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DEGREE THEORY ON ORIENTED INFINITE DIMENSIONAL VARIETIES AND THE MORSE NUMBER OF MINIMAL SURFACES SPANNING A CURVE IN $\mathbb{R}^n$

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The algebraic number of disc minimal surfaces spanning a wire in $\mathbb{R}^3$ is defined and shown to be equal to one.

Part II : n = 3

In [16] the author developed a limited "Morse theory" for minimal surfaces spanning a smooth curve Γ in $\mathbb{R}^n$, $n \geq 4$. It was shown that for almost all Γ a Morse number of minimal surfaces was defined and this number, independent of Γ was always one. The reason why the results of [16] were not applicable to $\mathbb{R}^3$, is, roughly speaking, the existence in $\mathbb{R}^3$ of formally degenerate minimal surfaces which are stable under perturbation of the boundary wire Γ. Thus generic non-degeneracy breaks down. However in [4] the author and Reinhold Böhme introduced for $\mathbb{R}^3$ a new notion of non-degeneracy and showed, that relative to this new concept, almost all wires Γ in $\mathbb{R}^3$ had the property that only a finite number of "non-degenerate" minimal surfaces spanned Γ.

As stated in Section 7 of [16] we shall prove that a Morse number of minimal surfaces spanning a generic wire Γ in $\mathbb{R}^3$ is still defined and is, as expected, still one.

Finally we indicate a proof of a strengthed version of Morse-Shiffman-Tompkin's famous mountain pass theorem for wires in $\mathbb{R}^3$.

STATEMENT OF MAIN RESULT

Let D be the unit disc in $\mathbb{R}^2$, $\partial D = S^1$ and $\alpha : S^1 \to \mathbb{R}^3$ an embedding of Sobolev calss H^r with $\Gamma^\alpha = \alpha(S')$ the image wire. Let $N(\alpha)$ be the space of all H^s maps $u : D \to \mathbb{R}^3$, $u = (u^1, u^2, u^3)$ such that for all i, $\Delta u^i = 0$ (each u^i is harmonic) and $u : S^1 \longrightarrow \Gamma^\alpha$ is homotopic to α. We shall assume that $s \geq 7$, $r > 2s + 4$ and the total curvatures of Γ^α, $K(\Gamma^\alpha) \leq \pi(s-2)$ {the choice of regularity class depends on the wire}.

A minimal surface of disc type is an element $u \in N(\alpha)$ satisfying

(0.1)

(1) $$\langle u_x, u_y \rangle_{\mathbb{R}^3} = 0 = \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle$$

(2) $$\|u_x\|_{\mathbb{R}^3} = \|u_y\|_{\mathbb{R}^3}$$

(3) $u : S^1 \to \Gamma$ is a homeomorphism.

A map $u \in N(\alpha)$ satisfying (o.1) is a critical point of Dirichlet's functional $E_\alpha : N(\alpha) \to R$ defined by

(0.2) $$E_\alpha(u) = {}^1/2 \sum_{j=1}^{n} \int_D \nabla u^i \cdot \nabla u^i .$$

Let G be the three dinemsional conformal group of the disc. Each $g \in G$ is of the form $g(z) = c \cdot \frac{z-a}{1-az}$, $|c| = 1$, $|a| < 1$.

G acts on $N(\alpha)$ via $g_{\#}(u)(z) = u(g(z))$ and E_α is equivalent with respect to this action. Consequently if u is any critical point of E_α the orbit of G through u, $O_u(G)$ consists of critical points. For such a critical point let $D^2E_\alpha(u) : T_u N(\alpha) \times T_u N(\alpha) \to R$ denote the Hessian or second derivature, where $T_u N(\alpha)$ is the tangent space to $N(\alpha)$ at u. By equivariance, it follows that $T_u \mathcal{O}_u(G)$ the tangent space to $\mathcal{O}_u(G)$ at u, will always be the kernel of the quadratic form $D^2E_\alpha(u)$. The surface u is said to be <u>non-degenerate</u> if this is the only kernel. If u is non-degenerate there is (modulo this kernel) a maximal subspace on which the Hessian is strictly negative, and this dimension is called the Morse index of u .

We should remark at this point that in [14] the author, and independently Reinhold Böhme derived an expression for $D^2E_\alpha(u)$, namely for $h,k \in T_u N(\alpha)$, u a critical point

$$(0.3) \qquad D^2E_\alpha(u)(h,k) = \sum_{j=1}^{3} \int_D \nabla h^i \cdot \nabla k^i - \int_{\Gamma^\alpha} k_g \langle h,k\rangle_{\mathbb{R}^3} ds$$

where k_g is the geodesic curvature of Γ^α .

Unfortunately, by the Fredholm index theorem of Böhme-Tromba [4] it is <u>false</u> that almost all wires (open dense) α in $\mathbb{R}^3$ enjoy the property that there are only a finite number of non-degenerate minimal surfaces spanning Γ^α , although this is true for $\mathbb{R}^n$, $n \geq 4$. This is the main distinction between the theory in Euclidian three space and $\mathbb{R}^n$, $n \geq 4$.

This finiteness result enabled us to define a <u>Morse number</u> for a generic Γ^α in $\mathbb{R}^n$, $n \geq 4$, namely let $u_1,\ldots,u_m$ be the finite number of minimal surfaces which span Γ^α with Morse numbers $\mu_1,\ldots,\mu_m$ respectively. Then

$$(0.4) \qquad \sum_{j=1}^{m} (-1)^{\mu_j} = 1$$

the right hand side of (0.4) is by definition the Morse number of minimal surfaces spanning Γ^α . Thus, independent of Γ^α the Morse number is always one.!

