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HYPERBOLICITY OF THE COMPLEMENT OF PLANE CURVES

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Dedicated to Karl Stein

In this paper it is proved that the complement in $\mathbb{P}_2$ of a holomorphic curve D of genus $g \geq 2$ is a hermitian hyperbolic complex manifold provided that certain conditions on the singularities of the dual D^* of D are satisfied and that every tangent at D^* intersects D^* in at least two distinct points.

Introduction

1. We want to give a proof of the following theorem in a somewhat more general form:

<u>Theorem</u>: <u>Assume that</u> $D \subset \mathbb{P}_2$ <u>is a curve of genus</u> $g \geq 2$ <u>whose dual curve</u> $D^* \subset \mathbb{P}_2^*$ <u>has only ordinary double points and</u> i <u>cusps such that</u> $i + \chi(D) < 0$. <u>Then any holomorphic map</u> $f : \mathbb{C} \to \mathbb{P}_2 \setminus D$ <u>is constant</u>.

In [CG] a differential geometric proof is given. This proof has a gap. The error is produced by multiplying metrics which leads to zeros. Instead, we have to sum metrics.

2. We always denote by $\mathbb{P}_2$ the 2-dimensional complex projective plane and by $\mathbb{P}_2^*$ its dual. D is any irreducible curve in $\mathbb{P}_2$ which may contain singularities.

For every $z_0 \in D$ the analytic set D decomposes in z_0 into irreducible components A_i, $i = 1,\dots,l$ which are analytic sets in a neighborhood of z_0. We denote by m_i the minimal intersection number of A_i with a complex line through z_0. We say that A_i has a singularity of order $m_i - 1$ in z_0 and that D has l irreducible singularities of order $m_1 - 1,\dots,m_l - 1$ in z_0. A singularity of order 0 is a smooth point. The dual D^* of D has the tangents at D for points. The normalizations of D and D^* are isomorphic. Their joint genus is denoted by g. Furthermore we denote by d the degree of D and by d^* the degree of D^*, by n, n^* the number of knots (with multiplicity) of D, D^* respectively and by c, c^* the number of cusps (with multiplicity) of D, D^* respectively. In the classic case where the curves D, D^* have only ordinary singularities we have the Plücker formulas:

$$d^* = d(d-1) - 2n - 3c, \qquad d = d^*(d^*-1) - 2n^* - 3c^*$$

$$c^* = 3d(d-2) - 6n - 8c, \qquad c = 3d^*(d^*-2) - 6n^* - 8c^*.$$

The second part of these formulas is superfluous since $D^{**} = D$. For the genus we have the equality:

$$g = \frac{(d-1)(d-2)}{2} - n - c = \frac{(d^*-1)(d^*-2)}{2} - n^* - c^* .$$

By $\chi(D)$ we denote the Euler number $2 - 2g$ of the normalization of D. If D is singularityfree we have $c^* + \chi(D) = d(2d-3)$ which is never negative. However, the conjecture is that $\mathbb{P}_2 \setminus D$ is hyperbolic in many of that cases. So the condition of the theorem is very strong.

In the general case of curves we have not only to take the cusps of D^* (with higher multiplicities) into account but all irreducible singularities of D^*. Let b^* be the number of all irreducible singularities of D^* counted with their orders $m - 1$.

3. We assume that $G \subset \mathbb{C}$ is a domain and that h is a real nowhere negative function in G. We consider the hermitian metric $ds^2 = h\, dz d\bar{z}$ in G. In the points $z \in G$ where h is C^∞ and $h(z) \neq 0$ the expression $K(h,z) = -\frac{1}{2h} \Delta \log h$ is the Gaussian curvature. We say that the Gaussian curvature of a general hermitian metric ds^2 is smaller or equal $-K$ and smaller 0, if $2\, hK \leq \Delta \log h$ in the sense of distribution.

