, we have

$$\left| \int_\Omega f(Du_h)dx - \sum_i \int_{\Omega_i} f(Dv_h)dx \right| \leq$$

$$(2.15) \qquad \leq C_7 \sum_i \int_{\Omega_i} (1 + |Du_h|^{p-1} + |Dv_h|^{p-1})|Du_h - \xi_i| dx$$

$$\leq C_9 (\int_\Omega (1+|Du_h|+|Dv_h|)^p dx)^{p-1/p} (\sum_i \int_{\Omega_i} |Du-\xi_i|^p)^{1/p} < C_{10}\, \varepsilon^{1/p}.$$

For the same reason

$$(2.16) \qquad |\int_\Omega f(Du)dx - \sum_i \int_{\Omega_i} f(\xi_i)dx| < C_{10}\, \varepsilon^{1/p}.$$

Our semicontinuity result is a consequence of (2.14), (2.15), (2.16).

3. Approximation of quasiconvex functions

In this section we assume, as in (1.4), that $f(x,s,\xi)$ is a Carathéodory function, quasiconvex with respect to ξ, and satisfying the growth conditions

$$(3.1) \qquad C_0|\xi|^p \leq f(x,s,\xi) \leq g(x,s)(1 + |\xi|^p),$$

where $p \geq 1$, $C_0 > 0$, and g is a Carathéodory function.

For every integer i, let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $\phi_i(t) = 1$ for $0 \leq t \leq i-1$, and $\phi_i(t) = 0$ for $t \geq i$. Let us define

$$(3.2) \qquad g_i(x,s,\xi) = \phi_i(|s|)f(x,s,\xi) + (1 - \phi_i(|s|))\, C_0|\xi|^p.$$

Let A be a subset of Ω, with zero measure, such that $g(x,s)$ is continuous with respect to s for every $x \in \Omega \setminus A$. For i,j integers ($j \geq C_0$), we define

$$(3.3) \qquad A_{ij} = \{x \in \Omega \setminus A: \max\,\{g(x,s) : |s| \leq i\} < j\}\ .$$

A_{ij} is a measurable set. We define $\psi_{ij}(x) = 1$ if $x \in A_{ij}$, and $\psi_{ij}(x)= =0$ otherwise. We define also

$$(3.4) \qquad g_{ij}(x,s,\xi) = \psi_{ij}(x) g_i(x,s,\xi) + (1 - \psi_{ij}(x)) C_0 |\xi|^p .$$

LEMMA 3.1 - *For every* i,j,g_{ij} *is a Carathéodory function, quasiconvex with respect to* ξ, *satisfying:*

$$(3.5) \qquad C_0 |\xi|^p \leq g_{ij}(x,s,\xi) \leq j(1 + |\xi|^p) \; ;$$

$$(3.6) \qquad g_{ij}(x,s,\xi) = C_0 |\xi|^p \; \text{for} \; |s| \geq i;$$

$$(3.7) \qquad \sup_{ij} g_{ij}(x,s,\xi) = f(x,s,\xi) \quad , \quad \forall\, x \in \Omega \setminus A, \; \forall\, s, \; \forall\, \xi.$$

Proof. g_{ij} is a Carathéodory function, since $\phi_i(|s|)$ is continuous and $\psi_{ij}(x)$ is measurable. With respect to ξ, g_i and g_{ij} are quasiconvex functions: in fact they are convex combination of quasiconvex functions. If $|s| \geq i$ then $g_{ij} = C_0 |\xi|^p$; $g_{ij} = C_0 |\xi|^p$ also if $x \notin A_{ij}$; while, if $|s| < i$ and $x \in A_{ij}$ we have

$$(3.8) \qquad g_{ij} \leq g_i \leq f \leq g(x,s)(1+|\xi|^p) \leq j(1 + |\xi|^p).$$

Thus (3.5), (3.6) are proved. To obtain (3.7) we observe that g_{ij} is nondecreasing with respect to i and j separately, since $f \geq g_i \geq C_0 |\xi|^p$; moreover $\lim_i \lim_j g_{ij} = f$.

For every integer $m \geq 1$ let us define in $\Omega \setminus A \times \mathbb{R}^N \times \mathbb{R}^{nN}$:

$$(3.9)\qquad G_{ijm}(x,s,\xi) = \begin{cases} g_{ij}(x,s,\xi) & \text{for } |\xi| \le m\,, \\ C_0|\xi|^p & \text{for } |\xi| > m\,; \end{cases}$$

$$(3.10)\qquad g_{ijm}(x,s,\xi) = \sup\{\, G(x,s,\xi) : G \text{ is quasiconvex with respect to } \xi \text{ and } G \le G_{ijm} \}.$$

LEMMA 3.2 - g_{ijm} *is quasiconvex with respect to* ξ *and satisfies:*

$$(3.11)\qquad C_0|\xi|^p \le g_{ijm}(x,s,\xi) \le j(1+|\xi|^p)\ ;$$

$$(3.12)\qquad g_{ijm}(x,s,\xi) = C_0|\xi|^p \ \textit{either for}\ |\xi| \ge m \ \textit{or}\ |s| \ge i.$$

Proof. Since the supremum of a family of quasiconvex functions is quasiconvex, in (3.10) we have a maximum and g_{ijm} is quasiconvex. Since the convex function $G = C_0|\xi|^p$ is less than or equal to G_{ijm}, we have $C_0|\xi|^p \le g_{ijm}$, and thus (3.12).

Fixed $x \in \Omega\setminus A$ and $s \in \mathbb{R}^N$, we consider the infimum

$$(3.13)\ \inf\{\frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s\ \xi + D\phi(y))dy:\ \phi \in H_0^{1,p}(\Omega;\mathbb{R}^N)\}.$$

LEMMA 3.3 - *The infimum in (3.13) is a continuous function of* (s,ξ).