As already mentioned in Euclidian three space a new notion of non-degeneracy was introduced which enabled one to prove that for a generic wire Γ^α there is only a finite number $u_1\ldots u_m$ of minimal surfaces spanning Γ^α although these may well be degenerate in a sense previously described. Thus the "old" formula (0.4) for the Morse number does not seem to hold. However, since for the generic wire one has only isolated solutions another strategy for defining this number will work, and it roughly goes as follows. For each α it was shown in [14] that there is a vector field X_α : $N(\alpha) \to TN(\alpha)$ whose zeros wire precisely the minimal surfaces spanning Γ^α . This vector field had the wonderful property that at every zero u, the derivature $DX_\alpha(u)$: $T_u N(\alpha) \to T_u N(\alpha)$ was a linear map of the form identity plus compact. This fits in with the theory of Euler-characteristics for vector fields on Banach manifolds as developed in [15]. In any case if a zero u of X_α is isolated one can define (and we shall do it in this paper) a local degree of X_α about u, say $\deg(X_\alpha, u)$.

Now if Γ^α has only finitely many minimal surfaces $u_1, \ldots, u_m$ which span it we define the Morse number Morse (Γ^α) of minimal surfaces spanning Γ^α by the formula

(0.5) $$\text{Morse } (\Gamma^\alpha) = \sum_{j=1}^{m} \deg(X_\alpha, u_j)$$

REMARK (0.6): This generalizes the definition of Morse number for a curve in $\mathbb{R}^n$, $n \geq 4$ since it works for curves Γ^α such that the minimal surfaces which span it are only isolated. Of course it can be shown that in the generic case both formulas yield the same number.

The main result of this paper is the following.

THEOREM: For a generic curve Γ^α in $\mathbb{R}^3$,

$$\sum_{j=1}^{m} \deg(X_\alpha, u_j) = 1 = \text{Morse } (\Gamma^\alpha) .$$

The basic methods used to prove this result will be the Fredholm index theory of the author and Reinhold Böhme, the degree theory of Elworthy and the author [5] as generalized in [15], and of course the basic results of [4].

A REVIEW OF THE LOCAL WINDING NUMBER OF A ROTHE VECTOR FIELD

In this section we follow essentially [15]. Let $\mathbb{E}$ be a Banach space and let $GL(\mathbb{E})$ denote its general linear group. Let S denote those linear operators $S : \mathbb{E} \circlearrowleft$ such that for $0 \leq t \leq 1$, $tS + (1-t)I$ belong to $GL(\mathbb{E})$, where I denotes the identity mapping. Let $R_C(\mathbb{E})$ denote all those linear operators of the form $S + K$ where $S \in S$ and

K is a compact linear operator. Denote by $GR_C(\mathbb{E}) = R_C(\mathbb{E}) \cap GL(\mathbb{E})$. Then from [15] we have

THEOREM 1.1: $\pi_0(GR_C(\mathbb{E})) = 2$. Thus $GR_C(\mathbb{E})$ has two components.

Denote by $GR_C^+(\mathbb{E})$ the component of the identity and $GR_C^-(\mathbb{E})$ the other component. These will be of importance momentarily.

Let U be an open subset of E with $Y : \overline{U} \to E$ a C^2 mapping.

DEFINITION 1.2: Such a field Y is said to be a Rothe-field on U if for each $x \in U$, $DY(x) \in R_C(\mathbb{E})$.

Since every element of $R_C(\mathbb{E})$ is a linear Fredholm operator of index zero we get immediately from the definition of non-linear Fredholm operator (one whose linearization is Fredholm)

THEOREM 1.3: A Rothe-field is a Fredholm map of index zero.

DEFINITION 1.4: Let $M \subset E$ be any subset. A map $f : M \to E$ is proper if the inverse image of a compact set is compact in M, or equivalently if $f(x_n) \to y$, $\{x_n\}$ has a subsequence which converges to a point in M.

The next result is due to Smale [12].

THEOREM 1.5: Fredholm maps are locally proper.

Let x_0 be an isolated zero of a Rothe field $Y : \overline{U} \to \mathbb{E}$ $(Y(x_0) = 0)$. We wish to define the local winding number or degree of Y about x_0. Choose a neighborhood B of x_0 so that $Y|\overline{B}$ is proper and no other zero of Y is in $\overline{B}$. By properness $\mathbb{E} - Y(\partial\overline{B})$ will be open and so let $\mathcal{O}$ be the component of $\mathbb{E} - Y(\partial\overline{B})$ containing 0 and let $M = Y^{-1}(\mathcal{O})$ and $Y_M = Y|M$. The map $Y_M : M \to \mathcal{O}$ is a proper Fredholm map of index zero.

By the Smale-Sard theorem we can find a regular value

$y \in \mathcal{O}$ for Y_M . Then $Y_M^{-1}(y)$ contains only a finite number of points $x_1,\ldots,x_m$, and since y is regular

$$DY_M(x_j) \in GR_C(\mathbb{E})$$

for all j . This permits us to define the signum of $DY_M(x_j)$ by

$$\operatorname{sgn} DY_M(x_j) = \begin{cases} +1 & \text{if } DY_M(x_j) \in GR_C^+(\mathbb{E}) \\ -1 & \text{if } DY_M(x_j) \in GR_C^-(\mathbb{E}) \end{cases}$$

We now use this for the next basic

DEFINITION 1.6: The local degree or winding number of Y about x_o , $\deg(Y,x_o)$ is defined by

$$\deg(Y,x_o) = \sum_{j=1}^{m} \operatorname{sgn} DY_M(x_j)$$

Using the methods of Elworthy and the author [5] it follows that this definition is independent of the choice of B and the choice of the regular value y .