Now let X be a complex manifold and $ds^2 = \sum_{i,j} h_{ij}\, dz_i d\bar{z}_j$ a semipositive definite continuous hermitian metric on X. We say that ds^2 is hyperbolic, if there is a constant $K > 0$ such that for every holomorphic map $\psi : G \to X$ the Gaussian curvature of $ds^2 \circ \psi = h\, dz d\bar{z}$ is smaller or equal $-K$ on G.

<u>Definition</u>: <u>$X \subset \mathbb{P}_n$ is a hermitian hyperbolic complex manifold, if there exists a positive definite continuous hyperbolic hermitian metric on X which is bigger than a positive multiple of the Fubini-Study metric</u>.

We shall prove in our paper

<u>Theorem</u>: <u>Assume that for $D \subset \mathbb{P}_2$ of genus $g \geq 2$ the inequality $b^* + \chi(D) < 0$ is valid. Moreover assume that every tangent at D^* intersects D^* in at least two distinct points. Then $X = \mathbb{P}_2 \setminus D$ is a hermitian hyperbolic complex manifold</u>.

The additional condition that every tangent at D^* intersects in two distinct points, at least is true for all classical curves of genus $g \geq 2$, since the greatest intersection number of a line and D^* in one point is 3 and $d^* \geq 4$.

Our proof will be differential geometry.
(For notations in the introduction see [GH].)

§ 1. Preparations

1. We prove the Ahlfors Lemma for continuous hermitian metrics.

Lemma 1: Assume that $ds^2 = h\, dz d\bar{z}$ is a continuous hermitian metric in the unit disk $E = \{z : |z| < 1\} \subset \mathbb{C}$ with curvature $K(h,z) \leq -K < 0$. Then we have the inequality

$$h(z) \leq \frac{4}{K} \frac{1}{(1 - z\bar{z})^2} .$$

Proof: We take an arbitrary number R with $0 < R < 1$

and put $E_R = \{z : |z| < R\} \subset E$ and $H(z) = \frac{4}{K} \frac{R^2}{(R^2 - z\bar{z})^2}$.

We only have to prove that $h(z) \leq H(z)$ in E_R for all R .

Assume that this is not the case. We define $M = \{z \in E_R : h(z) > H(z)\}$. Then we have $M \neq \emptyset$. Since $\lim\limits_{|z| \to R} H(z) = \infty$ the set M is relatively compact contained in E_R . The functions h, H are positive in $\bar{M}$. So $\log h, \log H$ are continuous in $\bar{M}$. We have $F := \log h - \log H = 0$ on ∂M and $F > 0$ in the interior of M . There is a point $z_0 \in M$ where F takes its maximum. Moreover we have $2Kh \leq \Delta \log h$ and $2KH = \Delta \log H$ and therefore $\Delta F \geq 2K(h - H) \geq 0$ in M (in the sense of distribution). So F is subharmonic in M, hence constant in the connected component $M(z_0)$ of M containing the point z_0. This is a contradiction since $F|\partial M = 0$.

2. We denote by S a connected compact Riemann surface of genus g and by $\omega = a(z)\, dz d\bar{z}$ a continuous differential form on S of type $(1,1)$. We say that ω has a zero of order $\alpha > 0$ in a point $x_0 \in S$, if there is a holomorphic coordinate system z in a neighborhood U of x_0 with $z(x_0) = 0$ and a continuous function $b(z) \neq 0$ in

U such that $a(z) = |z|^{2\alpha} b(z)$ in U. We say that ω is positive semidefinite, if $a(z)$ is always real and $a(z) \geq 0$. Of course such a $(1,1)$-form ω represents a (positive semi-definite) hermitian metric. We prove

Lemma 2: *Assume that $x_1,\dots,x_l \in S$ are distinct points with rational multiplicities $\alpha_1,\dots,\alpha_l > 0$, such that $\sum_{i=1}^{l} \alpha_i < 2g-2$. Then there is a positive semidefinite differential form ω on S which is different from 0 on $S \setminus \{x_1,\dots,x_l\}$ and which has in x_i a zero of order α_i, such that the Gaussian curvature $K(\omega)$ is smaller or equal -1 everywhere.*