Proof. For some fixed $x \in \Omega\setminus A$, the function G_{ijm} is uniformly continuous for $s \in \mathbb{R}^N$ and $|\xi| \le m$. Thus, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that, if $|s-t|+|\xi-\eta| < \delta$, we have (we decompose the integral over Ω into two integrals, and we use inequality (2.9) for the function $|\xi|^p$):

$$\left| \frac{1}{|\Omega|} \int_\Omega G_{ijm}(x,s,\xi+D\phi(y))dy - \frac{1}{|\Omega|} \int_\Omega G_{ijm}(x,t,\eta + D\phi(y))dy \right|$$

$$\text{(3.14)} \quad \leq \varepsilon + \frac{C_0}{|\Omega|} \left| \int_\Omega (|\xi+D\phi(y)|^p - |\eta+D\phi(y)|^p)dy \right|$$

$$\leq \varepsilon + C_{11} \{|\xi|^{p-1} + |\eta|^{p-1} + \frac{1}{|\Omega|} \int_\Omega |D\phi(y)|^{p-1}dy\}\delta \ .$$

Since $G_{ijm} \geq C_0|\xi|^p$, in the infimum (3.13) we can limit ourself to consider test functions ϕ that are uniformly bounded in $H_0^{1,p}(\Omega;\mathbb{R}^N)$ as ξ varies in a bounded set of $\mathbb{R}^{nN}$. For all such functions ϕ. if $|s-t|+|\xi-\eta| <$ min $\{\varepsilon, \delta\}$ we have

$$\text{(3.15)} \quad \left|\frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+D\phi(y))dy - \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,t,\eta+D\phi(y))dy\right| < C_{12}\varepsilon$$

Of course, this implies that the infimum in (3.13) is a continuous function of (s,ξ).

REMARK 3.4 - The previous result does not hold if the integrand G_{ijm} does not satisfy some properties of structure, such as, for example, coercivity with respect to ξ or continuity on s uniformly with respect to ξ (as suggested by corollary 3.12 of [21]). In fact, if we consider, as in [21], $G_{ijm} = (1+|\xi|)^{|s|}$, then the infimum in (3.13) is equal to G_{ijm} if $|s| \geq 1$, while is equal to 1 if $|s| < 1$. Thus, in this case, the infimum is not continuous (and not even lower semicontinuous) with respect to s.

LEMMA 3.5 (Dacorogna) - $g_{ijm}(x,s,\xi)$ *is a Carathéodory function, and is equal to the infimum in* (3.13).

Proof. By lemma 3.3 the infimum in (3.13) is a continuous function of $\xi \in \mathbb{R}^{nN}$. It is necessary to use this fact as the first step in the argument of Dacorogna ([7], theorem 5; or [8], pag. 87). Then, like in steps 2,3,4 of [7], [8] we obtain that g_{ijm} is the infimum in (3.13). Again by lemma 3.3 g_{ijm} is continuous in (s,ξ). g_{ijm} is measurable in x, since it is infimum of a family of measurable functions.

LEMMA 3.6 (Ekeland) - *There exists a sequence* u_m *that, for every* m, *minimizes the functionals*

$$(3.16)\qquad \phi \to \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+D\phi(y))dy + \frac{1}{m}\int_\Omega |D\phi(y)-Du_m(y)|dy \quad ,$$

on $H_0^{1,p}(\Omega;\mathbb{R}^N)$, *and satisfies*

$$(3.17)\qquad \frac{1}{|\Omega|}\int_\Omega G_{ijm}(x,s,\xi+Du_m(y))dy < g_{ijm}(x,s,\xi) + \frac{1}{m}\,.$$

Proof. This lemma is a particular case of a variational principle given by Ekeland [14] in the general setting of a complete metric space V and a lower semicontinuous functional $F : V \to \mathbb{R} \cup \{+\infty\}$ $(F \not\equiv +\infty)$. Here $V = H^{1,1}(\Omega;\mathbb{R}^N)$, and F is the integral of G_{ijm}, that is *strongly* lower semicontinuous in $H^{1,1}(\Omega;\mathbb{R}^N)$, by Fatou's lemma. In (3.17) we use the characterization of g_{ijm} given in lemma 3.5.

The fact stated in (3.16), that u_m is a minimum function, allows us to get a Mayer's type result [23], introduced in the context of minimum problems for vector valued functions by Giaquinta and Giusti [17] (see also [4] for the convex case and [18] for quasi-minima). Like in lemma 3.2 of Marcellini and Sbordone [22], from (3.16) we deduce:

LEMMA 3.7 - *If* $p>1$, *there exists an* $\varepsilon>0$ *such that the sequence* u_m *is bounded in* $H^{1,p+\varepsilon}_{loc}(\Omega;\mathbb{R}^N)$.

LEMMA 3.8 - *If* $p>1$, *the nondecreasing sequence* g_{ijm} *converges* to g_{ij} as $m \to +\infty$.