More generally let U be any open subset of $\mathbb{E}$ with $Y : \bar{U} \to E$ proper and Rothe on U . If $0 \notin Y(\partial\bar{U})$ we may then repeat the above construction to define a degree $\deg(Y,\bar{U})$. If no zeros of Y exist in U this degree will be zero. Moreover if Y has finitely many zeros $x_1,\ldots,x_m$ in U , then

$$\deg(Y, U) = \sum_j \deg(Y,x_j) \tag{1.7}$$

Finally, we have the basic property of all degree theories, namely

THEOREM 1.8: If Y_o , Y_1 are two proper Rothe fields on $\bar{U}$ such that there exists a homotopy Y_t , $0 \le t \le 1$ with the property that

(i) each Y_t is Rothe on $\overline{U}$
(ii) $0 \notin \overline{Y}_t(\partial\overline{U})$ for all t
(iii) $(t,x) \longrightarrow Y_t(x)$ is a proper map.

Then

$$\deg(\overline{Y}_1,\overline{U}) = \deg(Y_0,\overline{U})$$

THE INDEX THEOREM FOR CLASSICAL MINIMAL SURFACES

Our main reference for this section is the Index theorem of Böhme-Tromba [4]. Let Γ^α be a smooth wire in $\mathbb{R}^n$ which is the image of a differentiable embedding $\alpha : S^1 \to \mathbb{R}^n$, $\alpha(S^1) = \Gamma$. A solution to the classical Plateau problem for Γ^α is a mapping $u : D \to \mathbb{R}^n$, D the closed unit disc in $\mathbb{R}^2$, $\partial D = S^1$ such that (c.f. 0.1)

$$\text{(2.1)} \qquad \begin{aligned} &(0)\ \Delta u = 0 \\ &(1)\ u_x \cdot u_y = 0 \\ &(2)\ \|u_x\| = \|u_y\| \\ &(3)\ u : S^1 \to \Gamma^\alpha \text{ is a homeomorphism.} \end{aligned}$$

Conditions (1) and (2) imply that the surface is conformally parameterized. Moreover these are the Euler equations for critical points of Dirichlet's integral on the space of all surfaces spanning Γ^α.

A point z_0 in the interior of D where $F(z) = u_x - iu_y$, $i = \sqrt{-1}$ vanishes is called an interior branch point of u. Since F is holomorphic z_0 has a finite order $\lambda\Big(F(z) = (z - z_0)^\lambda G(z)$, $G(z_0) \neq 0\Big)$.

A point $\xi_0 \in S^1$ where F vanishes is called a boundary branch point and by the results of Heinz and Tomi [7] ξ_0 also has a well defined order. Equations (1) and (2) also imply that all points where a minimal surface u fails to be immersed are branch points. Moreover the monotonicity assumption (3) implies that boundary branch points will all be of even order.

For integers r, and s, $r \geq 2s+4$, $s \geq 7$ define

$$\mathcal{D} = \mathcal{D}^s = \left\{ \varphi : S^1 \to S^1 \mid \deg\varphi = 1 \text{ and } \varphi \in H^s(S^1, \mathbb{C}) \right\} .$$

where H^s denotes the Sobolev space of s-times differentiable (in the distribution sense) functions with values in the complexes $\mathbb{C}$.

(2.0) $\quad A = \left\{ \alpha : S^1 \to \mathbb{R}^n \mid \alpha \in H^r(S, \mathbb{R}^n) \text{ a differentiable embedding} \right\}$

By differentiable embedding we mean α is one to one and $\alpha'(p) \neq 0$ for all $p \in S^1$. Furthermore we shall assume that the total curvature of the image curve Γ^α is bounded by $\pi(s-2)$. (This π is not to be confused with projection maps to follow).

Let $\pi : A \times \mathcal{D} \to A$ denote the projection map onto the first factor. A minimal surface $u : D \to \mathbb{R}^n$ spanning Γ^α can be viewed as an element of $A \times \mathcal{D}$, since u is harmonic and therefore determined by its boundary values

$$u|\partial D = u|S^1 = \alpha \circ \varphi \text{ , where } (\alpha, \varphi) \in A \times \mathcal{D} .$$

The classical approach to minimal surfaces was to understand the set of minimal surfaces in $\pi^{-1}(\alpha)$. Our approach is first to understand the structure of the set of minimal surfaces as a subset of the bundle $N = A \times \mathcal{D}$ as a fibre bundle over A and then to approach the questions of the set of minimal surfaces in the fibre $\pi^{-1}(\alpha)$ in terms of the singularities of the projection map π restricted to a suitable subvariety of N. This is the spirit of Thom's original approach to unfoldings of singularities.

