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Stein Quotients of Connected Complex Lie Groups

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Dedicated to Prof. Karl Stein

Various algebraic and geometric conditions on connected complex Lie groups G and H are shown to characterize the quotient G/H as a Stein manifold. Among these conditions are analytic analogues of the algebraic notions of observable or strongly observable subgroups and cohomological conditions expressed in terms of equivariant maps. A specific group theoretic condition on H, generalizing Matsushima's criterion for reductive groups, is shown to be necessary for G/H to be Stein and the sufficiency of this condition is proven when G is solvable or when H satisfies a dimension restriction. Also included is a geometric description of a Stein quotient G/H as a bundle space over an orbit of a maximal reductive subgroup of G, and a theorem on the orbits of solvable groups in $\mathbb{P}^n$.

In this paper we study the problem of characterizing Stein quotients G/H of connected complex Lie groups in terms of various algebraic and geometric properties of the groups G and H. The most important of these properties involves how a simply-connected normal solvable subgroup V of H intersects reductive subgroups of G, cf. (1.4) below. This property is the natural generalization of Matsushima's criterion, cf. [15], that if G is reductive then G/H is Stein if and only if H is reductive. We prove that this group theoretic condition, which is necessary for G/H to be Stein, is also sufficient when G is solvable (generalizing Matsushima's result for nilpotent groups, [16]), or when dim V = 1. We conjecture that it is sufficient in general.

In §1 we show how the conjecture can be reduced to *linear algebraic* *groups*, where it is still an unsolved problem, cf. [4], and how it would follow from the conjecture that every *free* (algebraic or holomorphic) $\mathbb{C}$ action on $\mathbb{C}^{n+1}$ has $\mathbb{C}^n$ as a geometric quotient. We also give some general criteria for G/H to be Stein, cf. Theorem 1.11. For example, G/H is Stein if and only if $H^1(G/V,\mathcal{O}) = 0$. Another computationally useful criterion is the following: Let $V = V_k \triangleright V_{k-1} \triangleright \dots \triangleright V_0 = 1$ be a composition series for V. Then G/H is Stein if and only if there exist holomorphic functions $f_1,\dots,f_k \in \mathcal{O}(G)$ such that $f_i(gv) = f_i(g) + \nu_i(v)$ for all $g \in G$, $v \in V_i$, where ν_i is the homomorphism $V_i \longrightarrow V_i/V_{i-1} \cong \mathbb{C}$, $1 \leq i \leq k$.

The algebraic notions of an "observable" or "strongly observable" subgroup of an algebraic group have geometric analogues for subgroups of complex Lie groups in terms of induced G-spaces. In §2 we draw conclusions similar to those in [4] for affine quotients of linear algebraic groups based on these "obsevability" criteria. The holomorphic version of observability, however, implies the group-theoretic condition (1.4) mentioned above and is therefore stronger than its algebraic counterpart. We also give a geometric description of a Stein quotient G/H as a homogeneous bundle over an M-orbit M/L of minimal dimension in G/H, where M (resp. L) is a maximal reductive subgroup of G (resp. H). The bundle is "almost" an algebraic vector bundle over the affine variety M/L in the sense that the fiber S is an L-invariant subspace of a rational L-module E such that $F \oplus S \cong E$ for some rational L-module F. Here, and elsewhere in this paper, we use the fact that a holomorphic representation of a reductive complex Lie group is already a rational representation with respect to its natural structure as a linear algebraic group, cf. [9].

We define the notion of a *quasi-algebraic* subgroup of a complex Lie group in §3. A subgroup H of G is quasi-algebraic if and only if G/H can be equvariantly mapped into some projective space $\mathbb{P}^n$. We show that if H is a (not necessarily connected) quasi-algebraic subgroup of a connected solvable complex Lie group G, then G/H is biholomorphic to $(\mathbb{C}^*)^s \times \mathbb{C}^t$. In particular, G/H is Stein and $\pi_1(G/H) \cong \mathbb{Z}^s$. Note that π_1 can then be used as an obstruction to finding equivariant maps into $\mathbb{P}^n$. These techniques are used to prove the above mentioned result that if G is solvable and H is connected then G/H is Stein if and only if V satisfies (1.4). This theorem is also a consequence of the reduction to linear algebraic groups given in §1.

All of our charcterizations of Stein quotients G/H for H connected can be easily modified for subgroups H with finitely many components, since in that case $G/H^0 \longrightarrow G/H$ is a finite holmorphic covering map and so G/H is Stein if and only if G/H^0 is Stein.

1. Group-theoretic characterizations

In this section, we shall investigate conditions on the connected complex Lie groups $G \supset H$ for the quotient $X = G/H$ to be Stein. First of all, to prevent irrelevant groups from getting involved, we may assume that G acts effectively on X, i.e. that H contains no non-trivial normal subgroups of G. With this assumption we know that H has a semi-direct product decomposition

$$H = L \ltimes V \tag{1.1}$$

where L is a maximal reductive subgroup of H and V is a simply-connected normal solvable subgroup of H (cf. [16,p.154]; note that the existence of such a decomposition is equivalent to H having a faithful holomorphic representation [9]). Now if X is Stein then it follows easily that G is a Stein Lie group, cf. [17,p.146]. The smallest complex subgroup of G which contains a maximal compact subgroup $K \subset G$ is then a reductive subgroup of G isomorphic to the (abstract) complexification of K. We call such a subgroup a maximal reductive subgroup of G; any two maximal reductive subgroups of G are conjugate, cf. [17].

For any maximal reductive subgroup $M \subset G$ there is a decomposition

$$(1.2) \qquad G = M \cdot U$$

where U is a submanifold of G, the biholomorphic image of a subspace $\mathbf{u} = \mathbf{u}_1 + \ldots + \mathbf{u}_k$ in the Lie algebra of G under the map (cf. [16,p.161]):

$$(1.3) \qquad \varphi: \mathbf{u} \longrightarrow U, \ \phi(x) = \exp(x_1)\ldots\exp(x_k), \ \text{where } x = x_1 + \ldots + x_k, \ x_i \in \mathbf{u}_i.$$

Moreover, the subspaces $\mathbf{u}_i$ are ad(M) invariant so that U is invariant under conjugation by M. In general, U is not a subgroup of G. However, as mentioned above, if G is linear (i.e. has a faithful representation) then U is an intrinsically defined normal solvable subgroup of G, and for *any* maximal reductive subgroup $M \subset G$, (1.2) is a semi-direct product, $G = M \ltimes U$. If G is a linear algebraic group over $\mathbb{C}$, then U is just the unipotent radical of G. We shall use the above notation and the decompositions (1.1) and (1.2) frequently throughout this paper without further comment.

Keeping the above remarks in mind, we can formulate the following necessary condition: *If* $X = G/H$ *is Stein then*

(1.4) $V \cap M = 1$ *for every maximal reductive subgroup* M *of* G.