Proof. Let Ω_0 be a fixed open set compactly contained in Ω. The sequence u_m of the previous lemma is bounded in $H^{1,p+\varepsilon}(\Omega_0;\mathbb{R}^N)$. Thus, if we denote by Ω_m the set $\Omega_m = \{x \in \Omega_0 : |Du_m| \geq m\}$, we have

$$(3.18) \qquad \int_{\Omega_m} |Du_m|^p dx \leq |\Omega_m|^{\varepsilon/p+\varepsilon} \left(\int_{\Omega_0} |Du_m|^{p+\varepsilon} dx\right)^{p/p+\varepsilon}.$$

Therefore the left side in (3.18) converges to zero, as $m \to +\infty$. From (3.17) and (3.5) we obtain

$$g_{ijm} + \frac{1}{m} > \frac{1}{|\Omega|} \int_{\Omega_0} G_{ijm}(x,s,\xi+Du_m(y))dy$$

$$(3.19) \qquad \geq \frac{1}{|\Omega|} \int_{\Omega_0 \setminus \Omega_m} g_{ij}(x,s,\xi+ Du_m(y))dy$$

$$\geq \frac{1}{|\Omega|} \int_{\Omega_0} g_{ij}(x,s,\xi+Du_m(y))dy - \frac{j}{|\Omega|} \int_{\Omega_m} (1+|Du_m(y)|^p)dy .$$

Since u_m is bounded in $H_0^{1,p}(\Omega;\mathbb{R}^N)$, it has a subsequence that weakly converges. We still denote this subsequence by u_m, and we denote by $u \in H_0^{1,p}(\Omega;\mathbb{R}^N)$ their weak limit. Let $m \to +\infty$; we use (3.18), (3.19), the semicontinuity result of section 2, and quasiconvexity of g_{ij}:

$$\lim_m g_{ijm} \geq \liminf_m \frac{1}{|\Omega|} \int_{\Omega_0} g_{ij}(x,s,\xi+Du_m(y))dy$$

$$(3.20) \qquad \geq \frac{1}{|\Omega|} \int_{\Omega_0} g_{ij}(x,s,\xi+Du(y))dy$$

$$\geq g_{ij}(x,s,\xi) - \frac{1}{|\Omega|} \int_{\Omega \setminus \Omega_0} g_{ij}(x,s,\xi+Du(y))dy .$$

As $\Omega_0 \nearrow \Omega$, we obtain our result, since $g_{ijm} \leq g_{ij}$.

REMARK 3.9 - Several lemmas, from 3.4 to 3.8, are devoted to the study of the convergence of g_{ijm} as $m \to +\infty$. Let us show that this study is much easier if we know that f is *convex* with respect to ξ: for every m we can construct a function $G(x,s,\xi)$, convex with respect to ξ, that coincides with g_{ij} for $|\xi| \le m$ and grows linearly at ∞ (the supremum of all hyperplanes supporting g_{ij} where $|\xi| < m$). Since G grows linearly, there exists an $m' > m$ such that $G \le C_0 |\xi|^p$ for $|\xi| \ge m'$. By the same definition (3.10), $G \le g_{ijm'} \le g_{ij}$, and thus $g_{ijm'} = g_{ij}$ for $|\xi| \le m$. Note that also in this simple argument for the convex case we need $p > 1$.

Finally we obtain the approximation result stated in the introduction:

Proof of theorem 1.2 - For every integer k ($\ge 2 + C_0$) we define

$$(3.21) \qquad f_k(x,s,\xi) = \max \{ g_{ijm}(x,s,\xi) \ : \ i + j + m \le k \} .$$

(1.5) is consequence of (3.11), while (1.6) follows from (3.12). Finally the supremum of f_k is f, by lemma 3.8 and formula (3.7).

4. Semicontinuity in the quasiconvex case

In this section we will prove theorem 1.1. It will be consequence of the approximation theorem 1.2., by proving a semicontinuity result for the functions f_k as in theorem 1.2. In the following lemmas we assume k is fixed and f_k satisfies (1.5), (1.6).

LEMMA 4.1 (Scorza-Dragoni) - *For every positive* ε *there exists a compact set* $C \subset \Omega$, *with* $|\Omega \setminus C| < \varepsilon$, *such that* $f_k(x,s,\xi)$ *is continuous in* $C \times \mathbb{R}^N \times \mathbb{R}^{nN}$.

Proof. See i.e. lemma 1 of pag. 37 of [10], or [15], chap. VIII, section 1.3.

LEMMA 4.2 - *There exists a continuous bounded function* $w:\mathbb{R}^+ \to \mathbb{R}^+$, *with* $w(0)=0$, *such that, for* $x,y \in C$, $s,t \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{nN}$ *we have*

$$|f_k(x,s,\xi)-f_k(y,t,\xi)| \leq w(|x-y| + |s-t|) \ . \tag{4.1}$$

Proof. For $|\xi| \leq k$, by (1.5), we have

$$|f_k(x,s,\xi)-f_k(y,t,\xi)| \leq 2k(1 + k^p) \ . \tag{4.2}$$

Inequality (4.2) holds also if $|\xi| > k$ since the left side is zero. The function f_k is continuous in the compact set $C \times \{|s| \leq k+1\} \times \{|\xi| \leq k+1\}$. Thus (4.1) holds on this set with w equal to the oscillation of f_k. By (4.2) the function w is bounded, and we can assume that $w(r) = 2k(1+k^p)$ for $r \geq 1$. By (1.6), formula (4.1) holds also if either $|\xi| \geq k$ or $|s|$ and $|t| \geq k$. It remains to consider the case $|s| < k$, $|t| > k+1$ and $|\xi| < k$. In this case $w = 2k(1+k^p)$, and thus (4.1) follows from (4.2).

LEMMA 4.3 - *The integral*

$$\int_\Omega f_k(x,u(x),Du(x))dx \tag{4.3}$$

is sequentially lower semicontinuous in the weak topology of $H^{1,p}(\Omega;\mathbb{R}^N)$.