Proof: We take a positive integer n such that $n\alpha_i$ is integral for all $i = 1,\dots,l$ and denote by L the line bundle of the divisor $-n\alpha_1 x_1 - \dots - n\alpha_l x_l$. If K is the canonical bundle of S the Chern class of $K^n \otimes L^n$ is positive. Hence this bundle is positive. There is a positive integer m such that $\bigotimes_m (K^n \otimes L^n) = K^{nm} \otimes L^{nm}$ is very ample. For each point $x \in S$ there are global cross-sections s_1, s_2 with $s_1(x) \neq 0$ and s_2 vanishing in x of order one. Since L^{nm} belongs to a holomorphic divisor, s_1, s_2 may be considered as holomorphic cross sections in K^{nm} where s_1 vanishes in x of order $nm\alpha_i$ and s_2 of order $nm\alpha_i + 1$ precisely, if $x = x_i$. If x is different from all x_i, the order of vanishing is the old one.

We take a basis $s_1,\dots,s_k \in \Gamma(S, K^{nm} \otimes L^{nm})$ and put $\omega = \left(\sum_{i=1}^{k} s_i \bar{s}_i\right)^{\frac{1}{nm}}$. We obtain a positive definite $(1,1)$-form on S which vanishes in x_i of order α_i and nowhere else. We still have to prove $K(\omega) \leq -1$, that means

$2a \leq \Delta \log a$ for each local coefficient a of ω.

We put $s_i = f_i(z)\, dz^{nm}$, locally. Then we have

$$a = \Big(\sum_{i=1}^{k} |f_i(z)|^2\Big)^{\frac{1}{nm}} \quad \text{and} \quad \Delta \log a = \frac{1}{nm} \Delta \log \sum_{i=1}^{k} |f_i(z)|^2 .$$

We wish to prove that $\Delta \log a > 0$. First we note that we may divide $\sum_{i=1}^{k} |f_i(z)|^2$ by a holomorphic function, since $\Delta \log a$ is not changed by this. Thus we may assume that $(f_1(z),\dots,f_k(z)) \neq 0$. Moreover $(f_1(z),\dots,f_k(z))$ and $(f_1'(z),\dots,f_k'(z))$ are linearly independent. So by [GR] (p. 113,III) we obtain $\Delta \log a > 0$. This is true in every point of S. After having multiplied ω by a small positive factor we also have the inequality $2a < \Delta \log a$. That completes the proof.

§ 2. Proof of the theorem

1. We denote by S the normalization of D^* and by $\{\hat{x}_1,\dots,\hat{x}_l\} \subset S$ the inverse image of the irreducible singularities x_i, $i = 1,\dots,l$ of D^* under the normalization. If $m_i - 1$ are the orders of the irreducible singularities x_i respectively, the condition of the theorem is that $b^* = \sum_{i=1}^{l} (m_i - 1) < 2g - 2$. According to Lemma 2 there is a $(1,1)$-form ω on S which has a zero in $\hat{x}_i$ of order $\alpha_i := m_i - 1$. We prove that ω leads to a positive definite, hyperbolic hermitian metric on $X = \mathbb{P}_2 \setminus D$ with Gaussian curvature smaller or equal -1.

