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ON INTERIOR REGULARITY AND LIOUVILLE'S THEOREM FOR HARMONIC MAPPINGS

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It is well known that the weakly harmonic mapping U: M → N (M,N: Riemannian manifolds) is regular if the image U(M) is contained in some sufficiently small ball and for this case Liouville's theorem is valid. In this paper we show that the smallness condition for U(M) can be released if U minimizes the energy functional and the sectional curvatures of the target manifold N are bounded by some suitable function of the distance from some fixed point of N.

0. Introduction

This paper deals with the problem of the regularity and the theorem of Liouville-type for weakly harmonic mapping U: M → N, whose energy is minimal.

Let M and N be Riemannian manifolds of dimention m and n, and class C^1 and C^3 respectively. Furthermore we assume that N is complete. For every C^1 map U: M → N we can define the energy

$$(0.1) \qquad E(U) = \int_M e(U)\, d\mu$$

where $d\mu$ denotes the volume element on M and

$$e(U) = \frac{1}{2} tr_M \langle U_*, U_* \rangle_N$$

the energy density of U, is the trace of the pull-back of metric tensor of N under the mapping U with respect to the metric tensor of M. In local coordinates it can be written in the form

$$e(U) = \frac{1}{2} h^{\alpha\beta} D_\alpha u^i D_\beta u^j g_{ij}(u)$$

where $(h^{\alpha\beta})$ is the inverse matrix of the metric tensor $(h_{\alpha\beta})$ of M, and (g_{ij}) is the metric tensor of N. Moreover we write $u = u(x)$ for a representation of the map $U: M \to N$ in local coordinates $x=(x^1,\dots,x^m)$ and $u=(u^1,\dots, u^n)$ on M and N respectively.

A mapping $U: M \to N$ is said to be harmonic if it is of class C^2 and satisfies Euler equation of the energy functional. In local coordinates it can be written in the form

$$(0.2)\qquad \Delta_M u^i + \Gamma_{j\,k}^{\,i}(u) D_\alpha u^j D_\beta u^k h^{\alpha\beta} = 0 \qquad 1 \leqq i \leqq n$$

where

$$\Delta_M = h^{-\frac{1}{2}} D_\alpha (h^{\frac{1}{2}} h^{\alpha\beta} D_\beta)$$

is the Laplace-Beltrami operator on M. Here $h = \det(h_{\alpha\beta})$, $\Gamma_{j\,k}^{\,i}$ are the components of Christoffel symbols for (g_{ij}).

For the general case there are the papers by Hildebrandt-Kaul-Widman[11] for the regularity, and by Hildebrandt-Jost-Widman[9] for the theorem of Liouville-type (see also Giaquinta-Hildebrandt[4]). Roughly speaking these works say that if the range U(M) of a weakly harmonic mapping U: M → N is contained in some geodesic ball B(Q,r) in N with radius $r \leqq \pi/2\sqrt{k}$ (k = maximum of the sectional curvatures of N) which does not meet the cut locus of the center Q, then U is regular and the theorem of Liouville-type is valid.

In this paper we show that the assumption that U(M) is contained in some small ball can be released if we suppose that the sectional curvatures K(P) is bounded by a sufficiently rapid decreasing function of the distance dist(P,Q) from some fixed point Q∈N. With this assumption we prove existence and interior regularity of a weakly harmonic mapping which minimizes the energy and a Liouville-type result for such mappings.

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1. Auxilary differential geometric estimates

Let K(P) be the maximum of the sectional curvatures of N at P. We consider the following condition for K(P):

(ASS. Q_0, f_0, r_0)

For some point $Q_0 \in N$, some positive constant $r_0 > 0$ and some function $f_0 \in C^2([0, r_0))$ such that

$$\lim_{t \to 0} f_0(t)/t = 1,\ f_0(t) > 0,\ f_0'(t) > 0 \text{ for all } t \in (0, r_0)$$