Proof. Let $u \in H^{1,p}(\Omega;\mathbb{R}^N)$. Let us consider a partition of Ω into open cubes Ω_i, with $\Omega_i \cap \Omega_j = \emptyset$. $\cup_i \bar{\Omega}_i = \bar{\Omega}$. Like in Morrey

[25] we define the vector valued functions (constant in each Ω_i):

$$(4.4)\qquad x_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} x\,dx \quad ; \quad u_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} u(x)\,dx\ .$$

By the dominated convergence theorem, for every $\varepsilon > 0$, we can choose the partition so that

$$(4.5)\qquad \int_\Omega w(|x-x_i| + |u(x)-u_i(x)|)dx < \varepsilon.$$

Let u_h be a sequence of $H^{1,p}(\Omega;\mathbb{R}^N)$ that converges to u in the weak topology. We have

$$\begin{aligned}\int_\Omega f_k(x,u_h,Du_h)dx &= \int_{\Omega\setminus C}\{f_k(x,u_h,Du_h) - f_k(x_i,u_i,Du_h)\}\ dx\\ (4.6)\qquad &+ \int_C \{f_k(x,u_h,Du_h) - f_k(x,u,Du_h)\}\ dx\\ &+ \int_C \{f_k(x,u,Du_h) - f_k(x_i,u_i,Du_h)\}dx + \int_\Omega f_k(x_i,u_i,Du_h)dx\end{aligned}$$

We use (4.2), (4.1) and (4.5):

$$\begin{aligned}\int_\Omega f_k(x,u_h,Du_h)dx &\geq -\ 2k(1+k^p)|\Omega\setminus C|\\ (4.7)\qquad &- \int_C w(|u_h-u|)dx - \varepsilon + \sum_i \int_{\Omega_i} f_k(x_i,u_i,Du_h)dx\ .\end{aligned}$$

As $h \to +\infty$, by the semicontinuity theorem 2.1, we have

$$(4.8)\qquad \liminf_h \int_\Omega f_k(x,u_h,Du_h)dx \geq -\ C_{13}\varepsilon + \int_\Omega f_k(x_i,u_i,Du)dx\ .$$

We obtain the proof as the sides of the cubes Ω_i and ε go to zero.

<u>Proof of theorem 1.1</u> - Let us assume first that $p > 1$. Similarly

to Serrin [31], for $\varepsilon > 0$, let us define

$$(4.9) \qquad g_\varepsilon(x,s,\xi) = f(x,s,\xi) + C_2|s|^t + C_3(x) + \varepsilon|\xi|^p + C_\varepsilon .$$

Since $p > r$, we can choose the constant C_ε to obtain $g_\varepsilon(x,s,\xi) \geq$ $\geq \varepsilon/2 \; |\xi|^p$. Let $f_{\varepsilon k}$ be the sequence of quasiconvex functions that converges, as $k \to +\infty$, to g_ε, according to theorem 1.2. If u_h weakly converges to u in $H^{1,p}(\Omega;\mathbb{R}^N)$, by lemma 4.3 we have

$$(4.10) \qquad \begin{aligned} &\liminf_h \int_\Omega g_\varepsilon(x,u_h,Du_h)dx \geq \lim_k \liminf_h \int_\Omega f_{\varepsilon k}(x,u_h,Du_h)dx \\ &\lim_k \int_\Omega f_{\varepsilon k}(x,u,Du)\,dx = \int_\Omega g_\varepsilon(x,u,Du)\,dx . \end{aligned}$$

Let C_{14} be an upper bound for the $H^{1,p}$ norm of u_h. Since u_h converges to u in the strong topology of $L^t(\Omega;\mathbb{R}^N)$ (here we use the assumption that $\partial\Omega$ is smooth if $C_2 \neq 0$), we obtain

$$(4.11) \qquad \begin{aligned} \liminf_h \int_\Omega f(x,u_h,Du_h)\,dx &\geq \liminf_h \int_\Omega g_\varepsilon(x,u_h,Du_h)dx \\ &- \lim_h \int_\Omega \{C_2|u_h|^t + C_3(x) + C_\varepsilon\}\,dx - \frac{\varepsilon}{2} C_{14}^p \\ &\geq \int_\Omega f(x,u,Du)dx + \frac{\varepsilon}{2}\int_\Omega |Du|^p dx - \frac{\varepsilon}{2} C_{14}^p \quad . \end{aligned}$$

We complete the proof of the case $p > 1$ as $\varepsilon \to 0$. If $p = 1$, the proof is much simpler since, if u_h converges to u in the weak topology of $H^{1,1}(\Omega;\mathbb{R}^N)$, then the integrals of $|Du_h|$ are equiabsolutely continuous. We do not give the details; we can use the approximation lemma 3.1 and then the argument by Fusco [16], or the argument of section 2 of [22].

5. The polyconvex case

In this section we consider a particular case of quasiconvex functions. Following Ball [5], we say that a function $f(x,\xi)$ is *polyconvex* with respect to ξ if there exists a function $g(x,\xi)$, convex with respect to $\eta \in \mathbb{R}^m$, such that

$$(5.1) \qquad f(x,\xi) = g(x,\xi,\det_1\xi, \det_2\xi,\dots) \quad ,$$

where $\det_i \xi$ are subdeterminants (or adjoints) of the $n \times N$ matrix ξ. If $\xi = Du$, then each determinant is a divergence. For example, for $n = N = 2$, if $u \equiv (u^1,u^2) \in C^2(\Omega:\mathbb{R}^2)$, we have

$$(5.2) \qquad \det Du = u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1} = (u^1 u^2_{x_2})_{x_1} - (u^1 u^2_{x_1})_{x_2} \; .$$

Using (5.2) (and in general (5.4)), we can verify by Jensen's inequality that every polyconvex function is quasiconvex. By multiplying (5.2) by a test function $\phi \in C^\infty_0(\Omega)$ and by integrating by parts we have

$$(5.3) \qquad \int_\Omega \det Du \, \phi \, dx = - \int_\Omega \{ u^1 u^2_{x_2} \phi_{x_1} - u^1 u^2_{x_1} \phi_{x_2} \} \, dx \; .$$