Every point $x \in X$ is a line $l(x) \subset \mathbb{P}_2^*$. If $y_i \in l(x) \cap D^*$, the intersection number of $l(x)$ and the local irreducible component A_i of D^* containing y_i is m_i. If $y \in l(x) \cap D^*$ is a smooth point of D^*, the intersection in y is transversal. Now if we count the intersections of a line $l(x)$ and D^* with multiplicity, we have always $l(x) \cap D^* = \{y_1,\ldots,y_{d^*}\}$. The set $\{y_1,\ldots,y_{d^*}\}$ is the image of a unique set $\{\hat{y}_1,\ldots,\hat{y}_{d^*}\}$ of the normalization S. Hence we have a continuous map $F : X \to S^{d^*}$, where S^{d^*} denotes the d^*-th symmetric power of S which is a d^*-dimensional complex manifold. Outside the thin set R consisting of the points $x \in X$ for which $l(x)$ contains an irreducible singularity of D^* is F C^∞-regular. There are d^* projections $\pi_i : S \times \ldots \times S \to S$ of the d^*-fold Cartesian product on S. We lift ω to $\omega_1,\ldots,\omega_{d^*}$ by this maps and define the sum $\tilde{\omega} = \frac{1}{d^*}\sum_{i=1}^{d^*} \omega_i$. This hermitian metric is invariant against permutation, hence it comes from a hermitian form $\hat{\omega}$ on S^{d^*}. The pullback $\Omega = \hat{\omega} \circ F$ on X is outside the set R a C^∞-regular hermitian metric. Since ω vanishes in the points $x_i \in S$ of order $m_i - 1$, our metric Ω is completable to a continuous hermitian metric on X.

2. We shall prove now that Ω is a continuous, positive definite, hyperbolic hermitian metric on X with Gaussian curvature smaller or equal -1, which is bigger than a positive multiple of the Fubini-Study metric on $\mathbb{P}_2$. We take a point $x_0 \in X$ and a point $y_0 \in l(x_0) \cap D^*$ and decompose D^* in y_0 into irreducible components:

$$D^* \cap U(y_0) = A_1 \cup \ldots \cup A_l .$$

We assume that A_1 has a singularity of order $m-1$ in y_0. The intersection number of $l(x_0)$ and A_1 is m. There is a line $P(y_0)$ through x_0, such that all lines

$l(x)$, $x \in P$ pass through y_0 . We denote by T a singularity-free local curve through x_0 which intersects P transversally. Then the points $t \in T$, $z \in l(t)$ are holomorphic coordinates in a neighborhood $V(y_0) \subset U(y_0)$. The intersection of the curve $A_1 \cap V$ with $l(t)$ can be given by an equation

$$z = f(\sqrt[m]{t}) ,$$

where $f(w) = a_1 w + a_2 w^2 + \ldots$ is holomorphic. The metric ω is of the form

$$\omega = (w\bar{w})^{m-1} (b_{00} + b_{10}w + \bar{b}_{10}\bar{w} + b_{20}w^2 + \bar{b}_{20}\bar{w}^2 + \\ + b_{11}w\bar{w} + \ldots) dwd\bar{w}$$

with $b_{00} > 0$, $\frac{b_{11}}{b_{00}} - \frac{1}{2}\frac{b_{10}\bar{b}_{10}}{b_{00}} > 0$ and therefore $b_{11} > 0$.

According to paragraph 1 we have to consider the pullback on T . We have

$$dw = \frac{1}{m} t^{\frac{1}{m} - 1} dt \quad \text{and} \quad dwd\bar{w} = \frac{1}{m^2} |t|^{\frac{2-2m}{m}} dtd\bar{t} .$$

For one branch of $t^{\frac{1}{m}}$ it is

$$\omega' = \frac{1}{m^2} (b_{00} + b_{10}t^{\frac{1}{m}} + \bar{b}_{10}\bar{t}^{\frac{1}{m}} + b_{20}t^{\frac{2}{m}} + \bar{b}_{20}\bar{t}^{\frac{2}{m}} + b_{12}|t|^{\frac{2}{m}} \\ + \ldots) dtd\bar{t} .$$

