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A FIXED POINT INDEX THEORY FOR SYMMETRIC PRODUCT MAPPINGS

Nancy Rallis

In this paper we develop a fixed point index theory for symmetric product mappings of ENR-spaces. For such mappings we show that an index can be defined which is an extension of the usual integer-valued fixed point index. Further, we show that the classical properties of the index hold in this setting: The index is additive, multiplicative and commutative, the index is preserving under homotopy, and finally, the index is equal to the Lefschetz number as defined by Maxwell [9].

1. INTRODUCTION. Let X be a topological space and X^n the n^{th} cartesian product. Given a group G of permutations of the numbers $[1,2,\ldots,n]$, the n^{th} *symmetric product*, $SP_G^n X$, *of* X *with respect to* G is the orbit space of the action of G on X^n with the identification topology. A continuous map of the form

$f : X \longrightarrow SP^n_G X$ is called a <u>symmetric product mapping</u>. A point $x \in X$ is said to be a <u>fixed point of</u> f if x is a coordinate of $f(x)$.

Many of the results of classical fixed point theory generalize for fixed points of symmetric product mappings. C. N. Maxwell, for instance, defined the notion of a Lefschetz number for symmetric product mappings of compact polyhedra and established a Lefschetz fixed point theorem for such maps [9]. S. Masih [7] and C. Vora [14] each extends these results, respectively, to compact symmetric product mappings of metric ANRs and to compact symmetric mappings of metric manifolds.

Recently interest has turned towards generalizing the above Lefschetz theory to a fixed point index theory for symmetric product mappings. S. Masih, for example, defines by means of chain maps an index for symmetric product mappings of compact polyhedra and shows that most of the basic properties of the index hold [8]. S. Kwasik, appealing to the Poincaré duality principal and to the notion of transfer, develops a coincidence theory for symmetric product mappings of manifolds without boundary [6].

In this paper we develop a fixed point index theory for symmetric product mappings of euclidean neighborhood retracts (ENRs) which generalizes the work of Dold [3]. The main tool in defining the index--the trace homomorphism--involves the notion of the transfer. Having described this homomorphism in section 2, we apply it in section 3 in order to define the index for symmetric product maps of the form $g : V \longrightarrow SP^n_G\mathbb{R}^m$, where V is open in $\mathbb{R}^m$. We show that this index satisfies formulae and properties analogous to those in the classical case. One of these properties--the invariance of the index under commutativity--allows us to extend this theory to symmetric product mappings of ENRs. We conclude by showing that for compact symmetric product mappings of ENRs, the index is equal to the Lefschetz number as defined by Maxwell.

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2. <u>THE TRACE HOMOMORPHISM</u>. In this section we describe the trace homomorphism as well as introduce notation and terminology which shall be used throughout this paper; more detail may be found in [1, III], [9] and [10].

Let H_* denote singular homology with rational coefficients unless otherwise stated.

Let $\eta : X^n \longrightarrow SP^n_G X$ be the identification map with $\eta(x_1,\ldots,x_n) = [x_1,\ldots,x_n]$ and $\pi_i : X^n \longrightarrow X$ the i^{th} projection.

For $x \in X$ and $y \in SP^n_G X$, we say that x is a coordinate of y if $\eta(z) = y$ implies that $x = z_i$ (where $z = (z_1, \ldots, z_n) \in X^n$) for some $1 \leqslant i \leqslant n$. Let $x \in y$ read x is a coordinate of y. Then a point $x \in X$ is a fixed point of $f : X \longrightarrow SP^n_G X$ if $x \in f(x)$.

A map $f : X \longrightarrow Y$ induces maps $f^n : X^n \longrightarrow Y^n$ and $\tilde{f} : SP^n_G X \longrightarrow SP^n_G Y$ where $f^n(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$ and $\eta^Y f^n = \tilde{f}\eta^X$.

Let X be a compact polyhedron and $A \subset X$ a subpolyhedron which is closed under the action of G. From Bredon [1, p. 119] there exists a natural homomorphism

$$\tau_* : H_*(SP^n_G X, SP^n_G A) \longrightarrow H_*(X^n, A^n)$$

such that

$$\begin{aligned} \eta_*\tau_* &= |G| : H_*(SP^n_G X, SP^n_G A) \longrightarrow H_*(SP^n_G X, SP^n_G A) \\ \tau_*\eta_* &= \sum_{g \in G} g_* : H_*(X^n, A^n) \longrightarrow H_*(X^n, A^n) \end{aligned} \qquad (*)$$

where $|G|$ denotes the order of the group G. The homomorphism τ_* is referred to as the transfer.

We now define the following homomorphism,

$$\mu = \frac{\sum_{i=1}^{n} \pi_{i_*} \tau_*}{|G|} : H_*(SP_G^n X, SP_G^n A) \longrightarrow H_*(X,A),$$

which we shall call the trace homomorphism. We note that μ is natural. Further it can be shown that

$$\mu\eta_* = \sum_{i=1}^{n} \pi_{i_*}. \qquad (**)$$

3. <u>THE FIXED POINT INDEX IN EUCLIDEAN SPACE</u>. The notion of local degree of a self-mapping now readily extends to symmetric product mappings. For $f : V \longrightarrow SP_G^n S^m$, where V is open in S^m, we define the preimage set of a point $y \in S^m$ by $f_y^{-1} = \{x \in V : y \in f(x)\}$. Assuming that f_y^{-1} is compact and modifying the usual definition of local degree by the trace homomorphism we obtain the notion of the local degree of f about the point y. All properties of the local degree are preserved in this more general setting. In particular, additivity holds. Consequently, we obtain a measure of the "number of points in f_y^{-1}". So for a symmetric product mapping the fixed point set can be measured by finding the local degree of an appropriate difference map. This local degree will be the fixed point index. We proceed now to give a formal definition of this concept.