By continuity (5.3) holds for $u \in H^{1,2}(\Omega;\mathbb{R}^2)$. If $u \in H^{1,p}(\Omega;\mathbb{R}^2)$ for $p < 2$, then $u \in L^{2p/2-p}_{loc}(\Omega;\mathbb{R}^2)$ and thus the product $u^1 u^2_{x_2}$ is summable if $1/p + (2-p)/2p \leq 1$, i.e., $p \geq 4/3$. Thus, if $u \in H^{1,4/3}(\Omega;\mathbb{R}^2)$ we can define by (5.3) the determinant of Du as a distribution. Moreover for the same reasons the map $u \to \det Du$ is continuous in the following sense: if u_h converges to u in $H^{1,p}_{loc}(\Omega;\mathbb{R}^2)$ for $p > 4/3$ and if (5.3) holds for u_h and u, then $\det Du_h$ converges to $\det Du$ in the sense of distributions. In fact in this case u^1_h strongly converges to u in $L^4(\Omega)$ and we can go to the limit as $h \to +\infty$ in (5.3).

For general $n = N \geq 2$ we can still write det Du as a divergence (Morrey [26], pp. 122-123)

$$(5.4)\quad \det Du = \frac{\partial(u^1,\dots,u^n)}{\partial(x_1,\dots,x_n)} = -\sum_{\alpha=1}^{n}(-1)^{\alpha}\frac{\partial}{\partial x_\alpha}\left(u^1\frac{\partial(u^2,\dots,u^n)}{\partial(x_1,\dots,x_{\alpha-1},x_{\alpha+1},\dots,x_n)}\right)$$

This formula holds if $u \in H^{1,n}(\Omega;\mathbb{R}^n)$. Since if $u \in H^{1,p}$ $(n-1 \leq p < n)$ then $u^1 \in L^{np/(n-p)}_{loc}$ and the Jacobian of order $n-1$ belongs to $L^{p/(n-1)}$, the right side of (5.4) is well defined in the sense of distributions if $(n-1)/p+(n-p)/np \leq 1$, i.e., $p \geq n^2/(n+1)$. Thus we have proved, as in Ball [5] and Ball, Currie and Olver [6], the following result:

LEMMA 5.1 ([5],[6]) - *Let* $n = N \geq 2$ *and* $u \in H^{1,p}_{loc}(\Omega;\mathbb{R}^n)$. *If* $p \geq n^2/(n+1)$ *then* det Du *is defined by* (5.4) *as a distribution; while if* $p \geq n$, det Du *is defined as a* L^1_{loc} *-function and formula* (5.4) *holds. Moreover, if* u_h *converges to* u *in the weak topology of* $H^{1,p}_{loc}$ *for* $p > n^2/(n+1)$, *then* $\det Du_h$ *converges to* det Du *in the sense of distributions.*

To get a semicontinuity result for polyconvex functions, we consider a function $g(x,\eta)$ defined for $x \in \Omega \subset \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ with values in $[0,+\infty]$ (see in next section the reason to assume g continuous in x and independent of s) satisfying:

(5.5) *The set* $\{(x,\eta) : g(x,\eta) < +\infty\}$ *is open (and not empty) in* $\Omega \times \mathbb{R}^m$, *and* g *is continuous on this set.*

(5.6) $g(x,\eta)$ *is convex and lower semicontinuous with respect to* $\eta \in \mathbb{R}^m$.

REMARK 5.2 - Of course, the semicontinuity assumption in (5.6) is a nontrivial condition only at the boundary of the domain where g is finite. We assume (5.5) to simplify the next lemma, but we could consider also other cases. On assuming (5.5), (5.6) we have in mind the situation described by Ball [5], of interest in nonlinear elasticity (see also the paper by Antman [3]), where are considered functions $f(x,Du) = g(x,\det Du)$ that are finite if and only if $\det Du > 0$, and go to $+\infty$ if $\det Du \to 0$.

Let us begin with two approximation lemmas.

LEMMA 5.3 (De Giorgi) - *There exists a nondecreasing sequence of real nonnegative functions* g_k *that converge, as* $k \to +\infty$, *to* g. *For every* $k, g_k(x,\eta)$ *is uniformly continuous in* $\Omega \times \mathbb{R}^m$, *it grows linearly with respect to* η, *it is convex with respect to* η *and it is equal to zero if* $\operatorname{dist}(x,\partial\Omega) \leq 1/k$.

Proof. Let $\Omega_i = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > 1/i\}$, and let $\phi_i \in C_0^\circ(\Omega_i)$ be equal to one on Ω_{i-1}, and $\phi_i \geq 0$. For every $x \in \Omega$, the function g is lower semicontinuous on $\mathbb{R}^m$; thus it is the supremum of a sequence of affine functions $(a_j(x),\eta) + b_j(x)$. Like in pag. 31 of De Giorgi [10] (the argument of [10] can be applied, since g is finite in a neighbour of each supporting point), we can choose $a_j(x)$ and $b_j(x)$ to be continuous in Ω. Let us define $a_0, b_0 \equiv 0$, and

$$(5.7) \qquad g_k(x,\eta) = \max\{\phi_i(x)[(a_j(x),\eta) + b_j(x)] : i+j \leq k\}.$$

The sequence g_k has all the required properties.