To see why this is true, we first note that G/H is Stein if and only if G/V is Stein because $G/V \longrightarrow G/H$ is a principal L-bundle, cf. [17,p.147]. If M is any maximal reductive subgroup of G, then the M-orbits of minimal dimension in G/V are closed and hence Stein, cf. [18,p.82]. The isotropy subgroup in M of such an orbit must have finitely many connected components (cf.[18,§2]) and the connected component of the identity must be reductive, cf.[15]. Since $V^g = gVg^{-1}$, $g \in G$, contains no non-trivial compact subgroups, this isotropy subgroup is trivial. Therefore, the M-orbit of minimal dimension also has the maximum dimension, dim M. Thus all M-orbits have the same dimesnion and all are closed in G/V. In particular, $V \cap M = 1$.

We note in passing that any connected complex subgroup of G satisfying (1.4) must be *closed* (by Goto's criterion, cf. e.g. [15,p.207]), *simply connected*, and *solvable*.

Of course, if G is reductive, then condition (1.4) is another way of saying that H must be reductive; in this case, (1.4) is necessary and sufficient for X to be Stein, cf. [15]. Matsushima [16] showed that (1.4) is also necessary and sufficient when G is a *nilpotent* complex Lie group. In §3 we shall extend this latter result to *solvable* complex Lie groups. For now we shall concentrate on (1.4) and related conditions for general complex Lie groups and show how the sufficiency of (1.4) for Stein quotients would follow from the sufficiency of (1.4), in the category of linear algebraic groups, for X to be an *affine algebraic variety*. Even in this category, it remains an unsolved problem, cf. [4].

(1.5) Lemma (Reduction to Linear Groups): *Let* G *be a connected Stein Lie group and let* V *be a connected complex subgroup of* G *satisfying* (1.4). *Let* R *be the radical of* G *and let* M_0 *be a maximal reductive subgroup of the commutator group* R'. *Then* M_0 *is central and* $G_0 := G/M_0$ *is a linear complex Lie group. Moreover, the natural map* $G/V \longrightarrow G/VM_0$ *realizes* G/V *as a principal* M_0*-bundle over* $G/VM_0 = G_0/V_0$, *and* $V_0 = VM_0/M_0 \cong V$ *satisfies* (1.4) *in* G_0. *In particular,* G/V *is Stein if and only if* G_0/V_0 *is Stein.*

Proof: The first part follows from the fact that a complex Lie group is linear if and only if R' contains no compact subgroups, cf.[8,p.220]. In particular, $M_0 \subset \ker \mathrm{ad}_G = Z(G)$, the center of G. To see why V_0 satisfies (1.4), which shows incidentally that M_0V is closed in G, note that a maximal reductive subgroup in G_0 is $M_1 := M/M_0$ where M is a maximal reductive subgroup of G containing M_0. If $V_0 \cap M_1{}^g \neq 1$ for some $g \in G_0$, then $VM_0 \cap M^g \neq M_0$ for some $g \in G$. But this implies $V \cap M^g \neq 1$, a contradiction. □

Every linear complex Lie group $G = M \ltimes U$ has a canonical imbedding into a linear algebraic group $\hat{G}$ such that $\hat{G} = T \ltimes G$ where $T \cong (\mathbb{C}^*)^t$ is a torus in $\hat{G}$ which commutes with M, cf. [10]. We call $\hat{G}$ the *abstract linear algebraic closure* of G. The *algebraic closure* of a complex subgroup of an algebraic group is defined to be the smallest algebraic subgroup which contains it.

(1.6) Proposition (Reduction to algebraic groups): *Let* G_0 *be a* *connected linear complex Lie group and let* V_0 *be a connected complex subgroup of* G_0 *satisfying* (1.4). *Let* G_1 *denote the abstract linear algebraic closure of* G_0 *and let* V_1 *be the unipotent radical of the algebraic closure of* V_0 *in* G_1. *Then* dim V_1 = dim V_0 *and* V_1 *satisfies* (1.4) *in* G_1. *Moreover,* G_0/V_0 *is Stein if and only if* G_1/V_1 *is Stein.*

Proof: If $G_0 = M \ltimes U$ then, as remarked above, $G_1 = T \ltimes G_0$ where T is a torus which commutes with M. A maximal reductive subgroup of G_1 is then $M_1 := T \times M$. By the conjugacy of maximal reductive subgroups we have that for all $p \in G_1$ there is a $q \in G_0$ such that $M_1{}^p \cap G_0 = M^q$. The algebraic closure W of V_0 in G_1 is of the form $W = T_1 \cdot V_0$ where $T_1 \cong (\mathbb{C}^*)^s$, cf. [10]. Now for some $p \in G_1$, $q \in G_0$:

$$V_0 \cap T_1 \subset V_0 \cap G_0 \cap M_1{}^p = V_0 \cap M^q = 1.$$

So, W is isomorphic to the semi-direct product $W = T_1 \ltimes V_0$. Note that W is also isomorphic to $T_1 \ltimes V_1$ which shows that dim V_1 = dim V_0, although this does not imply that V_1 is isomorphic to V_0. Thus, both G_1/V_0 and G_1/V_1 are principal T_1-bundles over G_1/W. Since $G_1/V_0 \cong T \times G_0/V_0$ we obtain:

$$G_0/V_0 \text{ is Stein} \Longleftrightarrow G_1/V_0 \text{ is Stein} \Longleftrightarrow G_1/V_1 \text{ is Stein.}$$

The proof that V_1 inherits property (1.4) requires a little more care.

For any $v_1 \in V_1$ there are unique elements $t_1 \in T_1$ and $v_0 \in V_0$ such that $v_0 = t_1 v_1 = v_1 t_1$; this is the Jordan decomposition of v_0, cf. [13,p.99]. Note that v_1 commutes with v_0 also. Conversely, by the above remarks, for every $v_1 \in V_1$ there exists a unique $t_1 \in T_1$ such that $v_0 := t_1 v_1 \in V_0$ and v_1 commutes with t_1 and v_0. Now, if $V_1 \cap M_1{}^g \neq 1$ for some $g \in G_1$, then there is a non-trivial 1-parameter subgroup $v_0 : \mathbb{C} \longrightarrow V_0$ and $t_1 : \mathbb{C} \longrightarrow T_1$ such that

$$v_0(z) = t_1(z) v_1(z), \quad z \in \mathbb{C}.$$

Define S to be the algebraic closure of $t_1(\mathbb{C}) \subset T_1$. Note that S is a subtorus of T_1 and for every $s \in S$ we have $sv_1(z) = v_1(z)s$, $z \in \mathbb{C}$. Therefore, $v_1(\mathbb{C}) \subset M_1{}^g$ is contained in the centralizer C_1 of S in $M_1{}^g$. Let C_2 be the centralizer of S in G_1; then $T_1 \subset C_2$. Since C_1 is reductive, cf. [13,p.159], and is contained in C_2, there is a $c \in C_2$ such that $v_1(\mathbb{C})^c$ is contained in a maximal reductive subgroup M_2 of C_2 with $M_2 \supset T_1$. But then $v_0(\mathbb{C})^c = (t_1(\mathbb{C}) v_1(\mathbb{C}))^c = t_1(\mathbb{C}) v_1(\mathbb{C})^c$ is contained in M_2 which implies that $v_0(\mathbb{C}) \subset M_1{}^p$ for some $p \in G_0$. This is a contradiction since $v_0(\mathbb{C}) \subset (M_1{}^p(G_0) \cap V_0 = M^q \cap V_0 = 1$ (for some $q \in G_0$) and $v_1(\mathbb{C})$ was assumed to be non-trivial. $\square$