Let us say that a minimal surface $u \in A \times \mathcal{D}$ has branching type (λ, ν), $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{Z}^p$, $\nu = (\nu_1, \ldots, \nu_q) \in \mathbb{Z}^q$, each λ_i, $\nu_i \geq 0$, if u has p distinct but arbitrarily located interior branch points $z_1, \ldots, z_p$ in D of integer orders $\lambda_1,\dots,\lambda_p$ and q distinct boundary branch points $\xi_1,\dots,\xi_q$ in S^1 of (even) integer orders $\nu_1,\dots,\nu_q$. In a formal sense the subset M of minimal surfaces in N is an algebraic subvariety of N [4] and is a stratified set, stratified by branching types. To be more precise let M_ν^λ denote the minimal surfaces in N of branching type (λ,ν). We can now state the index result in [4].

THEOREM 2.1: (Index Theorem for Disc Surfaces) The set M_0^λ is a C^{r-s-1} submanifold of N and the restriction of π, π^λ to M_0^λ is C^{r-s-1} Fredholm map of index $I(\lambda)+3 = 2(2-n)|\lambda| + 2p + 3$ where $|\lambda| = \sum\lambda_i$. Moreover, locally, for $\nu \neq 0$, $M_\nu^\lambda \subset W_\nu^\lambda$ such that W_ν^λ is a submanifold of N where the restriction π_ν^λ of π to W_ν^λ is Fredholm of index $I(\lambda,\nu)+3 = 2(2-n)|\lambda| + (2-n)|\nu| + 2p + q + 3$. The number 3 comes from the equivariance of the Dirichlet functional under the action of the three dimensional conformal group of the disc.

The index $I(\lambda,\nu)$ measures (in some sense) the stability of minimal surfaces of branching type (λ,ν) in $\mathbb{R}^n$ and the likelyhood of finding such surfaces; the more negative the index of π_ν^λ the less likely it is to find a wire admitting minimal surfaces of branching type (λ,ν) which span it.

It is easy to see that if $n \geq 3$ $I(\lambda,\nu) \leq 0$ and if $n \geq 4$, $I(\lambda,\nu) = 0$ if and only if $\lambda = 0$ and $\nu = 0$. However when $n = 3$ $I(\lambda,\nu) = 0$ also when $\nu = 0$ and $\lambda = (1,\dots 1,\dots 1)$; that is on the strata of minimal surfaces with only p simple branch points.

We would like to "factor out" the action of the conformal group G. Let Q_1, Q_2, Q_3 be three fixed points on ∂D. Let $\tilde{\mathcal{D}} = \{\varphi \in \mathcal{D} \mid \varphi(Q_i) = Q_i\}$ and let $S = A \times \tilde{\mathcal{D}}$. S is clearly a subbundle of N. Define $\Sigma_\nu^\lambda = S \cap M_\nu^\lambda$. Then one shows easily that the index of the projection map π restricted to the manifold Σ_0^λ is precisely $I(\lambda)$. For notational convenience denote this projection map again by π^λ. If $n \geq 4$ the index of π^λ will be less than or equal to negative-two. However if $n = 3$ and $\lambda = (1,\ldots 1)$ (simple branch points) the index of π^λ as computed above is zero. This astonishing fact permits us to give a new definition of non-degeneracy, namely $u \in \Sigma^\lambda$ is a <u>non-degenerate minimal surface in $\mathbb{R}^3$</u> if $D\pi^\lambda(u) : T_u\Sigma^\lambda \to T_\alpha A$ is an isomorphism. Such a u will be, by the inverse function theorem stable under perturbations by α yet will be <u>formally degenerate</u> according to the definition given in the introduction. However this definition of non-degeneracy certainly permits one to conclude that in $\mathbb{R}^3$ such formally degenerate surfaces are isolated.

These stratification and index results are the basis to prove the generic (open-dense) finiteness and stability of minimal surfaces of the type of the disc as announced in [13]; i.e. there exists an open dense subset $\hat{A} \subset A$ such that if $\alpha \in \hat{A}$ there exists only a finite number of minimal surfaces bounded by α, and these minimal surfaces are stable under perturbations of α. This open dense set will be the set of regular values of the map π.

Moreover if $n > 3$ the minimal surfaces spanning $\alpha \in \hat{A}$ are all immersed up to the boundary, and if $n = 3$ they are simply branched. It should be emphasized that in this theory we are considering not only area minimizing minimal surfaces bounded by $\alpha \in A$, but <u>all</u> critical points of the area functional defined on the space of surface spanning α which satisfy the classical monotonicity condition along the boundary.

Finally there are some other surprising consequences of this index formula. For example minimal surfaces in $\mathbb{R}^3$ are free of interior branch points if they minimize area [1,6,10], whereas most minimal surfaces with simple interior branch points are stable with respect to perturbations of the boundary [4.3]. Second, for $n \geq 4$ minimal surfaces in $\mathbb{R}^n$ may have branch points even if they are area minimizing but for such n <u>no</u> such minimal surface in $\mathbb{R}^n$ is stable as a branched surface under perturbation of the boundary.

THE MINIMAL SURFACE VECTOR FIELD X

Let $N = A \times D$ be the bundle over A introduced in the last section. Let $\alpha \in A$ and Γ^α its image. Consider the manifold of maps $H^s(S^1, \Gamma^\alpha)$. In [14] it is shown that $H^s(S^1, \Gamma^\alpha)$ is a C^{r-s} submanifold of $H^s(S^1, \mathbb{R}^n)$. Let $N(\alpha)$ denote the component of $H^s(S^1, \Gamma^\alpha)$ determined by α. (In [14], the notion N_α was used for $N(\alpha)$). Recall that the tangent space to $N(\alpha)$ at the point $x \in N(\alpha)$ can be identified with the H^s maps $h : S^1 \to \mathbb{R}^n$ with $h(\theta) \in T_{x(\theta)}\Gamma^\alpha$ (the tangent space to Γ^α at $x(\theta)$). By harmonic extension we can identify elements of $N(\alpha)$ with harmonic surfaces spanning α. We shall always assume this identification.