LEMMA 5.4 - *There exists a sequence* $h_k(x,\eta)$ *of* C^∞*-functions satisfying all the properties stated in the previous lemma (except the fact that* $h_k \geq -1$*).*

Proof. Let be a mollifier, i.e. $\alpha \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $\int \alpha \, dxdy = 1$,

$\alpha \geq 0$. Let us define $h_{k\varepsilon} = h_k * \alpha_\varepsilon$, where $\alpha_\varepsilon(x,\eta) = \varepsilon^{-(n+m)}\alpha(x/\varepsilon, \eta/\varepsilon)$. If ε is sufficiently small, $h_{k\varepsilon}$ is a nonnegative C^∞-function, and is zero if dist $(x, \partial\Omega) \leq 1/k+1$. By the uniform continuity of g_k, as $\varepsilon \to 0$ $h_{k\varepsilon}$ converges to g_k uniformly in $\Omega \times \mathbb{R}^m$. Thus we can choose $\varepsilon = \varepsilon(k)$ such that

$$(5.8) \qquad |h_{k,\varepsilon(k)} - g_k| < \frac{1}{2(k+1)^2} \quad .$$

For $k \geq 2$ let us define $h_k(x, \eta) = h_{k-1,\varepsilon(k-1)}(x,\eta) - (k-1)^{-1}$. The sequence h_k satisfies all the stated properties. For example, let us verify that h_k is increasing with respect to k:

$$(5.9) \qquad \begin{aligned} h_k &< g_{k-1} + \frac{1}{2k^2} - \frac{1}{k-1} \leq g_k + \frac{1}{2k^2} - \frac{1}{k-1} \\ &< h_{k,\varepsilon(k)} + \frac{1}{2(k+1)^2} + \frac{1}{2k^2} - \frac{1}{k-1} \\ &< h_{k+1} + \frac{1}{k} + \frac{1}{k^2} - \frac{1}{k-1} < h_{k+1} \ . \end{aligned}$$

Now we prove a semicontinuity result, that generalizes an analogous result of Reshetnyak [28].

<u>THEOREM 5.5</u> – *Let* $g: \Omega \times \mathbb{R}^n \to [0,+\infty]$ *be a function satisfying* (5.5), (5.6) . *Let* v_h *and* v *be functions of* $L^1_{loc}(\Omega;\mathbb{R}^m)$, *and assume that* v_h *converges to* v *in the sense of distributions, i.e.:*

$$(5.10) \qquad \lim_h \int_\Omega (v_h, \phi)dx = \int_\Omega (v, \phi)dx \ , \ \forall \phi \in C_0^\infty(\Omega;\mathbb{R}^m)$$

Then we have

$$(5.11) \qquad \liminf_h \int_\Omega g(x,v_h(x))dx \geq \int_\Omega g(x,v(x))\, dx \ .$$

<u>Proof.</u> By using our approximation lemma 5.4 in the usual way

like in (4.10), it is enough to prove the theorem assuming that g is $C^{\infty}(\Omega \times \mathbb{R}^m)$, and that there exists an open set Ω_0 compactly contained in Ω such that g is equal to zero for $x \notin \Omega_0$. Let α_ε be a mollifier and let $v_\varepsilon = v * \alpha_\varepsilon$. By the convexity of g, similarly to Serrin [31], we have

$$(5.12) \qquad g(x,v_h) \geq g(x,v_\varepsilon) + (D_\eta g(x,v_\varepsilon),\ v_h - v_\varepsilon) \quad .$$

Since $D_\eta g(x,v_\varepsilon(x)) \in C_0^\infty(\Omega;\mathbb{R}^m)$, by (5.10) we have

$$(5.13) \qquad \liminf_h \int_\Omega g(x,v_h)dx \geq \int_\Omega g(x,v_\varepsilon)dx + \int_\Omega (D_\eta g(x,v_\varepsilon), v-v_\varepsilon)dx \; .$$

We obtain the result for $\varepsilon \to 0$. In fact, in the first term of the right side we can use Fatou's lemma, while in the second term we can use the fact that $D_\eta g$ is bounded in $\Omega \times \mathbb{R}^m$ independently of ε.

By lemma 5.1 and theorem 5.5 we obtain two semicontinuity results for integrals of the type:

$$(5.14) \qquad F(u) = \int_\Omega f(x,Du)dx = \int_\Omega g(x,Du,\det_1 Du,\ \det_2 Du,\ldots)dx \; .$$

Here $f(x,\xi)$ is a polyconvex function like in (5.1), and $g(x,\eta)$ satisfies (5.5), (5.6).

<u>COROLLARY 5.6</u> - *Let* u_h *and* u *be functions of* $H^{1,p}_{loc}(\Omega;\mathbb{R}^N)$, *for* $p > \min\{n^2/(n+1); Nn/(n+1)\}$. *Assume that the subdeterminants of the Jacobians* Du_h *and* Du *are defined as* L^1_{loc} *-functions and that formula* (5.4) *holds for* u_h *and* u. *If* u_h *converges to* u *in the weak topology of* $H^{1,p}_{loc}(\Omega;\mathbb{R}^N)$, *then* $\liminf_h F(u_h) \geq F(u)$.

<u>COROLLARY 5.7</u> - *Let* u_h *and* u *be function of* $H^{1,r}_{loc}(\Omega;\mathbb{R}^N)$, *for* $r \geq$

$\geq \min\{n,N\}$. *If* u_h *converges to* u *in the weak topology of* $H^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ *for* $p > \min\{n^2/(n+1);\ Nn/(n+1)\}$ *then* $\liminf_h F(u_h) \geq F(u)$.

6. Some examples and remarks

Here we discuss the necessity of some assumptions of the semicontinuity theorems 1.1 and 5.5. Let us begin with theorem 5.5 and let us show that the result does not hold if $g(x,\eta)$ is only measurable with respect to x, or if $g = g(x,s,\eta)$ (with the usual meaning of s). Of course, to exhibit counterexamples, we must consider non coercive cases: in fact, if $g(x,\eta) \geq \text{cost}\ |\eta|^p$ for some $p > 1$, the semicontinuity theorem 5.5 reduces to the usual semicontinuity theorem in the weak topology of L^p.