We summarize the preceding lemma and proposition in the following:

(1.7) Theorem: *Let* G *be a connected Stein Lie group and let* V *be a connected complex subgroup of* G *satisfying* (1.4). *Then there are connected linear algebraic groups* G_1 *and* V_1 *canonically* *associated to* G *and* V *respectively such that* V_1 *satisfies* (1.4) *in* G_1. *Moreover,* G/V *is Stein if and only if* G_1/V_1 *is Stein*

Note that G_1/V_1 is Stein if and only if it is affine. (G_1/V_1 is already quasi-affine since V_1 is unipotent, cf. [2].) Thus, the sufficiency of (1.4) for Stein quotients would follow from the sufficiency of (1.4) for affine quotients of linear algebraic groups.

If $\dim V = 1$, property (1.4) is the precise group-theoretic condition for G/H to be Stein. This follows directly from Theorem (1.7) and the corresponding theorem for linear algebraic groups, cf. [4,p.12].

(1.8) Theorem: *Let* G *be a connected Stein Lie group and let* H *be a closed connected complex subgroup of* G *with a semi-direct product decomposition* $H = L \ltimes V$ *as in* (1.1). *Assume* $\dim V = 1$. *Then* G/H *is Stein if and only if V satisfies* (1.4).

A geometric interpretation of condition (1.4) on a subgroup V is that the maximal reductive subgroup M acts *freely* by left multiplication on G/V, or that V acts *freely* by right multiplication on $U \cong M\backslash G$. If G is linear, the induced right action of V on U is

$$u.v := m_v^{-1} u m_v u_v, \quad \text{where } u \in U,\ v = m_v u_v \in V \subset M \ltimes U.$$

Note that the geometric quotient of U by V exists (i.e. the topological quotient space is a complex space and the quotient map is holomorphic) if and only if the geometric quotient of G/V by M exists and in this case the two quotients are naturally isomorphic. A geometric quotient exists if and only if the quotient space is Hausdorff, cf. [11].

(1.9) Lemma: *Let* G *be a connected Stein Lie group and let* V *be a connected complex subgroup satisfying* (1.4). *If the geometric quotient* Y *of* G/V *by* M *exists, then* G/V *is a locally trivial principal* M*-bundle over* Y.

Proof: Let $\pi: G/V \longrightarrow Y$ be the quotient map. For each point $p \in G/V$ there is a locally closed analytic set S containing p such that $\pi(S)$ is open in Y and $\pi|S \longrightarrow \pi(S)$ is finite, i.e. a proper holomorphic map with finite fibers, cf. [11]. Since S is local, we may assume that S is Stein. Then the image of the holomorphic map $M \times S \longrightarrow G/V$, $(m,s) \longrightarrow msV$, is an open M-invariant Stein subset W of G/V containing p. But since the action of M on W is *free*, it follows from [18,p.92] that $\pi|W \longrightarrow \pi(W) = \pi(S)$ is a locally trivial principal M-bundle. □

(1.10) Remark: The generalization of Theorem 1.8 that one would hope for is that (1.4) implies G/V is M-equivariantly isomorphic to $M \times \mathbb{C}^n$, without a restriction on dim V. As we shall see in §2, this would also imply that G/H is a homogeneous *vector bundle* over an M-orbit $M/L \subset G/H$ of minimal dimension. We know of no examples where this is not the case. This generalization would in fact be true if it were known that every free holomorphic action of the additive group $\mathbb{C}$ on $\mathbb{C}^{n+1}$ had $\mathbb{C}^n$ as a geometric quotient. (This is an open problem even for algebraic actions.) To see this, we may assume by Lemma 1.5 that G is a linear complex Lie group. Let $V = V_k \triangleright V_{k-1} \triangleright \ldots \triangleright V_0 = 1$ be a *composition series* for V, i.e. a series of subgroups, each normal in the next, such that $V_{i+1}/V_i \cong \mathbb{C}$. Assuming the above statement were true, $V_2/V_1 \cong \mathbb{C}$ would act freely and holomorphically on the quotient $U_1 \cong \mathbb{C}^{n_1}$ of U by V_1 with quotient $U_2 \cong \mathbb{C}^{n_2}$. Continuing in this way with V_{i+1}/V_i acting on the quotient $U_i \cong \mathbb{C}^{n_i}$ of U_{i-1} by V_i/V_{i-1}, we would reach a quotient $Y \cong \mathbb{C}^n$ of U by V. By the above remarks and Lemma 1.9, $G/V \longrightarrow Y$ is a locally trivial principal M-bundle, and therefore is trivial: $G/V \cong M \times \mathbb{C}^n$. Note that if free algebraic $\mathbb{C}$ actions on $\mathbb{C}^{n+1}$ had $\mathbb{C}^n$ as a geometric quotient, then the above argument would also apply to linear algebraic groups. Hence by Theorem 1.7, condition (1.4) would imply G/H is Stein. □

Given a *free* action of a group G on a space X (in an appropriate category), a natural way to prove there is an equivariant splitting $X \cong G \times S$ is to produce a G-equivariant map (in the category) $\phi: X \longrightarrow G$, $\phi(g.x) = g\phi(x)$, $(x \in X, g \in G)$. The subspace $S = \phi^{-1}(1) \subset X$ intersects each G-orbit exactly once so the natural map $\mu: G \times S \longrightarrow X$, $\mu(g,s) = g.s$, is an isomorphism. Its inverse is defined by $\mu^{-1}(x) = (\phi(x), \phi(x)^{-1}.x)$. This construction will be used in the following theorem, which gives some general criteria (independent of condition (1.4)) for G/H to be Stein. With obvious modifications, these criteria are also valid in the algebraic category for G/H to be affine.