K(P) satisfy

$$K(P) \leqq \min\{-f_0''(r)/f_0(r),\ (1-f_0'(r)^2)/f_0(r)^2\}$$

where $r = \mathrm{dist}(P, Q_0) \leqq r_0$

Lemma 1.1

Assume that N satisfies (ASS. Q_0, f_0, r_0) and that the geodesic ball $B(Q_0, r_0)$ does not meet the cut locus of Q_0. Let $(u^1, \dots, u^n)$ be a normal coordinate system on $B(Q_0, r_0)$ such that Q_0 has the coordinates $(0, \dots, 0)$. Denote by $g_{ij}(u)$, $\Gamma_{ijk}(u)$ and $\Gamma_{j\ k}^{\ i}(u)$ the coordinates of the metric tensor and the Christoffel symbols in this coordinate system respectively. Then for all $\xi \in \mathbb{R}^n$ we have the following estimates

$$(1.2) \qquad 0 \leqq |u| \frac{f_0'(|u|)}{f_0(|u|)} g_{ij}(u)\xi^i \xi^j$$

$$\leqq g_{ij}(u)(\xi^i \xi^j + u^m \Gamma_{m\ k}^{\ i}(u)\xi^j \xi^k)$$

Proof

The estimates (1.2) follows from Rauch's comparison theorem, (as [10] Lemma 6) by comparing N with the n-dimensional manifold $\tilde{N}$ which has the following metric tensor with respect to some normal coordinate system (z^i)

$$\tilde{g}_{ij}(z) = \frac{z^i z^j}{|z|^2} + \frac{f_0(|z|)^2}{|z|^2}\left(\delta_{ij} - \frac{z^i z^j}{|z|^2}\right)$$

Writing $\tilde{f}_0(t)=f_0(t)/t$, Christoffel symbols of $\tilde{N}$ are

$$\tilde{\Gamma}_{ikj}(z) = \frac{1}{|z|^4}(1-\tilde{f}_0^{\,2}(|z|))(\delta_{ij}z^k|z|^2-z^iz^jz^k) + $$
$$+ \frac{1}{|z|^3}\tilde{f}_0\tilde{f}_0{}'(|z|)\{(\delta_{ik}z^j+\delta_{jk}z^i-\delta_{ij}z^k)|z|^2+$$
$$-z^iz^jz^k\}$$

$$\tilde{\Gamma}_{i\,j}^{\;k}(z) = \frac{1}{|z|^3}\left(\frac{1}{|z|}(1-\tilde{f}_0^{\,2}(|z|))-\tilde{f}_0\tilde{f}_0{}'(|z|)\right)\times$$
$$\times\ (\delta_{ij}z^k|z|^2-z^iz^jz^k) +$$
$$+ \frac{1}{|z|^3}\frac{\tilde{f}_0{}'(|z|)}{\tilde{f}_0(|z|)}\{(\delta_{ik}z^j+\delta_{jk}z^i)|z|^2-2z^iz^jz^k\}$$

and

$$\tilde{g}_{ij,kl}(z) = \partial^2\tilde{g}_{ij}(z)/\partial z^k\partial z^l$$
$$= (1-\tilde{f}_0^{\,2}(|z|))\frac{1}{|z|^6}\ \{(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})|z|^4+8z^iz^jz^kz^l+$$
$$-2(\delta_{ik}z^jz^l+\delta_{jl}z^iz^k+\delta_{jk}z^iz^l+\delta_{il}z^jz^k+\delta_{kl}z^iz^j)|z|^2\}+$$
$$+2\tilde{f}_0\tilde{f}_0{}'(|z|)\frac{1}{|z|^5}\{\delta_{ij}\delta_{kl}|z|^4+5z^iz^jz^kz^l-(\delta_{ik}z^jz^l+$$

$$+\delta_{jl}z^iz^k+\delta_{jk}z^iz^l+\delta_{il}z^jz^k+\delta_{kl}z^iz^j+\delta_{ij}z^kz^l)|z|^2\}+$$

$$+2(\tilde{f}_0\tilde{f}_0')'(|z|)\frac{1}{|z|^4}(\delta_{ij}z^kz^l|z|^2-z^iz^jz^kz^l)$$

Therefore

$$\tilde{R}_{hijk}(z)=\frac{1}{|z|^4}f_0^2(1-f_0')(\delta_{ij}\delta_{hk}-\delta_{hj}\delta_{ik})$$
$$+\frac{1}{|z|^4}\{f_0f_0''+\frac{f_0^2}{|z|^2}(1-f_0'^2)\}(\delta_{ik}z^hz^j+$$
$$+\delta_{jh}z^iz^k-\delta_{ij}z^hz^k-\delta_{hk}z^iz^j)$$