EXAMPLE 6.1 - Let $n = m = 1$ and let $g(x,\eta) = a(x)\,\eta^2$, with $a(x)$ nonnegative, bounded and measurable in $(0,1)$. It has been proved in theorem 5 of [19] that for every $p \in [1,+\infty]$ and for every $u \in H^{1,2}$ there exists in $H^{1,2}$ a sequence u_h that converges to u in L^p and satisfies

$$(6.1)\qquad \lim_h \int_0^1 a(x)\,(u_h')^2\,dx = \int_0^1 b(x)\,(u')^2\,dx\ ,$$

where

$$(6.2)\qquad b(x) = \lim_{\varepsilon\to 0^+} 2\varepsilon\left[\int_{x-\varepsilon}^{x+\varepsilon} a^{-1}(y)\,dy\right]^{-1} .$$

If we consider a function $a(x) \neq 0$ a.e. that is not locally summable a.e. in $(0,1)$, then $b(x)$ is zero a.e. Let us define $v_h = u_h'$ and $v = u'$. Then, for every $\phi \in C_0^\infty$, we have

$$(6.3)\qquad \lim_h \int_0^1 v_h\,\phi\,dx = -\lim_h \int_0^1 u_h\,\phi'\,dx = -\int_0^1 u\phi'\,dx = \int_0^1 v\phi\,dx.$$

Moreover v_h and v are in L^1, but they do not satisfy (5.11), since, if v is not identically zero,

$$\text{(6.4)} \qquad \lim_h \int_0^1 a(x)\, v_h^2(x)\, dx = 0 < \int_0^1 a(x)\, v^2(x) dx \ .$$

EXAMPLE 6.2 - We can adapt the counterexample of Eisen [12] to our situation. In fact, Eisen showed that there exists a sequence of Lipschitz continuous functions $u_h \equiv (u_h^1, u_h^2)$ defined in $(0,1)$, such that the product $u_h^1 (u_h^2)' = 0$ a.e., and u_h converges in L^1 to the function $u(x) \equiv (1,x)$. Let us define $v_h = (u_h^2)'$, $v = (u^2)' = 1$, $w_h = u_h^1$, $w = u^1 = 1$. Like in (6.3), v_h converges to v in the sense of distributions; v_h and v are in L^1, but

$$\text{(6.5)} \qquad \int_0^1 (w_h v_h)^2 dx = 0 \quad , \int_0^1 (wv)^2 dx = 1 \ .$$

This means that in general we cannot extend theorem 5.5 to integrals of the type $\int g(w(x),v(x))dx$, where $g(s, \eta)$ is continuous in (s, η), and convex with respect to η, and the topology considered is the product of the L^1 norm topology for w, and the topology of distributions for v.

EXAMPLE 6.3 - In theorem 1.1 the assumption $t < np/(n-p)$ if $p < n$ is necessary. We have a counterexample for $f(x,s,\xi) = |\xi|^p - \text{cost}|s|^{np/(n-p)}$, by choosing a sequence u_h that weakly converges in $H^{1,p}$, but does not converge in the norm topology of $L^{np/(n-p)}$.

EXAMPLE 6.4 - If n and N are greater than or equal to 2, the assumption $r < p$ in theorem 1.1 is necessary. In fact there is a counterexample by Murat and Tartar (see the counterexample in section 4.1 of [27]), for $n = N = p = r = 2$, where it is shown that the integral

(6.6) $\int_\Omega a(x) \det Du \, dx$,

is not continuous in the weak topology of $H^{1,2}(\Omega;\mathbb{R}^2)$, even if a is a nonzero constant. It is a consequence of theorem 1.1, but it is also well known (see [29], [5], [6]; and for more general functionals [16], [22]), that, if $a \in L^\infty(\Omega)$, the integral in (6.6) is weakly sequentially continuous in the weak topology of $H^{1,2+\varepsilon}(\Omega;\mathbb{R}^2)$, for every positive ε.

EXAMPLE 6.5 - To discuss the necessity of the upper bound in (1.1) let us summarize our results in the special case $n = N \geq 2$ for the integral

(6.7) $\int_\Omega g(x,u(x))|\det Du(x)|^\alpha \, dx$.

We distinguish two cases: $\alpha \geq 1$ or $\alpha < 0$; in the second case we define $|\eta|^\alpha = +\infty$ if $\eta \leq 0$. If g is a nonnegative Carathéodory function and $\alpha \geq 1$, then the integral (6.7) is sequentially lower semicontinuous in the weak topology of $H^{1,n}(\Omega;\mathbb{R}^n)$. This follows from theorem 1.1 in the general case (if $\alpha > 1$ we can approximate $|\eta|^\alpha$ with a nondecreasing sequence of convex functions on $\mathbb{R}$, each of them growing linearly at ∞), and from corollary 5.7, if g is independent of s and continuous. If $g = g(x)$ is a nonnegative continuous function, and either $\alpha \geq 1$ or $\alpha < 0$, then from corollary 5.7 it follows also that, if $p > n^2/(n+1)$:

(6.8) *If* u_h *and* u *are smooth (say* $u_h, u \in H^{1,n}(\Omega;\mathbb{R}^n)$) *and* u_h *weakly converges to* u *in* $H^{1,p}(\Omega;\mathbb{R}^n)$, *then*

$$\liminf_h \int_\Omega g(x)|\det Du_h|^\alpha \, dx \geq \int_\Omega g(x)|\det Du|^\alpha \, dx.$$

Let us mention explicitely that we have not proved that (6.8) is true if $1 \leq p \leq n^2/(n+1)$. Let us also mention that Acerbi, Buttazzo and Fusco [1] proved that (6.8) is not true if we replace the weak convergence in $H^{1,p}(\Omega;\mathbb{R}^n)$ with the strong convergence in $L^p(\Omega;\mathbb{R}^n)$, whatever $p\in[1,+\infty)$ may be.