(1.11) Theorem: *Let* G *be a connected Stein Lie group and let* H *be a closed connected complex subgroup of* G *with a semi-direct product decomposition* $H = L \ltimes V$ *as in* (1.1). *The following are equivalent:*

(1) G/H *is Stein.*

(2) G/V *is Stein.*

(3) *There exists a V-equivariant holomorphic map* $\phi: G \longrightarrow V$.

(4) *Let* $V = V_k \triangleright V_{k-1} \triangleright \dots \triangleright V_0 = 1$ *be a composition series for* V. *For*

each homomorphism $\nu_i:V_i \longrightarrow V_i/V_{i-1} \cong \mathbb{C}$ *there exists a holomorphic function* $f_i \in \mathcal{O}(G)$ *such that* $f_i(gv) = f_i(g) + \nu_i(v)$, *for all* $g \in G$, $v \in V_i$, $1 \leq i \leq k$.

(5) $H^1(G/V,\mathcal{O}) = 0$.

(6) $H^1(V,\mathcal{O}(G)) = 0$.

Proof: The implications (1) $\Longleftrightarrow$ (2) $\Rightarrow$ (5) are immediate.

(2) $\Longleftrightarrow$ (3): If G/V is Stein, then the topologically trivial principal V-bundle $G \longrightarrow G/V$ is holomorphically trivial and so there is a V-equivariant isomorphism $G \longrightarrow G/V \times V$. Define $\phi: G \longrightarrow V$ to be the V-equivariant composition of this isomorpism with the projection onto V. Conversely, given a V-equivariant holomorphic map $\phi: G \longrightarrow V$, we obtain a V-equivariant splitting $G \cong G/V \times V$ as described above.

(2) $\longleftrightarrow$ (4): If G/V is Stein, then G/V_i is Stein, $1 \leq i \leq k$, cf. (2.4.2). By (3), there exists a V_i equivariant holomorphic map $G \longrightarrow V_i$ which may be composed with ν_i to give the desired result. Conversely, f_{k-1} defines a V/V_{k-1}-equivariant holomorphic map of G/V_{k-1} to V/V_{k-1}. By the remarks preceding the theorem, we have $G/V_{k-1} \cong G/V \times \mathbb{C}$. Induction on k implies that G/V_{k-1} and hence G/V is Stein.

(5) $\Rightarrow$ (2): Let V_1 be a closed normal complex subgroup of V with $V/V_1 \cong \mathbb{C}$. The holomorphic isomorphism class of the principal $\mathbb{C}$-bundle $G/V_1 \longrightarrow G/V$ is represented by a cocycle in $H^1(G/V,\mathcal{O})$. Therefore, the

bundle is holomorphically trivial and we obtain $G/V_1 \cong G/V \times \mathbb{C}$. It follows that $H^1(G/V_1,\mathcal{O}) = 0$, cf. [14], and induction on dim V implies G/V is Stein.

(5) $\Longleftrightarrow$ (6): The isomorphisms $H^p(G/V,\mathcal{O}) \cong H^p(V,\mathcal{O}(G))$, $p\geq 0$, are well known, cf. [7,p.1127]. □

We shall now sketch the details of the isomorphism $H^1(G/V,\mathcal{O}) \cong H^1(V,\mathcal{O}(G))$ (for any closed complex subgroup V of the Stein group G) so that the connection between splitting cocycles $\beta\colon V \longrightarrow \mathcal{O}(G)$ and finding sections of the principal bundle $G \longrightarrow G/V$ will be more apparent.

Let $\xi \in H^1(G/V,\mathcal{O})$ be represented by a cocycle $(a_{ij}) \in H^1(\mathfrak{U},\mathcal{O})$ where $\mathfrak{U}=(U_i)_{i\in I}$ is an open cover of G/V. By choosing a refinement of $\mathfrak{U}$ we may assume the quotient map $\pi\colon G \longrightarrow G/V$ is trivial over each U_i. Let $\mathfrak{Y} = (Y_i)_{i\in I}$ be the open cover of G defined by $Y_i = \pi^{-1}(U_i)$. Since $H^1(\mathfrak{Y},\mathcal{O}) = 0$ the cocycle $(\pi^* a_{ij})$ splits, i.e. there are holomorphic functions $\alpha_i \in \mathcal{O}(Y_i)$ such that $\pi^* a_{ij} = \alpha_j - \alpha_i$ on $Y_i \cap Y_j$. Define $\beta_i\colon V \longrightarrow \mathcal{O}(Y_i)$ by $\beta_i(v)(g) := \alpha_i(g) - \alpha_i(gv) = (\alpha_i - v.\alpha_i)(g)$ $(v \in V,\ g \in Y_i)$. Since $\pi^* a_{ij}(gv) = \pi^* a_{ij}(g)$ we see that $\beta_i(v)(g) = \beta_j(v)(g)$ for all $v \in V$, $g \in Y_i \cap Y_j$. We thus obtain a map $\beta\colon V \longrightarrow \mathcal{O}(G)$ satisfying $\beta(v_1 v_2) = \beta(v_1) + v_1.\beta(v_2)$ for all $v_1, v_2 \in V$, i.e. β is a cocycle representing an element $\varphi(\xi) \in H^1(V,\mathcal{O}(G))$. It is clear from the construction that φ is a well-defined homomorphism. Now if β is a coboundary, i.e. if $\beta(v) = f - v.f$ for some $f \in \mathcal{O}(G)$, then $a'_i := \alpha_i - f \in \mathcal{O}(Y_i)$, $i \in I$, defines a splitting of $\pi^* a_{ij}$ by V-*invariant* holomorphic functions and hence a splitting of a_{ij} by holomorphic functions $a_i \in \mathcal{O}(U_i)$.

Therefore φ is injective. To see that φ is surjective, let $\beta: V \longrightarrow \mathcal{O}(G)$ be any cocycle, and let $S_i \subset Y_i$ be a local section of $\pi|Y_i \longrightarrow U_i$ so that $W_i \cong S_i \times V$, V-equivariantly, $i \in I$. For $g \in Y_i$ write $g = s_i.v_i$ ($s_i \in S_i$, $v_i \in V$) and define $\alpha_i(g) := \beta(v_i)(s_i)$. On $Y_i \cap Y_j$ define $a'_{ij} := \alpha_j - \alpha_i$. Then using the cocycle condition on β, it is not hard to verify that a'_{ij} is V-invariant. Thus, we get a cocycle $(a_{ij}) \in H^1(\mathfrak{U}, \mathcal{O})$ representing $\xi \in H^1(G/V, \mathcal{O})$ with $\varphi(\xi) = [\beta]$.