For $a,b\in\mathbb{R}^n$

$$\tilde{R}_{hijk}(z)a^hb^ia^jb^k=$$

$$=\frac{1}{|z|^4}f_0^2(1-f_0'^2)((a.b)^2-|a|^2|b|^2)+$$
$$+\frac{1}{|z|^4}\{f_0f_0''+\frac{f_0^2}{|z|^2}(1-f_0'^2)\}\{(a.z)^2|b|^2+(b.z)^2|a|^2+$$
$$-2(a.b)(a.z)(b.z)\}$$

and

$$\tilde{g}_{ij}(z)a^i b^j = \frac{1}{|z|^2}(a.z)(b.z) + \frac{f_0^2}{|z|^2}(a.b) + $$
$$ - \frac{f_0^2}{|z|^4}(a.z)(b.z)$$

where $(a.b) = \sum_{i=1}^{n} a^i b^i$, $|a|^2 = (a.a)$

Assuming that $(a.b)=0$ and writing $\theta = \frac{1}{|z|^2}\left(\frac{(a.z)^2}{|a|^2} + \right.$ $\left. + \frac{(b.z)^2}{|b|^2}\right)$, we get

$$\tilde{R}_{hijk}(z)a^h b^i a^j b^k = |a|^2|b|^2\left\{\frac{1}{|z|^2} f_0 f_0''\theta - \frac{f_0^2}{|z|^4}\times\right.$$
$$\left.\times(1-f_0'^2)(1-\theta)\right\}$$

$$\|a\|^2\|b\|^2 - \tilde{g}(a,b)^2 = \frac{f_0^2}{|z|^2}\left\{\theta + \frac{f_0^2}{|z|^2}(1-\theta)\right\}|a|^2|b|^2$$

where $\|a\|^2 = \tilde{g}(a,a)$

Therefore $\tilde{K}_{\pi_{ab}}(z)$, the sectional curvature of $\tilde{N}$ at $\dot{z}$ with respect to π_{ab} which is a plane section at $\dot{z}$ spanned by a and b is

$$\tilde{K}_{\pi_{ab}}(z) =$$

$$= \frac{|z|^{-2} f_0^{\,2}(|z|)\{1-f_0'^{\,2}(|z|)\}(1-\theta) - f_0 f_0''(|z|)\theta}{f_0^{\,2}(|z|)\{\theta + |z|^{-2} f_0^{\,2}(|z|)(1-\theta)\}}$$

$$=: \tilde{K}_z(\theta)$$

For any plane section π at z , we can choose $a, b \in \mathbb{R}^n$ which span π and $(a.b)=0$. Therefore

$$\min_{0 \leqq \theta \leqq 1} \tilde{K}_z(\theta) \leqq \tilde{K}_\pi(z) \leqq \max_{0 \leqq \theta \leqq 1} \tilde{K}_z(\theta)$$

But $\tilde{K}_z(\theta)$ is a monotone function of θ, thus we have

$$\min \left\{ -\frac{f_0''(|z|)}{f_0(|z|)} , \frac{1-f_0'^{\,2}(|z|)}{f_0^{\,2}(|z|)} \right\} \leqq$$

$$\leqq \tilde{K}_\pi(z) \leqq \max \left\{ -\frac{f_0''(|z|)}{f_0(|z|)} , \frac{1-f_0'^{\,2}(|z|)}{f_0^{\,2}(|z|)} \right\}$$

Now because N satisfies (ASS. Q_0, f_0, r_0), we can apply Rauch's comparison theorem for N and $\tilde{N}$, thus we get (1,2) as the proof of Lemma 6 of [10], remarking that for our case, for ξ such that $(\xi.z)=0$

$$\tilde{\Gamma}_{ikj}(z)z^i\xi^j\xi^k=(\frac{|z|f_0'(|z|)}{f_0(|z|)}-1)(\tilde{g}_{kj}(z)-\frac{z^jz^k}{|z|^2})\xi^j\xi^k$$

Examples

i) If the sectional curvaturesof N are bounded above by a constant k_0 we can take $f_0=(\sqrt{k_0}r)^{-1}\sin(\sqrt{k_0}r)$ and $r_0=\pi/2\sqrt{k_0}$ and get the estimates in [10] lemma 6.

ii) If we can take, for some Q_0,

$$f_0(r)=\int_0^r(1+t^\alpha)^{-1}\,dt \qquad 0\leqq\alpha\leqq 2$$

then we can take $r_0=\infty$.

2. Maximum principle and existence of weakly harmonic mappings

In order to give a precise statement of our result we introduce the notion of normal range $\mathfrak{N}(P)$ of a point $P\in N$ as a complement of the cut locus of P in N, i.e. the maximal domain of any normal coordinatesystem with center P.