Regard S^m as $\mathbb{R}^m \cup \{\infty\}$, $m \geq 1$. Let $i : \mathbb{R}^m \longrightarrow S^m$ be the inclusion with $i(0) = P \in S^m$.

From section 2 there is a homomorphism

$$\tau_* : H_*(SP^n_G S^m) \longrightarrow H_*((S^m)^n)$$

satisfying both conditions in (*). Using this homomorphism together with the long exact sequences of $(SP^n S^m, SP^n(S^m - P))$ and $(S^m, S^m - P)$ and the contractibility of $SP^n(S^m - P)$ and $(S^m - P)$, we construct a homomorphism

$$\tau_*' : H_*(SP^n_G S^m, SP^n_G(S^m - P)) \longrightarrow H_*((S^m)^n, (S^m - P)^n)$$

satisfying the conditions of (*). In turn, the trace homomorphism, described in section 2,

$$\mu : H_*(SP^n_G S^m, SP^n_G(S^m - P)) \longrightarrow H_*(S^m, S^m - P)$$

is obtained.

3.1 DEFINITION. *Let $V \subset \mathbb{R}^m$ be open and $g : V \longrightarrow SP^n_G \mathbb{R}^m$. Assume $F_g = \{x \in V : x \in g(x)\}$, the fixed point set of g, is compact. Consider the composition*

$$H_m(V,V-F_g) \xrightarrow{d_*} H_m(SP^n_G\mathbb{R}^m, SP^n_G(\mathbb{R}^m-0))$$

$$\xrightarrow{\tilde{i}_*} H_m(SP^n_GS^m, SP^n_G(S^m-P))$$

$$\xrightarrow{\mu} H_m(S^m, S^m-P) \overset{j_*^{-1}}{\cong} H_mS^m$$

where $d : V \longrightarrow SP^n_G\mathbb{R}^m$ *is defined by* $d(x) = [x-y_1,\ldots,x-y_n]$ *if* $g(x) = [y_1,\ldots,y_n]$, $\tilde{i} : SP^n_G\mathbb{R}^m \longrightarrow SP^n_GS^m$ *is induced by the inclusion* i, μ *is the trace homomorphism and* $j : S^m \longrightarrow (S^m, S^m-P)$ *is inclusion. Then for a generator* O_m *of* $H_m(S^m)$ *and the fundamental class* $O_{F_g} \in H_m(V,V-F_g)$ *associated with it*

$$j_*^{-1}\mu\tilde{i}_*d_*(O_{F_g}) = I_g \cdot O_m$$

where I_g *is a uniquely determined rational number. This number will be called the fixed point index of* g.

For $n = 1$, μ is the identity homomorphism and so the above definition coincides with Dold's fixed point index [3, 1.2].

We now list some properties of the index. In what follows V is open in $\mathbb{R}^m$ and d, μ, i and j are the maps described above. The first property is an immediate generalization of Dold [3, 1.3] and the proof will be omitted.

3.2 LOCALIZATION. *For* $g : V \longrightarrow SP^n_G\mathbb{R}^m$, *assume that* F_g *is compact and* $F_g \subset K \subset W \subset V$, *where* K *is compact and* W *is open. Then* $j_*^{-1}\mu\tilde{\iota}_*(d|(W,W-K))_*(o_k) = I_g \cdot o_m$.

Suppose next that the symmetric product mapping $g : V \longrightarrow SP^n_G\mathbb{R}^m$ factors through $\mathbb{R}^{mn}$. Then I_g is given as follows:

3.3 FACTORIZATION. *Let* $g : V \longrightarrow SP^n_G\mathbb{R}^m$ *and* $\psi_p : V \longrightarrow \mathbb{R}^m$, *for* $1 \leqslant p \leqslant n$, *be such that* $g = \eta(\psi_1,\ldots,\psi_n)$. *Assume each* F_{ψ_p} *is compact. Then* $F_g = \bigcup_{p=1}^{n} F_{\psi_p}$ *and* $I_g = \sum_{p=1}^{n} I_{\psi_p}$.

PROOF. By expression (**) of section 2 we have that

$$
\begin{aligned}
I_g \cdot o_m &= j_*^{-1}\mu\tilde{\iota}_* d_*(o_{F_g}) = j_*^{-1}\mu\eta_*\iota^n(i-\psi_1,\ldots,i-\psi_n)(o_{F_g}) \\
&= \sum_{p=1}^{n} j_*^{-1}\iota_*(i-\psi_p)_*(o_{F_g})
\end{aligned}
$$

where $i-\psi_p : V \longrightarrow \mathbb{R}^m$ is defined by $(i-\psi_p)(x) = x-\psi_p(x)$. Then by property 3.2 above it follows that

$$I_g \cdot O_m = \sum_{p=1}^{n} j_*^{-1} i_* (i-\psi_p)_* (O_{F_{\psi_p}}) = (\sum_{p=1}^{n} I_{\psi_p}) \cdot O_m. \quad \blacksquare$$

If the map g factors through $\mathbb{R}^{mn}$ via constants then the preceding property, together with Dold [3, 1.4], gives:

3.4 UNITS. *Let* $g : V \longrightarrow SP_G^n \mathbb{R}^m$ *be such that* $g = \eta(c_1, \ldots, c_n)$ *where* $c_i : V \longrightarrow \mathbb{R}^m$ *is a constant mapping, for* $1 \leqslant i \leqslant n$. *Then* I_g *is equal to the number of* c_i*'s intersecting* V *each counted with their multiplicity*.

The next property describes the local nature of the index; specifically, it tells us that the "global index is equal to the sum of the local indices". The property following states that the index remains invariant during a deformation provided the fixed points stay away from the boundary of V. Both properties are immediate generalizations of Dold [3, 1.5] and [3, 1.7] and so are presented without proof.