(1.12) Examples: Let U be an n-dimensional complex vector space and let G_n be the linear algebraic group $GL(U) \ltimes U$ which we may view as the group of matrices of the form

$$\begin{bmatrix} 1 & u_1, \ldots, u_n \\ & \\ 0 & A \end{bmatrix}, \qquad u_i \in \mathbb{C},\ A \in GL(n, \mathbb{C}).$$

(1) Let $G = G_3$ and let $V = V_3$ be the unipotent subgroup

$$\begin{bmatrix} 1 & r & s & t \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad r, s, t \in \mathbb{C}.$$

Let V_2 be the normal subgroup of V_3, $\{v \in V_3 \mid r = 0\}$, and let V_1 be the normal subgroup of V_2, $\{v \in V_2 \mid s = 0\}$. We obtain homomorphisms $\nu_i: V_i \longrightarrow \mathbb{C}$, $\nu_3(v)=r$, $\nu_2(v)=s$, $\nu_3(v)=t$. Define $f_i \in \mathcal{O}(G)$ by $f_3 = u_1$, $f_2 = u_2 - u_1u_3$, $f_1 = u_3$. Then the conditions of Theorem 1.11(4) are fulfilled and G/V is Stein.

(2) (cf. [4]) Let $G = G_4$ and let $V = V_2$ be the unipotent subgroup

$$\begin{bmatrix} 1 & 0 & 0 & t & s \\ 0 & 1 & 2s & s^2 & t \\ 0 & 0 & 1 & s & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad s, t \in \mathbb{C}.$$

Let V_1 be the normal subgroup of V_2, $\{v \in V_2 \mid s = 0\}$. We obtain homomorphisms $\nu_i: V_i \longrightarrow \mathbb{C}$, $\nu_2(v) = s$, $\nu_1(v) = t$. Define $f_i \in \mathcal{O}(G)$ by $f_1 = u_3$, $f_2 = (u_2^2)/4 - u_1 u_3 + u_4$. Then the conditions of the theorem are again fulfilled and G/V is Stein.

(1.13) Remarks: (1) These examples illustrate the fact that in general one need only construct a V-equivariant map from U to V in (3), or one need only find holomorphic functions $f_i \in \mathcal{O}(U)$ satisfying $f_i(u.v) = f_i(u) + \nu_i(v)$ for all $u \in U$, $v \in V_i$, $1 \leq i \leq k$, in (4), to conclude that G/H is Stein. We also have another way of attacking the sufficiency of property (1.4) for G/H to be Stein: One need only find *one* holomorphic function $f \in \mathcal{O}(U)$ such that $f(u.v) = f(u) + \nu(v)$ for all $u \in U$, $v \in V$, where $\nu: V \longrightarrow \mathbb{C}$ is some non-trivial holomorphic homomorphism. Then $V_1 := \ker \nu$ inherits (1.4) and we may apply induction on dim V to conclude that $G/V_1 \cong G/V \times \mathbb{C}$ is Stein, as in the proof of the theorem.

(2) Another way of looking at the problem "(1.4) $\Rightarrow$ Stein" is to carefully 'deform' V to a subgroup $V_0 \subset U$. Then we have a 'deformation' of G/V to a Stein or affine manifold G/V_0 and one can hope to transfer this Stein-ness

or affine-ness to G/V. It is not hard to show that (1.4) implies the existence of a one parameter subgroup $\lambda \subset G$ such that, as $\lambda \to 0$, $ad(\lambda).\mathbf{v}$ converges in the Grassmann manifold to a subalgebra $\mathbf{v}_0 \subset \mathbf{u}$ (here, $\mathbf{v}$ and $\mathbf{u}$ are the Lie algebras of V and U). So, we do in fact have a deformation of V through conjugates to a subgroup $V_0 \subset U$. Unfortunately, the following example shows that this approach may not always work. Let U be a 2-dimensional vector space over $\mathbb{C}$, and let $G = SL(U) \ltimes U \subset G_2$ as above. Let V be the unipotent subgroup

$$\begin{bmatrix} 1 & 0 & v \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}, \quad v \in \mathbb{C}.$$

Then easy calculations show that $V \cap M^u \neq 1$ for some $u \in U$, where $M \cong SL(U)$. In fact, $G/V \cong \mathbb{C}^2 \times \mathbb{C}^2 \backslash \{0\}$ which is not affine. Conjugating V by

$$\lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^{-1} \end{bmatrix}, \quad \mu \in \mathbb{C}^*,$$

we see that $V^\lambda \longrightarrow V_0 \subset U$ as $\mu \to 0$. Thus we obtain a deformation through homogeneous manifolds isomorphic to $\mathbb{C}^2 \times \mathbb{C}^2 \backslash \{0\}$ into the affine manifold $G/V_0 \cong \mathbb{C} \times SL(U)$. (Note that $SL(U) \cong SL(2,\mathbb{C})$ *is* homeomorphic to $\mathbb{C}^2 \times \mathbb{C}^2 \backslash \{0\}$.)

2. Induced G-spaces

Let X be a complex space and H a complex Lie group. We say that X is an H-*space* if H acts holomorphically on X. If H is a closed subgroup of a complex Lie group G, then, as is well-known, we may construct a natural G-space which contains X as an H-equivariant closed subspace in the following manner: Let $h \in H$ act on $G \times X$ by sending $(g,x) \in G \times X$ to $(gh^{-1}, h.x)$. The orbit space quotient of $G \times X$ by this action is denoted by $G \times_H X$, or simply by GX if no confusion will arise, and is easily seen to be a complex space which fibers holomorphically (as a locally trivial bundle) over the coset space G/H with fiber X. (In fact, GX is the natural bundle associated to the principal H-bundle $G \longrightarrow G/H$.) The image of $(g,x) \in G \times H$ in GX will be denoted by $[g,x]$ or sometimes by $[g,x]_H$ to emphasize the dependence on H. Thus, $[gh,x] = [g,hx]$ for all $h \in H$. Note that G acts holomorphically on GX by $g_1[g,x] = [g_1 g,x]$ for all $g_1 \in G$. We now imbed X into GX by simply mapping $x \in X$ to $[1,x]$ and observe that this map is holomorphic and H-equivariant: $[1,h.x] = h[1,x]$. We shall call GX the *induced* G-*space of* X. It has the following *universal property*:

(2.1) *Given a* G-*space* Y *and an* H-*equivariant holomorphic map* $f{:}X \longrightarrow Y$, *there exists a unique* G-*equivariant holomorphic map* $F{:}GX \longrightarrow Y$ *such that* $F \circ \iota = f$, *where* $\iota{:}X \longrightarrow GX$ *is the imbedding* $\iota(x) = [1,x]$, $x \in X$. *The map* F *is given explicitly by* $F[g,x] := g.f(x)$ *for all* $[g,x] \in GX$.