The Sobolev space $H^{1,2}(M,\mathbb{R}^n)$ is constituted by measurable mappings $u:M\to\mathbb{R}^n$ such that $u\circ\chi^{-1}\in H^{1,2}(W,\mathbb{R}^n)$ for every coordinate map χ of M with range $W\in R^m_+=\{x\in R^m;\ x^m\geqq 0\}$. $H^{1,2}(M,\mathbb{R}^n)$ is a Hilbert space with norm

$$\|u\|_{H^{1,2}} = \int_M |u|^2 d\mu + \int_M \tilde{e}(u)d\mu$$

where $|u|^2 = \sum_{i=1}^{n} (u^i)^2$ and

$$\tilde{e}(u) = \frac{1}{2} h^{\alpha\beta} D_\alpha (u^i \circ \chi^{-1}) D_\beta (u^i \circ \chi^{-1})$$

For the normal range $\mathfrak{N}(Q)$ for some $Q \in N$, we can define $H^{1,2}(M, \mathfrak{N}(Q))$ as follows : $H^{1,2}(M, \mathfrak{N}(Q))$ is constituted by measurable mappings $U: M \to N$ such that $U(M) \in \mathfrak{N}(Q)$ and its representation u with respect to the normal coordinate system centered at Q is contained in $H^{1,2}(M, \mathbb{R}^n)$.

Throughout this paper, for $U \in H^{1,2}(M, \mathfrak{N}(Q))$ we use the representation with respect to the normal coordinate system centered Q.

We introduce the notation $\mathfrak{B}_K(Q)$ for $K < \text{dist}(Q, \partial\mathfrak{N}(Q))$ by

$$\mathfrak{B}_k(Q) = \{u \in H^{1,2}(M, \mathfrak{N}(Q);\ \sup_M |u|^2 \leqq K^2\}$$

$\mathfrak{B}_K(Q)$ can be identified with convex, weakly sequentially closed subset of $H^{1,2}(M, \mathbb{R}^n)$. Indeed the energy functional E can be extended to $\mathcal{B}_K(Q)$ (also to $H^{1,2}(M,\mathfrak{N}(Q))$).

If M is isometrically imbeddable in $\mathbb{R}^m$ it is well known that E is lower semicontinuous on $\mathcal{B}_K(Q)$ with respect to weak convergence in $H^{1,2}(M, \mathbb{R}^n)$. Moreover on the geodesic ball $B(Q,K) = \{u \in \mathfrak{N}(Q);\ |u|^2 \leqq K^2\}$ the metric tensors $(g_{ij}(u))$ are bounded and positive definite and therefore for some positive constants λ_1 and λ_2

$$\lambda_1 |\xi|^2 \leqq g_{ij}(u)\xi^i\xi^j \leqq \lambda_2 |\xi|^2$$

for all $u \in \mathcal{B}_K(Q)$ and all $\xi \in \mathbb{R}^n$. then we can see that for $u \in \mathcal{B}_K(Q)$

$$\|u\|_{H^{1,2}} \leqq \text{const.}\{K^2 + E(u)\}$$

and this means that a minimizing sequence in $\mathcal{B}_k(Q)$ is bounded in $H^{1,2}(M, \mathbb{R}^n)$. Thus we get the follwing lemma:

<u>Lemma 2.1</u>

For every $\phi \in \mathcal{B}_K(Q_0)$, $K < \text{dist.}(Q_0, \partial\mathfrak{N}(Q_0))$, there exists a solution of variational problem

$$E(u) \to \min. \quad u \in \mathcal{B}_K(Q_0) \cap \{u - \phi \in H_0^{1,2}(M, \mathbb{R}^n)\}$$

A straight forward computation shows that the first variation of the functional E at $u \in \mathcal{B}_K(Q_0)$ in the direction of ψ, defined by

$$\delta E(u,\psi) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\{E(u + \varepsilon\psi) - E(u)\}$$

exists for all $\psi \in H_0^{1,2}\ L^{\infty}(M, \mathbb{R}^n)$ such that $u + \varepsilon\psi \in \mathcal{B}_K(Q_0)$ for all small enough $\varepsilon \geqq 0$, and is given by

$$\delta E(u,\psi) = \int_M \delta e(u,\psi)d\mu$$

where

$$\delta e(u,\psi) = g_{ij}(u)h^{\alpha\beta}D_\alpha u^i D_\beta \psi^j + \frac{1}{2}D_k g_{ij}(u)h^{\alpha\beta}D_\alpha u^i D_\beta u^j \psi^k$$

$$(2.1) \qquad = g_{ij}(u)h^{\alpha\beta}D_\alpha u^i D_\beta \psi^j + \Gamma_{kij}(u)h^{\alpha\beta}D_\alpha u^i D_\beta u^j \psi^k$$

$$= g_{ij}(u)h^{\alpha\beta}D_\alpha u^i\{D_\beta \psi^j + \Gamma_{k\ l}^{\ j}(u)D_\beta u^l \psi^k\}$$

The minimizing mapping u of <u>Lemma 2.1</u> satisfies

$$(2.2) \qquad \delta E(u,\psi) \geqq 0$$

for all $\psi \in H_0^{1,2} \cap L^\infty(M, \mathbb{R}^n)$ such that $u+\varepsilon\psi \in \mathcal{B}_K(Q_0)$ for all small enough $\varepsilon \geqq 0$. But we can not yet say that this u is a critical point of E and weakly harmonic.