In [14] it was shown[1)] that there exists a smooth C^{r-s-1} vector field X_α, on $N(\alpha)$ whose zeros are precisely the minimal surfaces spanning Γ^α. We should note here that each of these zeros are minimal surfaces in a more general sense than classically defined, since a zero u of X_α viewed as a harmonic map $u : D \to \mathbb{R}^n$ need not induce a homeomorphism of S^1 onto Γ^α. We have the following theorem which is of great importance to us.

THEOREM 3.1: If $X_\alpha(u) = 0$, then the Frechêt derivative of X_α at u maps $TN(\alpha)$ into itself and is of the form: identity + a compact linear operator. Thus each X_α is a Rothe field.

We can give a definition of the vector field X_α as follows. Let $T_i : S^1 \to \mathbb{R}^n$, $i = 1, \dots n$ be a smooth framing of Γ^α; i.e. for each $p \in \Gamma^\alpha$, $\{T_i(p)\}_{i=1}^n$ forms an orthonormal basis for $\mathbb{R}^n$. We shall assume that T_i is always tangential so that $T_1(p) \in T_p\Gamma^\alpha$. Then the vector field is

1) In this paper the author took $s = 2$; however there is vertually no difference in the proofs for $s > 2$

characterized by the following conditions. For each $u \in N(\alpha)$, $X(u) : D \to \mathbb{R}^n$ is harmonic so

$$\text{(i)} \quad \Delta X_\alpha(u) \equiv 0$$

satisfying the mixed Neumann-Dirichlet boundary conditions

$$(3.2) \qquad \begin{aligned} &\text{(ii)} \quad \frac{\partial X}{\partial r}\alpha(u) \cdot T_1(u) = \frac{\partial u}{\partial r} \cdot T_1(x) \\ &\text{(iii)} \quad X_\alpha(u) \cdot T_j(u) = 0 \ , \ j = 2, \ldots, n \end{aligned}$$

where $T_j(x)$ denotes the composition $T_j(u(\theta))$ and $\frac{\partial}{\partial r}$ denotes the normal or radial derivature along S^1 . We can paraphrase these boundary conditions as follows: Let $\Omega : \Gamma^\alpha \to OP(\mathbb{R}^n)$, the orthogonal projections on $\mathbb{R}^n$, be the C^{r-2} map such that $\Omega(p)$ is the projection of $\mathbb{R}^n$ onto $T_p\Gamma^\alpha$. Then (3.2) can be written as

$$(3.2') \qquad \Omega(u)\frac{\partial X_\alpha}{\partial r}(u) = \Omega(u)\frac{\partial u}{\partial r}$$

where $\Omega(u)$ again denotes the map $\theta \to \Omega(u(\theta))$.

Let $N^* = \bigcup_\alpha N(\alpha)$ and $\pi : N^* \to A$ be the natural projection map $\pi(N(\alpha)) = \alpha$. The space N^* (in [12] we used the notation N for N^*) has the structure of a smooth fibre bundle over A which is bundle equivalent to the product bundle $N = A \times D$ via the map $\omega : (\alpha, \varphi) \to \alpha \circ \varphi$ and hence N^* is globally trivial. One often identifies N and N^* via this trivialization.

The family of vector fields X_α induces a C^{r-s-1} vector field X on $N^*(N)$ by the rule $X(u) = X_\alpha(u)$ if $u \in N(\alpha)$. This vector field will be vertical in the sense that $X_\alpha(u) \in T_u N(\alpha)$. If $u \in N(\alpha)$ is a zero of X (and hence a zero of X_α) the Frechêt derivature of X can be viewed as a map $DX(u) : T_u N \to T_u N(\alpha)$.

The following facts concerning X and X_α are proved in [14].

<u>THEOREM 3.3:</u> If u is a zero of X_α then $DX_\alpha(u) : T_u N(\alpha) \to$

is of the form identity plus compact linear.

This result will eventually tie into the results of Section 1.

Before stating the next theorem we would like to discuss the induced action of the conformed group G on $N(\alpha)$. We have already stated that every $g \in G$ is of the form

$$g(z) = c \cdot \frac{z-a}{1-\bar{a}z} , \quad |z| \le 1 ,$$

where $|c| = 1$ and $|a| < 1$. G is a three dimensional Lie group which is not compact. G acts on $\mathcal{D}$ in the following manner. Let $\varphi : S' \to \mathbb{C}$. Define $g_{\#}(\varphi) \in H^s(S^1,\mathbb{C})$ by

$$g_{\#}(\varphi)(z) = \varphi(g(z)) .$$

For fixed $g, g_{\#}$ is a linear isomorphism of H^s to itself which fixes $\mathcal{D}$. Thus g induces a diffeomorphism of $\mathcal{D}$ to itself and hence of $N(\alpha)$ to itself. Again denote this diffeomorphism by $g_{\#}$. The correspondence $G \to \mathrm{Diff}(\mathcal{D})$ given by $g \to g_{\#}$ defines the action of G on $\mathcal{D}$.

This G-action is not smooth (c.f. [14], p. 39), and this, of course creates many technical difficulties for the theory. We now have

THEOREM 3.4: The vector field X_α (fixing α) is equivariant with respect to this G-action on $N(\alpha)$. This means that

$$\left[Dg_{\#}(u)\right] X_\alpha(g_{\#}(u)) = X_\alpha(u)$$

where D denotes the Frechêt derivature of the map $u \to g_{\#}(u)$.