REMARK 6.6 - In (6.8) we distinguish between the space where the functional is well defined, and the space where the sequence u_h weakly converges. This is a natural point of view and it is not new. In this context of polyconvex functions we refer to theorem 9.2.1. by Morrey [26], and to [1]. We refer also to the well known semicontinuity theorems by Serrin [31] (see also [26], section 4.1), where the considered functions u_h are required to be in the space $H^{1,1}$, but the convergence is in the space L^1. We refer also to the theory of De Giorgi [11], related to this subject.

REFERENCES

[1] ACERBI E., BUTTAZZO G., FUSCO N., Semicontinuity and relaxation for integral depending on vector-valued functions, J. Math. Pures Appl., 62 (1983), 371-387

[2] ACERBI E., FUSCO N., Semicontinuity problems in the calculus of variations, Arch. Rat. Mech. Anal., to appear

[3] ANTMAN S.S., The influence of elasticity on analysis: modern developments, Bull. Amer. Math. Soc., 9 (1983), 267-291

[4] ATTOUCH H., SBORDONE C., Asymptotic limits for perturbed functionals of calculus of variations, Ricerche Mat., 29 (1980), 85-124

[5] BALL J.M., Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rat. Mech. Anal., 63 (1977), 337-403

[6] BALL J.M., CURRIE J.C., OLVER P.J., Null Lagragians, weak continuity and variational problems of arbitrary order, J. Funct. Anal., 41 (1981), 135-175

[7] DACOROGNA B., Quasiconvexity and relaxation of nonconvex problems in the calculus of variations, J. Funct. Anal., 46 (1982), 102-118

[8] DACOROGNA B., Weak continuity and weak lower semicontinuity of nonlinear functionals, Lecture Notes in Math., 922 (1982), Springer-Verlag, Berlin

[9] DAL MASO G., MODICA L., A general theory of variational functionals, Topics in Func. Anal. 1980-81, Quaderno Scuola Norm. Sup. Pisa, 1981, 149-221

[10] DE GIORGI E., Teoremi di semicontinuità nel calcolo delle variazioni, Istituto Nazionale di Alta Matematica, Roma, 1968-69

[11] DE GIORGI E., Sulla convergenza di alcune successioni di integrali del tipo dell'area, Rendiconti Mat., 8 (1975), 277-294

[12] EISEN G., A counterexample for some lower semicontinuity results, Math. Z., 162 (1978), 241-243

[13] EISEN G., A selection lemma for sequence of measurable sets, and lower semicontinuity of multiple integrals, Manuscripta Math., 27 (1979), 73-79

[14] EKELAND I., Nonconvex minimization problems, Bull. Amer. Math. Soc., 1 (1979), 443-474

[15] EKELAND I., TEMAM R., Convex analysis and variational problems, North Holland, 1976

[16] FUSCO N., Quasi convessità e semicontinuità per integrali multipli di ordine superiore, Ricerche Mat., 29 (1980), 307-323

[17] GIAQUINTA M., GIUSTI E., On the regularity of the minima of variational integrals, Acta Math., 148 (1982), 31-46

[18] GIAQUINTA M., GIUSTI E., Quasi-minima, 1983, preprint

[19] MARCELLINI P., Some problems of semicontinuity and of Γ-convergence for integrals of the calculus of variations, Proc. Intern. Meet. on Recent Meth. in Nonlinear Anal., De Giorgi, Magenes, Mosco Edit., Pitagora Bologna, 1978, 205-221.

[20] MARCELLINI P., Quasiconvex quadratic forms in two dimensions, Appl. Math. Optimization, 11 (1984), 183-189

[21] MARCELLINI P., SBORDONE C., Semicontinuity problems in the calculus of variations, Nonlinear Anal., 4 (1980), 241-257

[22] MARCELLINI P., SBORDONE C., On the existence of minima of multiple integrals of the calculus of variations, J. Math. Pures Appl., 62 (1983), 1 - 9

[23] MEYERS N., An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa, 17 (1963), 189-206

[24] MEYERS N., Quasiconvexity and lower semicontinuity of multiple integrals of any order, Trans. Amer. Math. Soc., 119 (1965), 125-149

[25] MORREY C.B., Quasiconvexity and the lower semicontinuity of multiple integrals, Pacific J. Math., 2 (1952), 25-53

[26] MORREY C.B., Multiple integrals in the calculus of variations, 1966, Springer-Verlag, Berlin

[27] MURAT F., Compacité par compensation II, Proc. Inter. Meet. on Recent Meth. in Nonlinear Anal., De Giorgi, Magenes, Mosco Edit., Pitagora Bologna, 1978, 245-256

[28] RESHETNYAK Y.G., General theorems on semicontinuity and on convergence with a functional, Sibirskii Math. J., 8 (1967), 1051-1069

[29] RESHETNYAK Y.G., Stability theorems for mappings with bounded excursion, Sibirskii Math. J., 9 (1968), 667-684

[30] SBORDONE C., Su alcune applicazioni di un tipo di convergenza variazionale, Ann. Scuola Norm. Sup. Pisa, 2 (1975), 617-638

[31] SERRIN J., On the definition and properties of certain variational integrals, Trans. Amer. Math. Soc., 101 (1961), 139-167

[32] BALL J.M., MURAT F., $W^{1,p}$-quasiconvexity and variational problems for multiple integrals, to appear

Paolo Marcellini
Dipartimento di Matematica
II Università di Roma
Via Orazio Raimondo
00173 ROMA, Italy

(Received February 28, 1984)