Lemma 2.2

Suppose that N satisfies (ASS. Q_0, f_0, r_0) and the boundary condition ϕ of Lemma 2.1 satisfies $\phi \in \mathcal{B}_{K'}(Q_0)$ for some $K' < K < r_0$. Then the solution u satisfies

$$(2.3) \qquad \int_M h^{\alpha\beta} D_\alpha |u|^2 D_\beta \eta d\mu \leqq 0$$

for all $\eta \in H_0^{1,2}(M, \mathbb{R})$. Therefore for $|u|^2$ the maximum principle is valid, and u is also contained in $\mathcal{B}_{K'}(Q_0)$ and satisfies

$$(2.4) \qquad \delta E(u,\psi) = 0 \text{ for all } \psi \in H_0^{1,2} \cap L^\infty(M, \mathbb{R}^n)$$

i.e. u is a critical point and weakly harmonic.

Proof

For $\eta \in C_0^\infty(M, \mathbb{R})$, $\eta \geqq 0$ we can see that $|u-\varepsilon u\eta| = |1-\varepsilon\eta||u| < K$ for sufficiently small $\varepsilon > 0$, therefore we can use $\psi = -\eta u$ as a test function in (2.2) and thus

$$(2.5) \qquad \delta E(u, -\eta u) \geqq 0$$

Now using (1.2) and the Gauss'lemma (cf. [8] p.136) we can see that from (2.2) follows

$$-\delta e(u,-\eta u) = g_{ij}(u)h^{\alpha\beta}u^j D_\alpha u^i D_\beta \eta + \eta g_{ij}(u)h^{\alpha\beta}\times$$

$$\times\{D_\alpha u^i D_\beta u^j + u^k \Gamma^i_{k\,l}(u)D_\alpha u^l D_\beta u^j\}$$

$$\geqq \frac{1}{2}h^{\alpha\beta}D_\alpha|u|^2 D_\beta\eta + \eta|u|\frac{f_0'(|u|)}{f_0(|u|)}\times \tag{2.6}$$

$$\times g_{ij}(u)D_\alpha u^i D_\beta u^j h^{\alpha\beta}$$

$$\geqq \frac{1}{2}h^{\alpha\beta}D_\alpha|u|^2 D_\beta\eta$$

From (2.5) and (2.6) we obtain (2.3), and it follows from Stampacchia's maximum principle that

$$\sup_M |u|^2 \leqq \sup_{\partial M} |u|^2 \leqq \sup_{\partial M}|\phi|^2 \leqq K'^2$$

By this estimates now we can see that $u\pm\varepsilon\psi\in\mathcal{B}_{K'}(Q_0)$ for every $\psi\in H_0^{1,2}\cap L^\infty(M, \mathbb{R}^n)$ and for all small enough $\varepsilon\geqq 0$. Therefore we get (2.4) from (2.2).

By a simple computation we can see that

$$\delta e(u, g_{ij}\psi^j) = h^{\alpha\beta} D_\alpha u^i D_\beta \psi^i - \Gamma_{j\,k}^{\,i} D_\alpha u^j D_\beta u^k h^{\alpha\beta} \psi^i$$

therefore (2.4) is equivalent to

$$(2.7) \qquad \int_M h^{\alpha\beta}(D_\alpha u^i D_\beta \psi^i - \psi^i \Gamma_{j\,k}^{\,i} D_\alpha u^j D_\beta u^k) d\mu = 0$$

for all $\psi \in H_0^{1,2} \cap L^\infty(M, R^n)$

and this means that u is weakly harmonic.

Now we can formulate the results of this section:

Theorem 2.1

Assume that N satifies (ASS. Q_0, f_0, r_0). Then for any $\phi \in \mathcal{B}_{K'}(Q_0)$ for $K' < r_0$, there exists a weakly harmonic mapping $u \in \mathcal{B}_{K'}(Q_0)$ such that $u - \phi \in H_0^{1,2}(M, \mathbb{R}^n)$ and this u minimizes the energy functional.