We also have the following G-equivariant isomorphisms:

(2.2) *If* $H \subset K$ *are closed subgroups of* G, *then*

$$G \times_K (K \times_H X) \cong G \times_H X,$$
$$[g,[k,x]_H]_K \longrightarrow [gk,x]_H.$$

(2.3) *If* Y *is a* G-*space, then with the appropriate diagonal actions*

$$(G \times_H X) \times Y \cong G \times_H (X \times Y),$$
$$([g,x],y) \longrightarrow [g, (x, g^{-1}.y)].$$

If E is a *finite dimensional* vector space over $\mathbb{C}$ and H acts on E via a holomorphic representation into GL(E), then we shall call E a *holomorphic* H-*module*. Note that in this case the induced G-space GE is a holomorphic *vector bundle* over G/H on which G acts as a group of bundle automorphisms. Conversely, if $\pi: B \longrightarrow G/H$ is a holomorphic vector bundle on which G acts naturally as a group of bundle automorphisms, then H acts linearly on the vector space $E = \pi^{-1}(1H)$ and B is G-equivariantly isopmorphic to GE via the map $[g,x] \in GE \longrightarrow g.x \in B$.

We can now formulate some well-known theorems on when the quotient of two complex Lie groups is Stein in terms of induced G-spaces (cf. [17]):

(2.4.1) *If* H *is an arbitrary closed complex subgroup of* G *then* G/H *is Stein if and only if for every holomorphic* H-*module* E, *the induced* G-*space* GE *is Stein.*

(2.4.2) *If* H *is a closed connected complex subgroup of* G, *then* G/H *is Stein if and only if for every Stein* H-*space* X, *the induced* G-*space* GX *is Stein.*

If $p = [g,x]$ is a point in an induced G-space GX, then the isotropy subgroup G_p is just $H_x{}^g = gH_xg^{-1}$, i.e. G_p is conjugate to a subgroup of H.

Note also that the orbit $H.x$ is closed in X if and only if $G.p$ is closed in GX. In particular, if H has a fixed point in X then G/H appears as a closed orbit in GX. Thus, merely the existence of a Stein H-space X where H has a fixed point and such that GX is Stein is sufficient for G/H to be Stein. We shall see that each of these properties (of the isotropy subgroups and the closed orbits) is strong enough in its own right to imply statements similar to (2.4.1) and (2.4.2).

(2.5) Definition: Let Y be a G-space and let K be a normal subgroup of G. We say that a subgroup $H \subset K$ *controls the isotropy of* K *on* Y if every isotropy subgroup K_y, $y \in Y$, is conjugate in G to a subgroup of H. We also say that H *controls the closed orbits of* K *on* Y if for any $y \in Y$, $K.y$ is closed in Y if and only if there is a $k \in K$ such that $Hk.y$ is closed in Y.

It is easy to verify that these two properties are transitive on subgroups of G, i.e. if H controls the isotropy (resp. the closed orbits) of K on Y and K controls the isotropy (resp. the closed orbits) of L on Y, then H controls the isotropy (resp. the closed orbits) of L on Y. In the following lemma and elsewhere, G^0 will denote the connected component of the identity in a Lie group G.

(2.6) Lemma: *Let* Y *be a* G-*space and let* K *be a closed normal complex subgroup of* G. *Assume a closed connected complex subgroup* $H \subset K$ *controls the isotropy of* K *on* Y *and has a fixed point in* Y. *Then there is a closed* G-*invariant complex subspace* Y_0 *of* Y *such that:*

(1) *For every* $y \in Y_0$, K_y^0 *is conjugate in* G *to* H;

(2) Y_0 *is* G-*equivariantly isomorphic to* $G \times_N Z$ *where* Z *is the closed subspace of* H-*fixed points in* Y *and* N *is the normalizer of* H *in* G.

Furthermore, if Y *is a Stein space, then* G/H *is Stein.*

Proof: Let Y_0 be the closed subspace of Y consiting of the K-orbits in Y of complex dimension $\leq \dim K/H$ (cf. [11]). Since K_y is conjugate in G to a subgroup of H for all $y \in Y$, and $\dim K_y = \dim H$ for $y \in Y_0$, we have $K_y{}^0$ is conjugate in G to H for all $y \in Y_0$.

The map $G \times_N Z \longrightarrow Y_0$, $[g,z] \longrightarrow g.z$ is clearly well-defined and holomorphic. To see it is injective, we note that if $g.z = g_1.z_1$ then $g^{-1}g_1Hg_1{}^{-1}g \subset K_z{}^0 = H$, so $g^{-1}g_1 = n \in N$. Thus, $[g,z] = [g_1n^{-1},n.z_1] = [g_1,z_1]$. The map is also surjective, since for any $y \in Y_0$ there exists a $g \in G$ such that $(K_{g.y})^0 = (K_y{}^g)^0 = H$. Therefore, $g.y \in Z$ and $[g^{-1},g.y] \in G \times_N Z$ maps onto y.

Now assume Y is Stein and consider the principal bundle $G \times_H Z \longrightarrow G \times_N Z = Y_0$. Since Y_0 is closed in Y, it is Stein. The fiber N/H, is a complex Lie group which is holomorphically separable since it is isomorphic to an N-orbit in Y_0. This implies that N/H is a Stein Lie group (cf. [17, p.147]). By [17,Théorème 4], the total space $G \times_H Z$ is Stein. Now $G \times_H Z$ is G-equivariantly isomorphic to $G/H \times Z$ by (2.3). Hence G/H is Stein. □

The above mentioned analogue of (2.4) now follows easily:

(2.7) Theorem: *Let* K *be a closed normal complex subgroup of a complex Lie group* G *and let* H *be a closed connected complex subgroup of* K. *Then the following are equivalent:*

(1) G/H *is Stein.*

(2) *Every Stein* H-*space* X *can be holomorphically and* H-*equivariantly imbedded in a Stein* G-*space* Y *such that* H *controls the isotropy of* K *on* Y.

(3) *Every Stein* H-*space* X *can be holomorphically and* H-*equivariantly imbedded in a Stein* G-*space* Y *such that* H *controls the closed orbits of* K *on* Y.

Proof: (1) $\Longleftrightarrow$ (2): If G/H is Stein, then $Y = G \times_H X$ is Stein (2.4.2); and for $p = [g,x] \in G \times_H X$, we have $K_p = K \cap G_p = (K \cap H_x)^g = H_x{}^g$.

(1) $\rightarrow$ (3): Since $G \times_H X = G \times_K (K \times_H X)$ by (2.2), we see that K.[g,x] is closed in $Y = G \times_H X \longleftrightarrow K.[1,x]$ is closed in $K \times_H X \longleftrightarrow H.[1,x]$ is closed in $K \times_H X \Longleftrightarrow Hg^{-1}[g,x]$ is closed in Y.

(2) $\Rightarrow$ (1): Let X be a point with trivial H-action. Then by assumption there is a Stein space Y where H controls the isotropy of K and has a fixed point. Lemma(2.6) implies that G/H is Stein.