We have to take the medium over the different values of $t^{\frac{1}{m}}$ and obtain a $(1,1)$-form

$$\bar{\omega}' = \frac{1}{m^2} (b_{00} + b_{11} |t|^{\frac{2}{m}} + b_{22}|t|^{\frac{4}{m}} + b_{33}|t|^{\frac{6}{m}} + \ldots \\ + b_{m0}t + b_{m+1,1}t|t|^{\frac{2}{m}} + b_{m+2,2}t|t|^{\frac{4}{m}} + \ldots$$

$$+ \bar{b}_{m0}\bar{t} + \bar{b}_{m+1,1}\bar{t}|t|^{\frac{2}{m}} + \bar{b}_{m+2,2}\bar{t}|t|^{\frac{4}{m}} + \dots$$

$$+ b_{2m,0}t^2 + b_{2m+1,1}t^2|t|^{\frac{2}{m}} + \dots$$

$$+ \bar{b}_{2m,0}\bar{t}^2 + \bar{b}_{2m+1,1}\bar{t}^2|t|^{\frac{2}{m}} + \dots \qquad)dtd\bar{t}$$

$:= d\, dtd\bar{t}$, which is continuous. It is $b_{11} > 0$. We get

$$\log d = d_0 + d_1|t|^{\frac{2}{m}} + d_2|t|^{\frac{4}{m}} + \dots + d_{m-1}|t|^{\frac{2}{m}(m-1)} + \dots$$

$$+ d_{10}t + d_{11}t|t|^{\frac{2}{m}} + d_{12}t|t|^{\frac{4}{m}} + \dots + d_{1,m-1}t|t|^{\frac{2}{m}(m-1)}$$

$$+ \dots$$

$$+ \bar{d}_{10}\bar{t} + \bar{d}_{11}\bar{t}|t|^{\frac{2}{m}} + \bar{d}_{12}\bar{t}|t|^{\frac{4}{m}} + \dots + \bar{d}_{1,m-1}\bar{t}|t|^{\frac{2}{m}(m-1)}$$

$$+ \dots$$

$$+ e_{11}t^2 + e_{12}t\bar{t} + \bar{e}_{11}\bar{t}^2 + \dots$$

with $d_1 > 0$, if $m \geqq 2$, and $e_{12} > 0$, if $m = 1$. We consider the distribution $\Delta \log d$. It is

$\Delta(t\bar{t})^{\frac{\nu}{m}} = 4 \frac{\partial}{\partial t} \frac{\partial}{\partial \bar{t}} (t\bar{t})^{\frac{\nu}{m}} = 4 \frac{\nu^2}{m^2} (t\bar{t})^{\frac{\nu}{m} - 1}$, $\Delta(t(t\bar{t})^{\frac{\nu}{m}}) =$

$4 (\frac{\nu}{m} + 1) \frac{\nu}{m} t^{\frac{\nu}{m}} \bar{t}^{\frac{\nu}{m} - 1}$ and $\Delta(\bar{t}\ (t\bar{t})^{\frac{\nu}{m}})$ similar, in the sense of distribution. The leading term is $\Delta(d_1(t\bar{t})^{\frac{2}{m}}) =$

$\frac{4}{m^2} \frac{1}{|t|^{2(1-\frac{1}{m})}}$ which converges to ∞ for $t \to 0$. This means that also in $t = 0$ the Gaussian curvature of $\overline{\omega}'$ is smaller equal -1.

3. If the local curve T does not intersect P transversally, we can approximate T by a sequence of curves T_ν which intersect P transversally. On T_ν, T we have a $\overline{\omega}' = d_\nu \, dtd\overline{t}$, $d \, dtd\overline{t}$ respectively with continuous non-negative functions d_ν, d. The sequence (d_ν) converges uniformly to d. Since always $2d_\nu \leqq \Delta \log d_\nu$ we get $2d \leqq \Delta \log d$ in the sense of distributions. The same procedure is possible, if T is not smooth. So the Gaussian curvature of $\overline{\omega}'$ is smaller or equal -1.

Now our hermitian metric Ω is obtained as a mean value of some $\overline{\Omega}'$, which are C^2-regular outside the set R. Here the formula for summing of metrics [GR] is valid. Thus the Gaussian curvature of Ω is smaller or equal -1.

4. Every line $l(x), x \in \mathbb{P}_2 \setminus D$ intersects D^* twice at least. Otherwise we would have $l(x) \cap D^* = \{y\}$ and the intersection number in y would be equal d^*. Since $l(x)$ is not a tangent at D^* there would be a line through y of higher intersection number. This is a contradiction.