3. Interior regularity of weakly harmonic mapping whose energy is local minimum

In this section we shall prove the interior regularity for the minimum u of the energy functional. For this case the method of M.Giaquinta - E.Giusti [3] is available. The proof of the following theorem is based on the method of [2] and [3].

Theorem 3.1

Let $u \in H^{1,2}_{loc}(M, B_K)$, $B_K = B(Q_0, K) = \{u \in N;\ dist(u, Q_0) \leqq K\}$, be a local minimum for the energy functional $E(u)$. Then for every $x_0 \in M$ and any $\alpha \in (0,1)$ there exist some positive numbers ε_0 and R_0 such that;

If for some $R_0 > R > 0$

$$R^{2-n} \int_{B(x_0,R)} |Du|^2 dx \leqq \varepsilon_0$$

then for all $\rho < R$

$$(3.1) \qquad \rho^{-n+2-2\alpha} \int_{B(x_0,\rho)} |Du|^2 dx \leqq CR^{-n+2-2\alpha} \int_{B(x_0,R)} |Du|^2 dx$$

Proof

Let R_0 be sufficiently small and take normal coordinate system around x_0 in $B(x_0, R_0)$ such that $x_0 = (0,\dots,0)$. We denote $B_r = B(0,r) = B(x_0, r)$.

For $R < R_0$ let $v \in H^{1,2}(B_R, \mathbb{R}^n)$ be the solution of the Dirichlet problem

$$\int_{B_R} \bar{h}^{\alpha\beta} D_\alpha v^i D_\beta \psi^i dx = 0 \text{ for all } \psi \in H^{1,2}_0(B_R, \mathbb{R}^n)$$

$$u - v \in H^{1,2}_0(B_R, \mathbb{R}^n)$$

where $\bar{h}^{\alpha\beta} = h^{\alpha\beta}\sqrt{h}$. Then by the theorem of De Giorgi - Nash we get

$$(3.2) \qquad \int_{B_\rho} |Dv|^2 dx \leqq c_1 \left(\frac{\rho}{R}\right)^{n-2+2\alpha} \int_{B_R} |Dv|^2 dx$$

therefore

$$(3.3) \qquad \int_{B_\rho} |Dv|^2 dx \leqq c_2 \left(\frac{\rho}{R}\right)^{n-2+2\alpha} \int_{B_R} |Du|^2 dx$$

On the other hand $u-v \in H_0^{1,2}(B_R, \mathbb{R}^n)$ satisfies

$$\int_{B_R} \bar{h}^{\alpha\beta} D_\alpha (u-v)^i D_\beta (u-v)^i dx = \int_{B_R} \bar{h}^{\alpha\beta} D_\alpha u^i D_\beta (u-v)^i dx =$$

(3.4)

$$= \int_{B_R} \Gamma_{j\,k}^{\,i}(u) \bar{h}^{\alpha\beta} D_\alpha u^j D_\beta u^k (u-v)^i dx$$

therefore

$$(3.5) \qquad \int_{B_R} |D(u-v)|^2 dx \leqq c_3 \int_{B_R} |Du|^2 |u-v| dx \leqq$$

$$\leqq c_4 \left(\int_{B_R} |Du|^{2q} dx \right)^{\frac{1}{q}} \left(\int_{B_R} |u-v|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}} \qquad \text{for any } q>1$$

Now because u is the minimum of E(u), by [3] theorem 4.1 we get

$$\left(\fint_{B_R} |Du|^r dx \right)^{\frac{1}{r}} \leqq C_5 \left(\fint_{B_R} |Du|^2 dx \right)^{\frac{1}{2}} \quad \text{for all } r \in (2, r_0] \text{ for some } r_0 > 2$$

taking $q = \frac{r}{2}$ we get

$$\left(\int_{B_R} |Du|^{2q} dx \right)^{\frac{1}{q}} \leqq C_5 R^{\frac{n}{q} - n} \int_{B_{2R}} |Du|^2 dx$$

Combining this with (3.5) we get

$$\text{(3.6)} \quad \int_{B_R} |D(u-v)|^2 dx \leqq C_6 \left(\int_{B_{2R}} |Du|^2 dx \right) \times \left(R^{-n} \int_{B_R} |u-v|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}}$$

Moreover because $u - v \in H_0^{1,2}(B_R, \mathbb{R}^n)$, and $|u-v| \leqq 2K$ we get by Poincarè inequality

$$\int_{B_R} |u-v|^{\frac{q}{q-1}} dx \leqq C_7 R^2 \int_{B_R} |D(u-v)|^2 dx \leqq C_8 R^2 \int_{B_R} |Du|^2 dx$$

taking q $(= \frac{r}{2} > 1)$ be sufficiently near to 1.