3.5 ADDITIVITY. *Let* $g : V \longrightarrow SP_G^n \mathbb{R}^m$. *Assume* V *is the finite union of open sets* V_p, $1 \leqslant p \leqslant r$, *such that every* $F_{g|V_p} = \{x \in V_p : x \in g(x)\}$ *is compact and*

$F_{g|V_p} \cap F_{g|V_q} = \emptyset$ <u>for</u> $p \neq q$. <u>Then</u> $F_g = \bigcup_{p=1}^{r} F_{g|V_p}$ <u>and</u> $I_g = \sum_{p=1} I_{g|V_p}$.

3.6 <u>HOMOTOPY INVARIANCE</u>. <u>Let</u> $g_t : V \longrightarrow SP^n_G\mathbb{R}^m$, $0 \leq t \leq 1$, <u>be a deformation such that</u> $\bigcup_{0\leq t\leq 1} F_{g_t} = \{x \in V : x \in g_t(x)$ <u>for some</u> $t\}$ <u>is compact</u>. <u>Then</u> $I_{g_0} = I_{g_1}$.

In formulating a multiplicative property we need to introduce the following map: For spaces X and Y define a map $\kappa : SP^r_G X \times SP^n_G Y \longrightarrow SP^{rn}_G(X\times Y)$ by $\kappa([x_1,\ldots,x_r],[y_1,\ldots,y_n]) = [(x_i,y_j) \mid 1 \leq i \leq r, 1 \leq j \leq n]$. We shall refer to maps of this type as <u>separation maps</u>.

3.7 MULTIPLICATIVITY. <u>Let</u> $U \subset \mathbb{R}^q$ <u>and</u> $V \subset \mathbb{R}^m$ <u>be open and</u> $f : U \longrightarrow SP^r_G\mathbb{R}^q$ <u>and</u> $g : V \longrightarrow SP^n_G\mathbb{R}^m$. <u>Assume that the fixed point sets</u> F_f <u>and</u> F_g <u>are both compact</u>. <u>Consider the composite</u>

$$U \times V \xrightarrow{f\times g} SP^r_G\mathbb{R}^q \times SP^n_G\mathbb{R}^m \xrightarrow{\kappa} SP^{rn}_G(\mathbb{R}^q\times\mathbb{R}^m)$$

<u>where</u> κ <u>is a separation map</u>. <u>Then</u> $F_{\kappa(f\times g)} = F_f \times F_g$ <u>and</u> $I_{\kappa(f\times g)} = I_f \cdot I_g$.

<u>PROOF</u>. The proof follows from the commutativity of Figure 1 below, where the α's are Künneth homomorphisms, the ρ's are induced by $S^q \times S^m \longrightarrow S^q \vee S^m \longrightarrow \dfrac{S^q \times S^m}{S^q \wedge S^m} \approx S^{q+m}$ [2, p. 19] and all other homomorphisms are as described above. We omit the straightforward details.

By applying the above properties to a set of appropriate deformations we next establish a commutativity property for the index. In order to obtain the necessary deformations we will need the following result.

3.8 <u>LEMMA</u>. $SP^n_G\mathbb{R}^m$ <u>is an absolute retract</u> (AR).

The proof of this lemma follows immediately from basic results in the theory of retracts. The space $SP^n_G\mathbb{R}^m$ is metrizable, moreover, locally finitely triangulable (see Maxwell [9]). Hence by Hu [5, p. 98] $SP^n_G\mathbb{R}^m$ is an ANR. But $SP^n_G\mathbb{R}^m$ is contractible. Therefore $SP^n_G\mathbb{R}^m$ is an AR.

$$
\begin{tikzcd}[column sep=small]
H_{q+m}(U\times V, U\times V-F_f\times F_g) \arrow[d, "d_*"] \arrow[drr, "(d\times d)_*"] & & & H_q(U,U-F_f)\otimes H_m(V,V-F_g) \arrow[lll, "\alpha"'] \arrow[d, "d_*\otimes d_*"] \\
H_{q+m}(SP_G^{rn}(\mathbb{R}^q\times\mathbb{R}^m), SP_G^{rn}(\mathbb{R}^q\times\mathbb{R}^m-0)) \arrow[d, "\tilde{\imath}_*"] \arrow[dr, "\widetilde{(\imath\times\imath)}_*"] & & \begin{array}{c} H_{q+m}(SP_G^r\mathbb{R}^q\times SP_G^n\mathbb{R}^m, (SP_G^r\mathbb{R}^q\times SP_G^n(\mathbb{R}^m-0) \\ \cup\ SP_G^r(\mathbb{R}^q-0)\times SP_G^n\mathbb{R}^m)) \end{array} \arrow[ll, "\kappa_*"'] \arrow[d, "(\tilde{\imath}\times\tilde{\imath})_*"] & H_q(SP_G^r\mathbb{R}^q, SP_G^r(\mathbb{R}^q-0))\otimes H_m(SP_G^n\mathbb{R}^m, SP_G^n(\mathbb{R}^m-0)) \arrow[l, "\alpha"', "\cong"] \arrow[d, "\tilde{\imath}_*\otimes\tilde{\imath}_*"] \\
H_{q+m}(SP_G^{rn}S^{q+m}, SP_G^{rn}(S^{q+m}-P\times P')) \arrow[d, "\mu"] & \begin{array}{c} H_{q+m}(SP_G^{rn}(S^q\times S^m), \\ SP_G^{rn}(S^q\times S^m-P\times P')) \end{array} \arrow[l, "\tilde{\rho}_*"'] \arrow[d, "\mu"] & \begin{array}{c} H_{q+m}(SP_G^rS^q\times SP_G^nS^m, \\ SP^rS^q\times SP_G^n(S^m-P')\cup(SP_G^r(S^q-P)\times SP_G^nS^m)) \end{array} \arrow[l, "\kappa_*"'] & H_q(SP_G^rS^q, SP_G^r(S^q-P))\otimes H_m(SP_G^nS^m, SP_G^n(S^m-P')) \arrow[l, "\alpha"', "\cong"] \arrow[d, "\mu\otimes\mu"] \\
H_{q+m}(S^{q+m}, S^{q+m}-P\times P') & H_{q+m}(S^q\times S^m, S^q\times S^m-P\times P') \arrow[l, "\rho_*"', "\cong"] \arrow[d, "j_*"] & & H_q(S^q,S^q-P)\otimes H_m(S^m,S^m-P') \arrow[ll, "\alpha"', "\cong"] \\
H_{q+m}(S^{q+m}) \arrow[u, "j_*"', "\cong"] & H_{q+m}(S^q\ S^m) \arrow[l, "\rho_*"', "\cong"] & & H_q(S^q)\otimes H_m(S^m) \arrow[ll, "\alpha"', "\cong"] \arrow[u, "j_*\otimes j_*"', "\cong"]
\end{tikzcd}
$$