The fact that each X_α is equivariant implies that the kernel of $DX_\alpha(u)$ at any zero u contains $T_u\mathcal{O}_u(G)$ the tangent space to the orbit of G through u. We must remark at this point that formally $T_u\mathcal{O}_u(G)$ is not a subspace of $T_uN(\alpha)$ unless u is of class H^{s+1}. However by the regularity result for minimal surfaces [8] all zeros are in fact of class H^r , the smoothness class of the boundary curve. Thus

$$\dim \mathrm{Ker}\, DX_\alpha(u) \geq 3 .$$

One can show that u will be non-degenerate if this is all the kernel. This is a consequence of a relationship between the Hessian of Dirichlet's functional and the derivative of X_α . Before stating this we need to introduce the weak Riemannian structure on the manifold $N(\alpha)$.

DEFINITION 3.4: Let $u \in N(\alpha)$ and $h,k \in T_u N(\alpha)$. Define $\langle\langle h,k\rangle\rangle_u$, the weak inner product of h and k over u by the formula

$$\langle\langle h,k\rangle\rangle_u = \int_{S^1} \langle \frac{\partial h}{\partial r} , k\rangle_{\mathbb{R}^3} = \sum_{i=1}^{3} \int_D \nabla h^i \cdot \nabla k^i$$

where again h and k are identified with harmonic extensions.

THEOREM 3.6: If $E_\alpha : N(\alpha) \to R$ denotes Dirichlet's functional (0.2) and u is a minimal surface then

(3.7) $$D^2 E_\alpha(u)(h,k) = \langle\langle DX_\alpha(u)[h],k\rangle\rangle$$

If $u \in M_\nu^\lambda$ (c.f. Section 2) then something more dramatic accurs.

THEOREM 3.8: If u is a minimal surface of branching type (λ,ν) then

$$\dim \mathrm{Ker}\, DX_\alpha(u) \geq 2|\lambda| + |\nu| + 3$$

COROLLARY 3.9: A branched minimal surface cannot be formally non-degenerate.

For a proof of (3.8) see [14] p. 92 or the appendix of [4].

We also have the following very important facts

THEOREM 3.10: The vector field X_α is the gradent of E_α with respect to $\langle\langle\ ,\ \rangle\rangle$. Thus

$$DE_\alpha(u)[h] = \langle\langle X_\alpha(u), h\rangle\rangle$$

and

THEOREM 3.11: The weak inner product $\langle\langle\ ,\ \rangle\rangle$ and Dirichlet's functional are G-invariant.

Let $\mathcal{O}_u(G)$ be the orbit of the conformal group through u . If u is a minimal surface $\mathcal{O}_u(G)$ will be a smooth manifold. Let $T_u\mathcal{O}_u(G)$ denote the tangent space to this orbit at u . Then $T_u[\mathcal{O}_u(G)]$ has an orthogonal complement $T_u[\mathcal{O}_u(G)]^\perp$ with respect to the weak Riemannian inner product $\langle\langle\ ,\ \rangle\rangle$. The equivariance of X_α , E_α and $\langle\langle\ ,\ \rangle\rangle$ under G implies the following.

THEOREM 3.12: At a zero u of X_α , $DX_\alpha(u) : T_u[\mathcal{O}_u(G)]^\perp \circlearrowleft$

COROLLARY 3.13: A minimal surface $u : D \to \mathbb{R}^3$ is non-degenerate if $DX_\alpha(u)$ restricted to $T_u[\mathcal{O}_u(G)]^\perp$ is an isomorphism.

THE MORSE NUMBER OF MINIMAL SURFACES SPANNING A WIRE IN $\mathbb{R}^3$

In this section we modify the vector field X_α in such a way that we can define a local rotation number about an isolated zero along the lines described in Section 2.

First it is obvious that due to the action of G no zero of X_α can be isolated. So by isolated, we of course mean isolated relative to some slice transverse to the orbists of G .

We begin with the following result again from [14].

THEOREM 4.1: The weak Riemannian structure $\langle\langle\ ,\ \rangle\rangle$ on $N(\alpha)$ induces a smooth C^2 geodesic spray. The corresponding exponential map $\exp_u : T_u N(\alpha) \to N(\alpha)$, $u \in N(\alpha)$ gives a local diffeomorphism of a neighborhood of zero in $T_u N(\alpha)$ onto a neighborhood of u in $N(\alpha)$. Moreover with respect to the G-action on $N(\alpha)$ and on $T_u N(\alpha)$

$$g_{\#}(\exp_u h) = \exp_{g_{\#}(u)} g_{\#} h$$

Let $u \in N(\alpha) \cap H^{s+1}(S^1, \mathbb{R}^3)$. Then the weak complement of $T_u \mathcal{O}_u(G)$, $T_u[\mathcal{O}_u(G)]^{\perp}$ is a subspace of $T_u N(\alpha)$. Let $E = \exp_u(V)$, V a neighborhood of 0 in $T_u[\mathcal{O}_u(G)]^{\perp}$. For V small E is a codimension three submanifold of $N(\alpha)$ transverse to the orbits of G .