(3) $\Rightarrow$ (1): Again, if X is a point, then by assumption there is a Stein space Y where G/H appears as a closed orbit and is therefore Stein. □

If H is an algebraic subgroup of a linear algebraic group G, we say that H is *observable* in G if every (finite dimensional) rational H-module is an H-submodule of some G-module, cf. [2]. By the existence of equivariant imbeddings for algebraic actions on affine varieties, this is equivalent to: every affine H-space can be H-equivariantly imbedded in an affine G-space. This latter definition carries over more naturally than the former to complex Lie groups. Thus we say that a complex subgroup H is *observable* in a Stein Lie group G if every Stein H-space can be H-equivariantly imbedded in a Stein G-space.

For linear algebraic groups, H being observable in G is equivalent to G/H being *quasi-affine*, cf. [2]. If one strengthens the observability hypothesis ("strongly observable" in [4], or "hyperobservable" in [7]) one obtains a criterion for G/H to be affine. The above Lemma and Theorem carry over directly to linear algebraic groups and affine quotients. Hence condition (2) or (3) may be used as the defintion of *strongly observable* for both algebraic groups and complex Lie groups.

There is an important difference, however, between *observable* subgroups in the two contexts. It appears that if a closed connected complex subgroup H is observable in a Stein Lie group G, then it may already be strongly observable in G, i.e. G/H is Stein. This is the case for dim V = 1 (Theorem 2.8), and would in fact be true for all H if condition (1.4) implies that G/H is Stein. This is a consequence of the following:

(2.8) Proposition: *Let* G *be a Stein Lie group and let* H *be a* *closed connected observable complex subgroup of* G. *Then* H *satisfies condition* (1.4).

Proof: Suppose $V \cap M^g \neq 1$ for some $g \in G$. Then there is a closed 1-dimensional complex subgroup C of $V \cap M^g$ isomorphic to the additive group $\mathbb{C}$. Now C acts holomorphically on $Y = \mathbb{C}^*$ via the exponential homomorphism with kernel $Z \cong \mathbb{Z}$. Since V is observable in G and $V/C \cong \mathbb{C}^k$, it follows from (2.4.2) that C is observable in G. Then C is also observable in $M_0 = M^g$. Therefore, there is a C-equivariant imbedding of Y into some Stein M_0-space X. By (2.1) we obtain an M_0-equivariant map $M_0 \times_C Y \longrightarrow X$. We conclude that there are enough holomorphic functions on $M_0 \times_C Y$ to separate the points of $Y = C/Z$. However, since $M_0 \times_C Y$ is M_0-equivariantly isomorphic to M_0/Z, this contradicts the main theorem of [1]. Therefore, $V \cap M^g = 1$ for all $g \in G$. □

We conclude this section with a brief description of the homogeneous bundle structure every Stein quotient G/H possesses.

(2.9) Theorem: *Let* G *be a connected complex Lie group and let* H *be a closed connected complex subgroup of* G. *If* $X = G/H$ *is Stein, then* X *is* M-*equivariantly biholomorphic to* $M \times_L S$ *where*

(1) $M \supset L$ *are connected reductive complex Lie groups,*

(2) S *is an* L-*invariant complex subspace of a rational* L-*module* E,

(3) *There is a rational* L*-module* F *such that* $S \times F$ *is* L*-equivariantly biholomorphic to* E.

Proof: We may assume $G = MU$ is Stein and H has a semi-direct product decomposition $H = L \ltimes V$, $L \subset M$, such that G/V is Stein and V satisfies (1.4), cf. §1. Moreover, by [18,p.92], a Stein geometric quotient Y of G/V by M exists and the quotient map $G/V \longrightarrow Y$ is a locally trivial principal M-bundle. Using local sections of $G \longrightarrow Y$ we see that the fibration of $U \cong M \backslash G \longrightarrow Y$ is a locally trivial principal V-bundle. Since the structure group V is contractible, the bundle is topologically trivial and therefore holomorphically trivial, cf. [5]. Thus, there exists a V-equivariant holomorphic map $\varphi : U \longrightarrow V$. Since V with conjugation by elements of L is isomorphic to a rational L-module F, the following integral over a maximal compact subgroup K of L

$$\phi(u) := \int k^{-1}\varphi(kuk^{-1})k \, dk, \quad u \in U,$$

makes sense (with respect to some invariant measure dk on K). Then ϕ is still V-equivariant and, moreover, standard arguments show that $\phi(\ell u \ell^{-1}) = \ell\phi(u)\ell^{-1}$, for all $u \in U$, $\ell \in L$. Then $S := \phi^{-1}(1)$ is an L-invariant subspace of U which, with conjugation by elements of L, is also isomorphic to a rational L-module E. By the remarks preceding Theorem 1.11, it is clear that E is L-equivariantly isomorphic to $S \times F$.

Finally, we note that the L-invariant section $S \subset U$ also provides a section of the principal M-bundle $G/V \longrightarrow Y$. Thus the (left) M-equivariant and (right) L-equivariant holomorphic map $M \times S \longrightarrow G/V$, $(m,s) \longrightarrow msV$, is an isomorphism. We conclude that G/H is M-equivariantly isomorphic to $M \times_L S$. □

3. Solvable groups

For any subgroup J of the general linear group $GL(n,\mathbb{C})$ we shall denote by J^* the smallest algebraic subgroup of $GL(n,\mathbb{C})$ which contains J.

(3.1) Definition: Let G be a complex Lie group and H subgroup of G. We say that H is a *quasi-algebraic* subgroup of G is there is a holomorphic representation of G, $\rho: G \longrightarrow GL(n,\mathbb{C})$, such that

$$H = \rho^{-1}(\rho(H)^* \cap \rho(G)).$$

(3.2) Lemma: *A subgroup* H *of* G *is quasi-algebraic if and only if* G/H *can be* G-*equivariantly mapped into some projective space* $\mathbb{P}^n$.

Proof: If H is quasi-algebraic then $G/H = \rho(G)/\rho(H)$ injects holomorphically and G-equivariantly into $\rho(G)^*/\rho(H)^*$. Now the quotient of two linear algebraic groups can always be realized as an orbit in some projective space, cf. [13,p.80]. Hence, the same is true for G/H. Conversely, if G acts on $\mathbb{P}^n$ via a holomorphic representation ρ, and if H is the isotropy subgroup G_p of some point $p \in \mathbb{P}^n$, then $\rho^{-1}(\rho(H)^* \cap \rho(G)) = \rho^{-1}(\rho(G)_p) = G_p = H$. □

Now suppose H_1 and H_2 are two quasi-algebraic subgroups of G. Then H_i is the isotropy subgroup of a point $p_i \in \mathbb{P}^{n_i}$ under the action of G defined by a holomorphic representation ρ_i, $i = 1,2$. The diagonal action of G on the product $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ imbeds equivariantly into $\mathbb{P}^m$, $m = (n_1+1)(n_2+1) - 1$, where G acts via the representation $\rho_1 \otimes \rho_2$. The isotropy subgroup of the point corresponding to (p_1,p_2) in $\mathbb{P}^m$ is then $H_1 \cap H_2$. Therefore, *the intersection of two quasi-algebraic subgroups of* G *is again quasi-algebraic.* If we define the quasi-algebraic closure of a subgroup H to be the the smallest quasi-algebraic subgroup H^* of G which contains H, we see from the above that H^* is the intersection of all the quasi-algebraic subgroups of G which contain H. (Of course a finite number of such subgroups would suffice to define H^*.)