Thus we get

$$(3.7)\qquad \int_{B_R} |D(u-v)|^2 dx \leqq c_9 \Big(\int_{B_R} |Du|^2 dx \Big) \Big(R^{2-n} \int_{B_R} |Du|^2 dx \Big)^{\frac{q-1}{q}}$$

From (3.3) and (3.7) we get

$$(3.8)\qquad \int_{B_\rho} |Du|^2 dx \leqq$$

$$\leqq c_{10} \Big\{ \Big(\frac{\rho}{R}\Big)^{n-2+2\alpha} + c_{11} \Big(R^{2-n} \int_{B_R} |Du|^2 dx \Big)^{\frac{q-1}{q}} \Big\} \int_{B_{2R}} |Du|^2 dx$$

From (3.8) the assertion of the theorem follows by the following well known lemma (for example cf.[6]).

<u>Lemma 3.1</u>

Let $\phi(t)$ be non-negative function satisfying

$$(3.9)\qquad \phi(\rho) \leqq A\Big\{ \Big(\frac{\rho}{R}\Big)^n + \omega\Big(\frac{1}{R^{n-2}} \phi(R)\Big) \Big\} \phi(R)$$

where $\omega(t) \to 0$ as $t \to 0^+$. Then for any $\alpha \in (0,1)$ there exists a constant $\varepsilon_0 > 0$ such that if for some $R > 0$

$$R^{2-n} \phi(R) \leqq \varepsilon_0$$

then

$$(3.10) \qquad \rho^{-n+2-2\alpha}\phi(\rho) \leqq A'R^{-n+2-2\alpha}\phi(R) \quad \text{for all } \rho \leqq R$$

where A' and ε_0 depend only on A,α and $\omega(t)$, but not on R.

Proof

Choose $\tau\in(0,1)$ in such a way that $2A\tau^{2-2\alpha}=1$. Let $\varepsilon_0 > 0$ be some constant such that

$$2\omega(\varepsilon_0) \leqq \tau^n$$

Then if $R^{-n+2}\phi(R) \leqq \varepsilon_0$ we get from (3.9)

$$\phi(\tau R) \leqq A\{\tau^n + \tau^n\}\phi(R) = 2A\tau^n\phi(R) = \tau^{n-2+2\alpha}\phi(R)$$

therefore

$$(\tau R)^{2-n}\phi(\tau R) \leqq \tau^{2\alpha}R^{2-n}\phi(R) \leqq \varepsilon_0$$

thus by induction we get for every k

$$\phi(\tau^k R) \leqq \tau^{k(n-2+2\alpha)}\phi(R)$$

Now for any $\rho\in(0,R)$ there exists k such that $\tau^{k+1}R < \rho \leqq \tau^k R$ then

$$\phi(\rho) \leq \phi(\tau^k R) \leq \tau^{k(n-2+2\alpha)}\phi(R) \leq$$

$$\leq \tau^{-n+2-2\alpha}\left(\frac{\rho}{R}\right)^{n-2+2\alpha}\tau(R)$$

Therefore we get (3.10) with $A' = \tau^{-n+2-2\alpha}$.

Theorem 3.2

Assume that N satisfies (ASS. Q_0, f_0, r_0). Then any local minimum u for the energy functional E(u) such that $\mathrm{dist}(u, Q_0) \leq r_0$ is of class $C^{0,\alpha}$ in the interior of M, int M.

Proof

In viaw of Theorem 3.1 and the integral characterization of Hölder continuous functions due to Morrey and Campanato (cf.[1]) it is sufficient to prove that for every $x_0 \in$ int M we have

$$(3.11) \qquad R^{2-n}\int_{B(x_0,R)} |Du|^2 dx < \varepsilon_0{}^2$$

for some $R < \frac{1}{2}\,\mathrm{dist.}(x_0, \partial M)$.

To prove (3.11) for an arbitrary point $x_0 \in$ int.M, we introduce the normal coordinate system in a suitably small ball $B(x_0, R_0)$, as in the Proof of Theorem 3.1 .