Let $\tau_\rho : N(\alpha) \to N(\rho)$ be the bundle trivialization $\tau_\rho(u) = \rho(\alpha^{-1}(u))$. Then using τ_ρ we may regard the family of vector fields $\tilde{X}_\rho$ as a family of vector fields X_ρ on a fixed $N(\alpha)$ via the formula

$$\tilde{X}_\rho(u) = D\tau_\rho^{-1} X_\rho(\tau_\rho(u)) \ . \tag{4.2}$$

Thus for $\rho = \alpha$, $\tilde{X}_\rho = X_\alpha$.

This new family of vector of $N(\alpha)$ will have the property that its zeros will represent via τ_α minimal surfaces spanning Γ^ρ . It follows from (4.2) that this family will also have the property that for fixed ρ its derivature at a zero will be of the form identity plus compact linear. However since the zeros of X_ρ are not isolated neither will the zeros of $\tilde{X}_\rho$ be isolated, and so the local degree theory described in Section 1 could not apply. We rectify this situation by defining a new family Y_ρ on E by the formula

$$Y_\rho(\omega) = P_\omega \tilde{X}_\rho(\omega) \tag{4.3}$$

where $P_\omega : T_\omega N(\alpha) \to T_\omega E$ is the weak Riemannian orthogonal projection. Then from [14] p. 102 we have

THEOREM 4.4: For $\mathcal{E}$ sufficiently small and ρ sufficiently close to α we have that the zeros of Y_ρ coincide with the zeros of $\tilde{X}_\rho$ on $\mathcal{E}$ and will be non degenerate iff the corresponding zeros of the minimal surface vector field X_ρ are non-degenerate. Moreover for fixed ρ the derivature $DY_\rho(u)$ at a zero u will be of the form identity plus compact.

Now suppose we have fixed an α and let u_o be an isolated zero of X_α. Construct as above the family of vector fields Y_ρ on $\mathcal{E}$.

By passing to a coordinate neighborhood we may regard $\mathcal{E}$ as an open subset U of a Banach space $\mathbb{E}$ and Y_ρ as a family of Rothe vector fields on U. By slightly shrinking U we may regard Y_ρ as a family

$$Y_\rho : \overline{U} \to E$$

as a family of Rothe vector field on $\overline{U}$. On the other hand since $DY_\alpha(u_o)$ is Fredholm of index zero Smale's local properness result for Fredholm maps yields

THEOREM 4.5: For $\mathcal{E}$ sufficiently small and ρ sufficiently close to α the family Y_ρ will be proper in the sense that if $\rho_n \to \rho$ and $Y_{\rho_n}(u_n) \to \nu$ then u_n has a convergent subsequence.

COROLLARY 4.6: For each fixed ρ the map $Y_\rho : \overline{U} \to E$ is proper and moreover for ρ sufficiently close to α and $\mathcal{E}$ sufficiently small $0 \notin Y_\rho(\partial\overline{U})$.

We are now sufficiently equipped to define the local degree of X_α about an isolated minimal surface.

DEFINITION 4.7: For u_o an isolated minimal surface choose $\mathcal{E}$ so that the consequences of (4.5) and (4.6) hold. We then define the local degree of X_α about u_o by

(4.8) $$\deg(X_\alpha, u_o) = \deg(Y_\alpha, u_o) = \deg(Y_\alpha, \bar{U})$$

If Γ^α has the (generic) property that only a finite number of minimal surfaces span it, say $u_1 \ldots u_m$ we define the Morse number of minimal surfaces spanning Γ^α by the formula

(4.9) $$\text{Morse}(\Gamma^\alpha) = \sum_{j=1}^{m} \deg(X_\alpha, u_j)$$

REMARK: Although we have (in this paper) mostly restricted our attention to curves in $\mathbb{R}^3$ formula (4.9) makes perfectly good sense for any $\mathbb{R}^n$ as long as the set of minimal surfaces spanning Γ^α is isolated.

We now conclude this section with

THEOREM 4.10: For any wire Γ^α (in $\mathbb{R}^3$ or $\mathbb{R}^n$, $n \geq 4$) which admits only finitely many minimal surfaces of disc type which span it

$$\text{Morse}(\Gamma^\alpha) = 1$$

PROOF: By the index theorem of Böhme and the author [4] one can find a ρ arbitrarily close to α (if Γ^α lies in $\mathbb{R}^3$, Γ^ρ may have to lie in $\mathbb{R}^4$) such that the zeros of X_ρ are non-degenerate (c.f. introduction and (3.13)).

Let $u_1, \ldots u_m$ be the zeros of X_α and construct as in (4.5) and (4.6) open sets U_i such that $\deg(Y_\alpha, \bar{U}_i)$ and $\deg(Y_\rho, \bar{U}_i)$ are defined as in Section 1. For ρ sufficiently close to α either the regularity apriori results of Hildebrandt [8] or the condition (CV) of the author [14] together with the apriori estimates implies that all the zeros of Y_ρ will be in $\bigcup U_i$.

Also for ρ sufficiently close to α we can clearly construct an isotopy of embeddings ρ_t, $0 \leq t \leq 1$ such that ρ_t is close to α and $\rho_o = \rho$ and $\rho_1 = \alpha$.