Figure 1

3.9 <u>COMMUTATIVITY</u>. *Let* $U \subset \mathbb{R}^q$ *and* $V \subset \mathbb{R}^m$ *be open and let* $f : U \longrightarrow \mathbb{R}^m$ *and* $g : V \longrightarrow SP^n_G\mathbb{R}^q$. *Consider the two composites*

$$gf : U' = f^{-1}(V) \longrightarrow SP^n_G\mathbb{R}^q$$

and

$$\widetilde{fg} : U'' = g^{-1}(SP^n_G U) \longrightarrow SP^n_G\mathbb{R}^m.$$

Assume F_{gf} *is compact and* $gf(F_{gf}) \subset SP^n_G U$. *Then* f *maps* F_{gf} *onto* $F_{\widetilde{fg}}$ *and* $I_{gf} = I_{\widetilde{fg}}$.

<u>PROOF</u>. First it is shown that f maps F_{gf} onto $F_{\widetilde{fg}}$. For any $x \in F_{gf}$ we have, by the assumption, that $f(x) \in \widetilde{fg}(f(x))$. Hence f maps F_{gf} into $F_{\widetilde{fg}}$. Next, take any $y \in F_{\widetilde{fg}}$. Then there exists a $x_i \in g(y)$ such that $f(x_i) = y$ and $x_i \in gf(x_i)$. So the first assertion is shown.

In order to prove that $I_{gf} = I_{\widetilde{fg}}$ we define a map

$$\Phi : U \times V \longrightarrow SP^n_G(\mathbb{R}^q \times \mathbb{R}^m),$$

$$(x,y) \longrightarrow \kappa(g(y),f(x))$$

where $\kappa : (SP^n_G \mathbb{R}^q) \times \mathbb{R}^m \longrightarrow SP^n_G(\mathbb{R}^q\ \mathbb{R}^m)$ is the separation map defined by $([x_1,\ldots,x_r],z) \longrightarrow [(x_i,z) \mid 1 \leqslant i \leqslant n]$. We show that $I_\Phi = I_{gf}$ and $I_\Phi = I_{\widetilde{fg}}$.

To begin we consider the following shaded region of $U' \times U'' \times [0,1] \times [0,1]$ (see Figure 2)

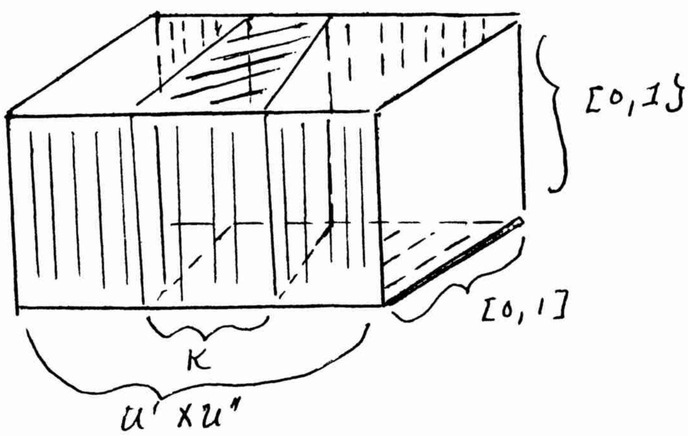

Figure 2

where $K = \{(x,f(x),t,s) : x \in U',\ f(x) \in U'',\ t \in [0,1],\ s \in [0,1]\}$. From this region, which is a closed subspace of $U' \times U'' \times [0,1] \times [0,1]$, we define a map into $SP^n_G \mathbb{R}^q$ as follows: on the front side of the cube $(x,y,0,s) \longrightarrow gf(x)$; on the back $(x,y,1,s) \longrightarrow g(y)$; along the bottom there runs a homotopy from $gf(x)$ to $g(y)$ through the diagonal element $[x,\ldots,x]$ defined by

$$(x,y,t,0) \longrightarrow \begin{cases} [(1-2t)z_1+2tx,\dots,(1-2t)z_n+2tx] & 0 \leqslant t \leqslant 1/2, \\ [(2t-1)y_1+(2-2t)x,\dots,(2t-1)y_n+(2-2t)x] & 1/2 \leqslant t \leqslant 1, \end{cases}$$

where $gf(x) = [z_1,\dots,z_n]$ and $g(y) = [y_1,\dots,y_n]$ for all $(x,y) \in U' \times U''$; and along K a deformation defined by

$$(x,f(x),t,s) \longrightarrow \begin{cases} [(1-2t)y_1+2t(sy_1+(1-s)x), \\ \quad \dots,(1-2t)y_n+2t(sy_n+(1-s)x)] & 0 \leqslant t \leqslant 1/2, \\ [(2t-1)y_1+(2-2t)(sy_1+(1-s)x, \\ \quad \dots,(2t-1)y_n+(2-2t)(sy_n+(1-s)x)] & 1/2 \leqslant t \leqslant 1, \end{cases}$$

where $gf(x) = [y_1,\dots,y_n]$ for $x \in U'$. This latter deformation runs from a homotopy on the bottom of K $(s = 0)$ that takes gf into itself to a map on the top of K $(s = 1)$ given by $(x,y,t) \longrightarrow gf(x)$ for all $t \in [0,1]$.