If $x \in \mathbb{P}_2 \setminus D$ runs on a smooth local curve T, there is at most one line P, for which every corresponding line $l(p), p \in P$ passes through a fixed singularity x_0 of D^*, that is a tangent at T in $t = 0$. So at least one of the intersection points of $l(t)$ and D^* changes with nonzero differential. Thus our metric Ω is positive definite even on R because of the order of vanishing of ds^2 in the points $\hat{x}_1, \ldots, \hat{x}_l$.

Now suppose that $x = x(t)$ runs on a smooth local curve T which intersects D in just one point $x_0 = x(0)$. The tangent at D in x_0 is a point $y_0 \in D^*$. We denote the minimal intersection number of a line with D^* in y_0 by $p \geqq 1$ and the intersection number of D^* with its tangent in y_0 by $q > p$. The lines $l(t) \subset \mathbb{P}_2{}^*$ corresponding to T tend to $l(x_0)$ for $t \to 0$. There are q intersection points $x_1(t),\ldots,x_q(t)$ of $l(t)$ and D^* which converge to y_0 for $t \to 0$ with orders $s_1,\ldots,s_q > 0$. If the intersection number n of T with the tangent y_0 is smaller than $\frac{q}{p}$, at least one of the orders s_i is smaller than 1. This implies that Ω grows to ∞ on T by approaching $t = 0$. In particular this is the case for every curve T that intersects D transversally. If $n \geqq \frac{q}{p}$, it is possible that the metric Ω tends to 0 on all branches $x_1,\ldots,x_q$ for $t \to 0$. But we have the condition that every tangent of D^* intersects in at least two distinct points. In the second point y_1 either (if $l(x_0)$ is also a tangent of D^* in y_1) case 1 is valid or the intersection points of $l(t)$ and D^* running to y_1 change with a nonzero differential such that Ω remains positive definite bounded away from zero by approaching $t = 0$. The boundary depends only on x_0 and not on the curve T. So we get: There is a constant $\delta > 0$, which depends on D only, such that $\Omega \geqq \delta\Omega_F$, where Ω_F denotes the Fubini-Study metric in $\mathbb{P}_2$.

5. The existence of Ω implies that the Kobayashi pseudometric is a metric, since the Kobayashi distance is maximal in the set of distance decreasing pseudo-distances on X. For every point $x_0 \in D$ there is a neighborhood U and a holomorphic map $f : U \to E := \{z \in \mathbb{C} : |z| < 1\}$ such

that $f|D = 0$. The pullback of the metric

$$ds^2 = \frac{1}{|z|^2(\log|z|^2)^2}$$ by f is a pseudometric on U of

Gaussian curvature -1 which converges to ∞ on every sequence $(x_\nu) \subset X$ that converges to the boundary of X.

The existence of Ω implies furthermore that the family $\mathcal{F}$ of the holomorphic maps of the disk E into X is equicontinuous with respect to the metric Ω . Since $\Omega \geqq \delta\Omega_F$, this family of maps considered as maps of E into $\mathbb{P}_2$ is also equicontinuous with respect to the Fubini-Study metric. Thus $\mathcal{F}$ is even ([R], p. 133). Therefore there exists an $\varepsilon > 0$ and a neighborhood $U' \subset U$ of x_0 such that for every $f \in \mathcal{F}$ we have $f(E_\varepsilon) \subset U$, where E_ε denotes the disk of radius ε , whenever $f(0) \in U'$. With the infinitesimal form of the Kobayashi metric $F_X(x,\xi) = \inf \frac{1}{R}$, where the infimum is taken of the set of all positive real numbers for which there is a holomorphic map $\phi : E_R \to X$ with $\phi(0) = x$ and $\phi'(0) = \xi$, we have on U' the inequality

$$F_X \geqq \varepsilon\, F_U \geqq \varepsilon\, ds^2 .$$

This implies that X is complete with respect to the Kobayashi metric.

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