By the homotopy property of degree

$$\deg(Y_\rho,\overline{U}_i) = \deg(Y_\alpha,\overline{U}_i) = \deg(Y_\alpha,u_i) \tag{4.11}$$

Let $v_1^i \dots v_{m_i}^i$ be the zeros of Y_ρ in U_i. Then by construction of degree

$$\deg(Y_\rho,\overline{U}_i) = \sum_{j=1}^{m_i}{}' \operatorname{sgn} DY_\rho(v_j^i)$$

By the author's theorem on the Morse number of minimal surfaces in $\mathbb{R}^n$, $n \geq 4$ (theorem (5.38) of [16]) it follows that

$$\sum_i \sum_{j=1}^{m_i} \operatorname{sgn} DY_\rho(v_j^i) = 1$$

which together with (4.11) yield the fact that $M(\Gamma^\alpha) = 1$.

A STRONG VERSION OF THE THEOREM OF MORSE SHIFFMAN TOMPKINS FOR $\mathbb{R}^3$

As before let $E_\alpha : N(\alpha) \to R$ be Dirichlet's functional with α fixed. In [14] the author proved using the result on the Morse number of minimal surfaces spanning a wire in $\mathbb{R}^n$, $n \geq 4$ that

THEOREM 5.1: Let $\alpha : S^1 \to \mathbb{R}^n$, $n \geq 4$ be an embedding. Suppose that there exists two minimal surfaces u_1 and u_2 spanning Γ^α which are isolated in $N(\alpha)$ and are strict minima for E_α. Then there exist at least one other minimal surface spanning Γ^α. If the set of minimal surfaces spanning Γ^α are isolated one of these other minimal surfaces cannot be a relative minimum.

This result generalizes that of Morse-Tompkins [9] and Shiffman [11] who proved it under the assumption of being isolated in the topology of $C^0 \cap H^1(D,\mathbb{R}^n)$.

It is not difficult to see that the same proof goes through if $\alpha : S^1 \to \mathbb{R}^3$. The essential lemma is the following and its proof can be found in Section 6 of [16].

LEMMA 5.2: If u_1 and u_2 are isolated minima for Dirichlet's integral E_α then

$$\deg(X_\alpha,u_1) = \deg(Y_\alpha,u_1) = \deg(Y_\alpha,u_2) = \deg(X_\alpha,u_2) = 1 \ .$$

From (5.2) the result follows easily since if there were no other minimal surface (5.2) implies that $\text{Morse}(\Gamma^\alpha) = 2$ a contradiction.

It also implies that if the rest, say $u_3 \ldots u_m$ are isolated not all could be strict minima, for if they were

$$\deg(X_\alpha,u_j) = 1$$

and then $M(\Gamma^\alpha) = M$, again a contradiction.

So we have

THEOREM 5.3: Theorem 5.1 holds for $n \geq 3$.

CONCLUDING REMARKS

In part I of this paper with the same title we described the Morse number of minimal surfaces spanning a generic wire Γ^α in $\mathbb{R}^n$, $n \geq 4$ in terms of the derivatives of our vector field X_α. To paraphrase this result let Γ^α be such a generic wire and let $u_1 \ldots u_m$ be the finite number of non-degenerate minimal surfaces which span Γ^α. Let $T_{u_i}[\mathcal{O}_{u_i}(G)]^\perp$ be the weak orthogonal complement of the tangent space to the orbit of the conformed group through u_i. Then

$$DX_\alpha(u_i) : T_{u_i}[\mathcal{O}_{u_i}(G)]^\perp \circlearrowleft$$

and is in (in the notation of Section 2)

$$GR_C\left(T_u[\mathcal{O}_{u_i}(G)]^\perp\right) \ .$$

Thus we have a well defined signum for $DX_\alpha(u_i)$ which is either ±1 according to whether $DX_\alpha(u_i)$ is in CR_C^+ or GR_C^-. Then

$$(6.1) \qquad \text{Morse}(\Gamma^\alpha) = \sum_{j=1}^{m} \text{sgn } DX_\alpha(u_i)$$

For curves in $\mathbb{R}^3$ we could only describe Morse(Γ^α) in terms of a local degree. Now let α be a curve in $\mathbb{R}^3$ and u be a minimal surface spanning Γ^α which has p-simple branch points $z_1 \ldots z_p$ in the interior of the disc D. Then we have that

$$(6.2) \qquad \dim \text{Ker } DX_\alpha(u) \geq 2p + 3$$

and (6.2) holds even if u is non-degenerate in the sense described in Section 2 ($\pi^\lambda : \Sigma^\lambda \to A$ is a local diffeomorphism about u).

The question is whether, in spite of the existence a forced kernel (6.2) a differential description of $\deg(X_\alpha, u)$ can be given only in terms of $DX_\alpha(u)$. The answer will require a much deeper study of the forced kernel, the so called forced Jacobi fields, and the relationship between these fields, the minimal surfaces vector field X_α, and the perturbation theory for minimal surfaces of disc type.

We conclude this section with a related question which fascinates the author and which he is, at this moment unable to answer.

Let u be a non-degenerate simply branched minimal surface spanning a wire Γ^α in $\mathbb{R}^3$. Then we know that u is isolated as a critical point of E_α in the manifold $N(\alpha)$.

We know from the index theorem that by perturbing α slightly to a wire ρ in $\mathbb{R}^4$ there will only be trully non-degenerate minimal surfaces spanning Γ^α, and certainly there will be a minimal surface v spanning Γ^α near u.

QUESTION: For any such perturbation ρ is there a unique V_ρ spanning Γ^α near u such that V_ρ depends differentiably on ρ ?

The answer is clearly yes for perturbations ρ which remain embeddings of S^1 into $\mathbb{R}^3$. But what happens if we perturb α to a curve in $\mathbb{R}^4$ is intimately connected with how the branch points might effect any bifurcation process.

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