By Lemma 3.8 $SP^n_G\mathbb{R}^q$ is an AR. So the map described above extends to the entire cube. Let $H : U' \times U'' \times [0,1] \longrightarrow SP^n_G\mathbb{R}^q$ be the restriction of this extension to the top face of the cube. Then $H(x,y,0) = gf(x)$, $H(x,y,1) = g(y)$ and $H(x,f(x),t) = gf(x)$ for all $t \in [0,1]$.

A continuous homotopy $\psi_t : U' \times U'' \longrightarrow SP^n_G(\mathbb{R}^q\ \mathbb{R}^m)$ can now be defined by $\psi_t(x,y) = \kappa(H(x,y,t),f(x))$ with $\psi_0(x,y) = \kappa(gf(x),f(x))$ and $\psi_1(x,y) = \kappa(g(y),f(x)) = \Phi(x,y)$. Further, since $\psi_t(x,f(x)) = \kappa(gf(x),f(x))$ for all t, $\bigcup_{0\leqslant t\leqslant 1} F_{\psi_t} = \{(x,y) \in U' \times U'' : x \in F_{gf}$ and $y = f(x)\}$. Hence, by assumption, the fixed point set of the deformation ψ_t is compact and so by Homotopy Invariance (3.6), $I_{\psi_0} = I_{\psi_1} = I_\Phi$. But ψ_0 is a restriction of the map $\gamma : U' \times \mathbb{R}^m \longrightarrow SP^n_G(\mathbb{R}^q \times \mathbb{R}^m)$ defined by $\gamma(x,y) = \kappa(gf(x),f(x))$. Further, $F_{\psi_0} = F_\gamma$ since by assumption $gf(F_{gf}) \subset SP^n_G U$. So, by Localization (3.2), $I_{\psi_0} = I_\gamma$.

We next define a deformation of γ by $\gamma_t(x,y) = \kappa(gf(x),(1-t)f(x))$ for $(x,y) \in U' \times \mathbb{R}^m$. The fixed point set of this deformation coincides with the image of $h : F_{gf} \times [0,1] \longrightarrow U' \times \mathbb{R}^m$ where $h(x,t) = (x,(1-t)f(x))$. This set is compact and so by Homotopy Invariance (3.6), $I_{\gamma_1} = I_{\gamma_0} = I_\gamma$. But $\gamma_1(x,y) = \kappa(gf(x),0)$. So by Units (3.4) and by Multiplicativity (3.7), $I_{\gamma_1} = I_{gf}\cdot I_{\text{constant}} = I_{gf}$. Consequently, $I_\Phi = I_{gf}$.

To show that $I_\Phi = I_{\widetilde{fg}}$ and thus complete the proof we use an argument similar to the one above (specifically, we use the deformations

$$[(y_1, tf(y_1)+(1-t)f(x), \ldots, (y_n, tf(y_n)+(1-t)f(x))]$$

and

$$[((1-t)y_1, f(y_1)), \ldots, ((1-t)y_n, f(y_n))]$$

when $g(y) = [y_1, \ldots, y_n]$). ■

4. <u>FIXED POINT INDEX FOR EUCLIDEAN NEIGHBORHOOD RETRACTS (ENRs)</u>. Let X be an ENR-space. Then there exist an open set W in some euclidean space $\mathbb{R}^q$ and mappings $h : X \longrightarrow W$ and $r : W \longrightarrow X$ such that $rh = 1_X$. The symmetric product mapping $g : U \longrightarrow SP^n_G X$, where $U \subset X$ is open, admits the decomposition $U \xrightarrow{h} V \xrightarrow{gr} SP^n_G X$ where $V = r^{-1}(U)$. If the fixed point set of g is compact then $F_g \overset{h}{\cong} F_{\widetilde{hgr}|V}$ and hence $I_{\widetilde{hgr}|V}$ is defined by Definition (3.1).

Suppose there exist another open set W' in, say, $\mathbb{R}^m$ and mappings $h' : X \longrightarrow W'$ and $r' : W' \longrightarrow X$ such that $r'h' = 1_X$. It is claimed that $I_{\widetilde{hgr}|V} = I_{\widetilde{h'gr'}|V'}$ where $V' = (r')^{-1}(U)$. To show this we proceed as follows.

Set $\alpha = h'r : W \longrightarrow \mathbb{R}^m$ and $\beta = \widetilde{hr'}\widetilde{h'}gr' : V' \longrightarrow SP^n_G \mathbb{R}^q$. Take the composites

$$\beta\alpha : \widetilde{h}\widetilde{r}'\widetilde{h}'gr'h'r : \alpha^{-1}(V') = V \longrightarrow SP^n_G\mathbb{R}^q$$

and

$$\widetilde{\alpha\beta} : \widetilde{h}'\widetilde{r}\widetilde{h}\widetilde{r}'\widetilde{h}'gr' : \beta^{-1}(SP^n_G W) = V' \longrightarrow SP^n_G\mathbb{R}^m.$$

Clearly, $I_{\widetilde{h}gr|V} = I_{\beta\alpha}$ and $I_{\widetilde{h}'gr'|V'} = I_{\widetilde{\alpha\beta}}$. Since $\beta\alpha(F_{\beta\alpha}) \subset SP^n_G W$ we have by Commutativity (3.9) that $I_{\beta\alpha} = I_{\widetilde{\alpha\beta}}$. Consequently, $I_{\widetilde{h}gr|V} = I_{\widetilde{h}'gr'|V'}$ and the claim is shown.