The most familiar examples of quasi-algebraic subgroups are given by the normalizers of connected subgroups: If H is a connected complex subgroup of G and N is the normalizer of H in G then N is precisely the subgroup of elements of G which leave the Lie algebra $\mathbf{h}$ of H invariant under the adjoint representation of G on its Lie algebra $\mathbf{g}$. If k = dim H, then the k^{th} exterior power of the adjoint representation of G defines a holomorphic action of G on the projective space of $\Lambda^k\mathbf{g}$, and N is the isotropy subgroup of the point corresponding to $\Lambda^k\mathbf{h}$. A consequence of this is that *the quasi-algebraic closure of an arbitrary complex subgroup* H *of* G *normalizes* H^0, *the connected component of the identity in* H.

(3.3) Theorem: *Let* G *be a connected solvable complex Lie group and let* H *be a quasi-algebraic subgroup of* G. *Then* G/H *is biholomorphic to* $(\mathbb{C}^*)^s \times \mathbb{C}^t$, *and is, in particular, Stein.*

Proof: By Lemma 3.2 we may identify G/H with a G-orbit in some projective space $\mathbb{P}^n$ where G acts via a holomorphic representation $\rho: G \longrightarrow GL(n+1,\mathbb{C})$. The commutator subgroup $\rho(G)' = \rho(G')$ is an algebraic unipotent subgroup of $GL(n+1,\mathbb{C})$, cf. [3]. Hence, its orbits in $\mathbb{P}^n$ are all isomorphic to euclidean spaces and are locally closed. It follows that the

$\rho(G')$-orbits are closed in $\rho(G)/\rho(H)$ and we obtain the fibration $G/H = \rho(G)/\rho(H) \longrightarrow G/G'H = \rho(G)/\rho(G'H)$ with fiber $G'H/H$ isomorphic to $\mathbb{C}^k$. Now $G'H$ is a quasi-algebraic subgroup of G; hence $G/G'H$ injects holomorphically into the affine variety $\rho(G)^*/\rho(G'H)^*$. Since $G/G'H$ is then a holomorphically separable abelian complex Lie group, it is isomorphic to $(\mathbb{C}^*)^s \times \mathbb{C}^{t'}$, cf. [17,p.147]. Also, $G'H/H$ is contractible and $G/G'H$ is Stein, so that the structure group of the bundle $G/H \longrightarrow G/G'H$ can be reduced to H. cf. [5]. Now H acts via conjugation on $G'H/H$ and thus factors through the adjoint representation, showing that the action of H on the fiber $\mathbb{C}^k$ is biholomorphic to a *linear* action. It follows from (2.4.1) that G/H must be Stein. This also follows directly from Theorem 2.7, since H controls the isotropy of G on the Stein space (in fact affine variety) $\rho(G)^*/\rho(H)^*$. We now show that the vector bundle $G/H \longrightarrow G/G'H$ is holomorphically trivial. For this, we may assume that G is simply-connected. Since the pull-back bundle $G/H^0 \longrightarrow G/(G'H)^0 \cong \mathbb{C}^{s+t'}$ is then holomorphically trivial, we see that the original bundle comes from a representation of the fundamental group of $G/G'H$. But this implies that $G/H \longrightarrow G/G'H$ has a holomorphic connection, cf. [0], and so all the Chern classes of this vector bundle vanish. Finally, since the base is Stein, this topologically trivial bundle must also be holomorphically trivial, cf.[5]. □

(3.4) Corollary: *If* H *is a quasi-algebraic subgroup of a connected solvable complex Lie group* G, *then* $\pi_1(G/H) \cong \mathbb{Z}^p$.

(3.5) Example: Let G be the unipotent complex Lie group consisting of the upper triangular complex matrices with 1's on the diagonal, and let H be the closed subgroup of G consisting of the matrices with integer entries. Then easy arguments show that G/H is Stein. However, since H is non-abelian, G/H does not admit an equivariant map into $\mathbb{P}^n$ (or a linear equivariant map into $\mathbb{C}^n$).

The above theorem can be used to prove the sufficiency of (1.4) for Stein quotients of connected solvable complex Lie groups. We also need the following:

(3.6) Lemma: *Let* G *be a Stein Lie group and let* H *be a connected normal complex subgroup of* G *with a semi-direct product decomposition* $H=L\ltimes V$ *as in* (1.1). *Then* G/H *is a Stein Lie group if and only if* V *satisfies* (1.4).

Proof: We already know that (1.4) is necessary. To show it is sufficient, let $G = MU$ be the decomposition (1.2) of G with $M \supset L$, and consider the fibration $G/H \longrightarrow G/MH$. The base G/MH is biholomorphic to $\mathbb{C}^n$ (this follows from [16, Théorème 3]), and hence the bundle is holomorphically trivial, $G/H \cong MH/H \times \mathbb{C}^n \cong M/L \times \mathbb{C}^n$. It follows that G/H is Stein. □

(3.7) Theorem: *Let* G *be a connected solvable Stein Lie group and let* H *be a closed connected complex subgroup of* G *with a semi-direct product decomposition* $H = L\ltimes V$ *as in* (1.1). *Then* G/H *is Stein if and only if* V *satisfies* (1.4).

Proof: Let N be the normalizer of H in G. Then G/N is Stein by Theorem 3.3. Since a covering space of a Stein manifold is Stein, G/N^0 is also Stein. Moreover, N^0/H is Stein by Lemma 3.6. Thus, the base and fiber of $G/H \longrightarrow G/N^0$ are Stein; so G/H is Stein, cf. (2.4.2). □

(3.8) Remarks: (1) By a result of Huckleberry and Oeljeklaus [12,Theorem 1], the Stein manifold G/H in Theorem 3.7 is biholomorphic to $(\mathbb{C}^*)^s \times \mathbb{C}^t$.

The proof of Theorem 1 in [12] was in fact the inspiration for Theorem 3.3 in this paper.

(2) Theorem 3.7 is also a consequence of Theorem 1.7: The linear algebraic group G_1 is easily seen to be solvable and the quotient of G_1 by the unipotent subgroup V_1 is clearly an affine variety.

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