So we define the index of g to be the rational number $I_{hgr|V}$ and denote it by I_g as before.

Properties 3.2 through 3.7 and 3.9 of the preceding section hold in this more general setting. The reformulations are clear and the proofs are obvious reductions of 3.2-3.7 and 3.9; hence are omitted.

We continue this section by extending the Lefschetz-Hopf Fixed Point Theorem to symmetric product mappings of ENRs. We recall that this theorem states that under suitable conditions the index of a self-mapping is equal to the Lefschetz number of that mapping. For symmetric product mappings the notion of a Lefschetz number is given as follows: *Let* X *be a compact* ENR *and* $g : X \longrightarrow SP^n_G X$. *The Lefschetz number of* g *is defined by*

$$L(g) = \sum_{p\in\mathbb{Z}} (-1)^p \operatorname{trace}(\mu_p g_{*p})$$

<u>where</u> $g_* : H_*(X;\mathbb{Q}) \longrightarrow H_*(SP^n_G X;\mathbb{Q})$ <u>and where</u> $\mu :$ $H_*(SP^n_G;\mathbb{Q}) \longrightarrow H_*(X;\mathbb{Q})$ <u>is the trace homomorphism</u> [7] <u>and</u> [10]. The Lefschetz-Hopf Fixed Point Theorem for symmetric product mappings states that $L(g)$ is equal to I_g. Before proving this theorem some algebraic preliminaries are given. A general reference for what follows can be found in [4, VII].

For every graded vector space $V = \{V_p\}_{p\in\mathbb{Z}}$ over $\mathbb{Q}$ and its dual $V^* = \{V_p^*\}_{p\in\mathbb{Z}}$, define homomorphisms $\theta : V^* \otimes V \longrightarrow \operatorname{Hom}(V,V)$ and $e : V^* \otimes V \longrightarrow Q$ respectively by $[\theta_p(\psi\otimes v)]v' = (-1)^p\psi(v')\cdot v$ and $e(\psi\times v) = \psi(v)$ where $\psi \in V_p^*$ and $v, v' \in V_p$. If V is a finitary graded vector space over $\mathbb{Q}$ and $\alpha : V \longrightarrow V$ an endomorphism of degree zero then θ is an isomorphism and

$$e(\theta^{-1}(\alpha)) = \sum_{p\in\mathbb{Z}} (-1)^p \operatorname{trace}(\alpha_p).$$

Next, suppose W and W' are finitely generated vector spaces over $\mathbb{Q}$ and $h : W \longrightarrow W$, $\phi : W \longrightarrow W'$, and $\psi : W' \longrightarrow W$ are homomorphisms such that $\psi\phi :$ $W \longrightarrow W$ is the identity; then $\operatorname{trace}(h) = \operatorname{trace}(\phi h\psi)$.

The Lefschetz-Hopf Fixed Point Theorem for symmetric product mappings of ENRs is now given. The methods used in the proof of this theorem are based on Dold [3, 4.1 and 4, VII, 6.13].

4.1 THEOREM. *Let* X *be a compact* ENR *and* $g : X \longrightarrow SP^n_G X$. *Then* F_g, *the fixed point set of* g, *is compact and* $I_g = L(g)$.

PROOF. For the ENR-space X there exist an open set W in some Euclidean space $\mathbb{R}^m$ and $X \xrightarrow{h} W \xrightarrow{r} X$ such that $rh = 1_X$. Then the index of g is equal to the index of $\widetilde{hgr} : W \longrightarrow SP^n_G \mathbb{R}^m$. Since $H_*(X)$ and $H_*(h(X))$ are finitary graded vector spaces and since $(r|h(X))_* h_* = 1_{H_*(X)}$ we have that $\operatorname{trace}(\mu g_*) = \operatorname{trace}(h_* \mu g_* (r|h(X))_*)$. So by the naturality of μ, $L(g) = L(\widetilde{hgr}|h(X))$. Set $Y = h(X)$. It suffices then to show that $I_{\widetilde{hgr}} = L(\widetilde{hgr}|Y)$.

Consider the following diagram (Figure 3),

$$
\begin{tikzcd}
\bigoplus\limits_{p+q=m} H_p(W,W-Y)\otimes H_q(W) \arrow[r, "1\otimes(\widetilde{hgr})_*"] & \bigoplus\limits_{p+q=m} H_p(W,W-Y)\otimes H_q(SP^n_G Y) \arrow[r, "1\otimes\mu"] \arrow[d, "\tilde{\iota}_*\gamma_{1_*}"] & \bigoplus\limits_{p+q=m} H_p(W,W-Y)\otimes H_q(Y) \arrow[r, "\beta\otimes 1"] \arrow[d, "\iota_*\gamma_{2_*}"] & \bigoplus\limits_{q\in\mathbb{Z}} (H_q(Y))^*\otimes H_q(Y) \arrow[d, "e"] \\
H_m(W,W-Y) \arrow[u, "\Lambda_*"] \arrow[r, "\tilde{\iota}_* d_*"] \arrow[ur, phantom, "(1)"] & H_m(SP^n_G S^m, SP^n_G(S^m-P)) \arrow[r, "\mu"] \arrow[ur, phantom, "(2)"] & H_m(S^m,S^m-P) \arrow[ur, phantom, "(3)"] & \mathbb{Q} \arrow[l, "O_m"', "\cong"]
\end{tikzcd}
$$

Figure 3

where Δ is the diagonal map, $d : (W, W-Y) \longrightarrow SP^n_G(\mathbb{R}^m, \mathbb{R}^m - 0)$ is defined by $x \longrightarrow [x-z_1, \ldots, x-z_n]$ when $\widetilde{hgr}(x) = [z_1, \ldots, z_n]$, $\gamma_1 : W \times SP^n_G Y \longrightarrow SP^n_G \mathbb{R}^m$ and $\gamma_2 : W \times Y \longrightarrow \mathbb{R}^m$ are difference maps defined respectively by $(x, [y_1, \ldots, y_n]) \longrightarrow [x-y_1, \ldots, x-y_n]$ and $(x,y) \longrightarrow x-y$, the i's inclusions of $\mathbb{R}^m$ into S^m, e is the homomorphism described above and β is defined by $([\beta(w)]\kappa)O_m = i_*\gamma_{2_*}(w \otimes \kappa)$ for $w \in H_*(W, W-Y)$ and $\kappa \in H_*(Y)$.

Subdiagram (1) commutes since it is induced by a commutative diagram at the level of spaces and maps. The homomorphism β is so defined as to make subdiagram (3) commute. We now turn to the commutativity of subdiagram (2). Let $\gamma_3 : W \times Y^n \longrightarrow \mathbb{R}^{mn}$ be defined by $(x, (y_1, \ldots, y_n)) \longrightarrow (x-y_1, \ldots, x-y_n)$. Then by expression (**) of section 2 we have that $\mu\eta_* i^n_* \gamma_{3_*} = i_*\gamma_{2_*}(1 \otimes \mu\eta_*)$. But $\eta_* i^n_* \gamma_{3_*} = \tilde{i}_*\gamma_{1_*}(1 \otimes \eta_*)$. So $\mu\tilde{i}_*\gamma_{1_*} = i_*\gamma_{2_*}(1 \otimes \mu)$ since the homomorphism $\eta_* : H_*(Y^n) \longrightarrow H_*(SP^n_G Y)$ is onto to expression (*) of section 2. Thus subdiagram (2) commutes, that is,

$$I_{\widetilde{hgr}} = e(\beta \otimes \mu \tilde{h}_* g_* r_*)\Delta_*(O_Y).$$

Therefore,

$$I_{\widetilde{hgr}} = \sum_{p\in\mathbb{Z}} (-1)^p \mathrm{trace}(\theta_p(\beta\otimes\mu\widetilde{h}_*g_*r_*)\Delta_*(O_Y)).$$

In order to complete the proof we need to show that $\theta(\beta\otimes\mu\widetilde{h}_*g_*r_*)\Delta_*(O_Y)) = \mu(\widetilde{hgr}|Y)_*$. To do this we consider the diagram (Figure 4) below, where $t(x,y) = (y,x)$. The commutativity of the left square is immediate. The commutativity of the right square is obtained by tensoring subdiagram (3) of diagram (Figure 3) with $\mu(\widetilde{hgr})_*$. So by the commutativity of this diagram we have that

$$\theta(\beta\otimes\mu\widetilde{h}_*g_*r_*)\Delta_*(O_Y) = \mu\widetilde{h}_*g_*r_*\Phi(Y,W)$$

where $\Phi(Y,W) = \{\Phi_i(Y,W)\}_{i\in\mathbb{Z}}$ is the composition

$$\begin{aligned}
\Phi_i(Y,W) : H_i(Y) &\xrightarrow{O_Y\times} H_{i+m}(W\times Y,(W-Y)\times Y)\\
&\xrightarrow{(\Delta\times 1)_*} H_{i+m}(W\times W\times Y,(W-Y)\times W\times Y)\\
&\xrightarrow{(1\times t)_*} H_{i+m}(W\times Y\times W,(W-Y)\times Y\times W)\\
&\xrightarrow{(i\gamma_2\times 1)_*} H_{i+m}(S^m\times W,(S^m-P)\times W) \xrightarrow{(O_m\times)^{-1}} H_m W.
\end{aligned}$$

In Dold [4, pp. 210-211] it is shown that $\Phi_i(Y,W)$ is induced by the inclusion map of Y into W. Therefore

$$
\begin{tikzcd}
\bigoplus_{p+q=m} (H_p(W,W-Y)\otimes H_q(W))\otimes H_\ell(Y) \arrow[r, "1\otimes t_*"] \arrow[d, "\beta\otimes\mu\otimes(\widetilde{hgr})_*\otimes 1"'] & \bigoplus_{p+q=m} H_p(W,W-Y)\otimes H_\ell(Y)\otimes H_q(W) \arrow[r, "i_*\gamma_{2_*}\otimes 1"] \arrow[d, "\beta\otimes 1\otimes\mu(\widetilde{hgr})_*"'] & H_m(S^m,S^m-P)\otimes H_\ell(W) \arrow[d, "\mu(\widetilde{hgr})_*"] \\
\bigoplus_{q\in\mathbb{Z}} (H_q(Y))^*\otimes H_q(Y)\otimes H_\ell(Y) \arrow[r, "1\otimes t_*"] & \bigoplus_{q\in\mathbb{Z}} (H_q(Y))^*\otimes H_\ell(Y)\otimes H_q(Y) \arrow[r, "e\otimes 1"] & \mathbb{Q}\otimes H_\ell(Y)
\end{tikzcd}
$$

Figure 4

$$\theta(\beta\otimes\mu\tilde{h}_*g_*r_*)\Delta_*(O_Y) = \mu(\tilde{h}gr|Y)_*$$

and the theorem is shown. ∎

Theorem 4.1, together with the Lefschetz Fixed Point Theorem [7] and [9], imply that if X is a compact ENR with $g : X \longrightarrow SP^n_G X$ fixed point free then $I_g = 0$.

As an analog to Dold [4, 6.22, p. 212] we relate the index to the Euler-Poincare characteristic. Let Y be a compact connected ENR-space and y_0 any arbitrary point of Y. For an integer k, $1 \leqslant k \leqslant n$, let $d_k : Y \longrightarrow Y^n$ be the identity on the first k factors and the constant value y_0 on the remaining $n-k$ factors. Set $\bar{d}_k = \eta d_k$. From properties 3.3 and 3.4 and Theorem 4.1 above we obtain

4.2 COROLLARY. *Let Y be a compact connected ENR and let $f : Y \longrightarrow SP^n_G Y$ be a mapping homotopic to $\bar{d}_k$, where d_k is as described above. Then $I_f = k\chi(Y) + (n-k)$ where $\chi(Y)$ is the Euler-Poincaré characteristic of Y.*

We note that in Masih [7] and Maxwell [9] a similar result is expressed in terms of the Lefschetz number of a symmetric product mapping.

If Y is a compact ENR such that $\tilde{H}(Y) = 0$ then for any map $f : Y \longrightarrow SP_G Y$, $I_f = n\chi(Y) = n$. Spaces satisfying this condition include contractible spaces and real projective spaces of even dimension.

For a further application of Theorem 4.1 we consider symmetric product mappings of complex projective space. We denote complex projective n-space by $P_n\mathbb{C}$ and we shall view $P_n\mathbb{C}$ as the set of all nonzero complex polynomials where two polynomials are identified if and only if they are proportional. As indicated in Dold [4, p. 193] the mapping

$$(P_1\mathbb{C})^n \longrightarrow P_n\mathbb{C},\ (az_1+b_1, a_2z+b_2, \ldots, a_nz+b_n) \longrightarrow \prod_{\nu=1}^{n} (a_\nu z+b_\nu)$$

induces a homeomorphism $SP_G^n(P_1\mathbb{C}) \overset{h}{\approx} P_n\mathbb{C}$, where G is the symmetric group.

Suppose $f : P_1\mathbb{C} = S^2 \longrightarrow P_n\mathbb{C}$ is a map. By the Simplicial Approximation Theorem there exists a map

$$S^2 \xrightarrow{\phi} S^2 = P_1\mathbb{C} \overset{i}{\subset} P_n\mathbb{C}$$

such that f and ϕ are homotopic. The <u>degree of</u> f, $\deg f$, can be defined by $\deg \phi$.

For the map

$$P_1\mathbb{C} = S^2 \xrightarrow{f} P_n\mathbb{C} \xrightarrow[\approx]{h^{-1}} SP^n_G P_1\mathbb{C}$$

say $\mu(h^{-1}f)_* : H_2(P_1\mathbb{C}) \longrightarrow H_2(P_1\mathbb{C})$ is multiplication by m. Then by Theorem 4.1

$$I_{h^{-1}f} = L(h^{-1}f) = n + m.$$

However, $I_{h^{-1}f} = I_{h^{-1}i\phi}$ and by Factorization (3.3) above, $I_{h^{-1}i\phi} = \deg \phi + (n-1) + n$. So $\deg f + (n-1) = m$.

4.3 <u>COROLLARY</u>. *Suppose* $f : P_1\mathbb{C} = S^2 \longrightarrow P_n\mathbb{C}$ *is a map and* h *is as described above. If* $\deg f \neq -2n+1$ *then* $h^{-1}f$ *has a fixed point.*

In the case that $n = 1$ we obtain the well known result that if $f : S^2 \longrightarrow S^2$ and $\deg f \neq -1$ then f has a fixed point.

We conclude by applying the above index theory to vector fields on manifolds. In what follows all manifolds are closed, that is, compact without boundary. Let M be a smooth manifold and $\tilde{M}$ a smooth n-sheeted covering of M with $\rho : \tilde{M} \longrightarrow M$ the covering map. Take a smooth vector field ν on $\tilde{M}$. For each $x \in \tilde{M}$, the tangent space $T_x\tilde{M} = T_{\rho(x)}M$. We define the map

$$\bar{\nu} : M \longrightarrow SP^n_G TM, \qquad \bar{\nu}(y) = [\nu(y_1),\ldots,\nu(y_n)]$$

where $\rho^{-1}(y) = [y_1,\ldots,y_n] \in SP^n_G\tilde{M}$. A point $y \in M$ is said to be a *singularity of* $\bar{\nu}$ if and only if for some i, $1 \leqslant i \leqslant n$, $\nu(y_i) = 0$ where $\rho^{-1}(y) = [y_1,\ldots,y_n]$. If the singularities of $\bar{\nu}$ are isolated we may associate with $\bar{\nu}$ a map $f : \tilde{M} \longrightarrow \tilde{M}$, homotopic to the identity, having as its fixed points the isolated set of points $\{z \in \tilde{M} : v(z) = 0\}$. We then define the map

$$\bar{f} : M \longrightarrow SP^n_G M, \qquad \bar{f}(y) = [\rho f(y_1),\ldots,\rho f(y_n)]$$

where $\rho^{-1}(y) = [y_1,\ldots,y_n]$. If $y \in M$ is an isolated singularity of $\bar{\nu}$ in an open neighborhood U we define the *index of* $\bar{\nu}$ *at* y, i_y, by $i_{\bar{f}|U}$. If $n = 1$ we obtain the classical definition of the index of a vector field at an isolated singularity. From property 3.5 and Corollary 4.2 we obtain

4.4 *COROLLARY. Let* M *be a closed smooth manifold and* M *a closed smooth n-sheeted covering of* M. *For a smooth vector field* ν *on* $\tilde{M}$ *let* $\bar{v}$ *be as described* above. *If the singularities of* $\bar{v}$ *are isolated then the sum of the indices of* $\bar{v}$ *is*

$$\sum_y i_y = I_{\bar{f}}$$

$$= n\chi(M).$$

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Nancy Rallis
Department of Mathematics
Boston College
Chestnut Hill, MA 02167

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