Let $R < \frac{1}{2} R_0$. Taking $\psi = \eta u$ in (2.4), $\eta \in C_0^\infty(B(x_0, 2R))$,

$\eta \geqq 0$, we get as the proof of Lemma 2.2

$$0 = \int_{B_{2R}} \frac{1}{2} D_\alpha |u|^2 D_\beta \eta h^{\alpha\beta} \sqrt{h} dx +$$

$$+ \int_{B_{2R}} \eta g_{ij}(u) \{ D_\alpha u^i D_\beta u^j + u^k \Gamma_{k\,l}^{\,j} D_\alpha u^i D_\beta u^l \} h^{\alpha\beta} \sqrt{h} dx$$

therefore by Lemma 1.1

$$(3.12) \quad \int_{B_R} |u| \frac{f_0'(|u|)}{f_0(|u|)} g_{ij}(u) D_\alpha u^i D_\beta u^j h^{\alpha\beta} \sqrt{h} dx$$

$$\leqq -\frac{1}{2} \int_{B_{2R}} D_\alpha |u|^2 D_\beta \eta h^{\alpha\beta} \sqrt{h} dx$$

By assumptions (ASS. Q_0, f_0, r_0) and $|u| < r_0$, we can see that

$$\inf_{x \in B_{2R}} |u| \frac{f_0'(|u|)}{f_0(|u|)} (x) \geqq \delta_0$$

for some constant $\delta_0 > 0$. Therefore the integral of the left-hand side of (3.12) is bounded below by $\lambda |Du|^2$ for some positive constant λ.

Thus we have

$$(3.13)\qquad \lambda \int_{B_{2R}} |Du|^2 dx \leqq -\frac{1}{2} \int_{B_{2R}} D_\alpha |u|^2 D_\beta \eta h^{\alpha\beta} \sqrt{h} dx$$

The function $z = m^2(2R) - |u|^2$, where $m(t) = \sup_{B_t} |u|$, is a non-negative super solution of an elliptic operator. From the weak Harnack inequality (cf. [7]) we have

$$(3.14)\qquad R^{-n} \int_{B_{2R}} z\, dx \leqq C_{11} \inf_{B_R} z$$

Let $w \in H_0^{1,2}(B_{2R})$ be the solution if the equation

$$\int_{B_{2R}} h^{\alpha\beta} \sqrt{h} D_\beta w D_\alpha \phi dx = -R^{-2} \int_{B_{2R}} \phi\, dx \quad \text{for all } \phi \in H_0^{1,2}(B_{2R})$$

Taking $\phi = wz$ we get

$$\frac{1}{2} \int_{B_{2R}} h^{\alpha\beta} \sqrt{h} D_\beta w^2 D_\alpha z dx + \int_{B_{2R}} z h^{\alpha\beta} \sqrt{h} D_\alpha w D_\beta w dx$$

$$= R^{-2} \int_{B_{2R}} wz\, dx$$

The second integral on the left-hand side is non-negative, moreover we have $w \leqq \alpha_1$ in B_{2R} and from the weak Harnack inequality $w \geqq \alpha_2 > 0$ in B_R since w is a positive super solution of an elliptic equation. We note that α_1 and α_2 do not depend on R. In conclusion, taking $\eta = w^2$ we get

$$\int_{B_{2R}} h^{\alpha\beta}\sqrt{h}D_\alpha zD_\beta\eta dx \leqq C_{12}R^{-2}\int_{B_{2R}} z\, dx$$

which together with (3.13) and (3.14) gives

$$(3.15)\quad \int_{B_R} |Du|^2 dx \leqq C_{13}R^{n-2}\inf_{B_R} z = C_{14}R^{n-2}\{m^2(2R) - m^2(R)\}$$

On the other hand we have

$$(3.16)\quad \sum_{k=0}^{\infty} \{m^2(2^{1-k}R) - m^2(2^{-k}R)\} \leqq m^2(2R) \leqq$$

$$\leqq \sup_M |u|^2 \leqq r_0$$

and therefore inequality (3.16) implies (3.11) with $\rho = 2^{-k}R$ for some k $\{0,1,2,\ldots, \varepsilon_0^{-1}\sup|u|^2 +1\}$. Thus the theorem is proved.

From Theorem 3.2 by the method which now are standard (cf. [4]) we can see that u is of class $C^{1,\alpha}$ interior M, and by the theory of linear equation we can also see that u is of class $C^{2,\alpha}$ interior M, and is harmonic.

4. Liouville-type result for the minima of energy functional

In this section we shall prove a Liouville-type result for a bounded minimum u: M → N of the energy functional E(u), under the condition that N satisfies (ASS. Q_0, f_0, r_0). For the case that the sectional curvatures of N is bounded above by a constant, see [5] and [9].

Theorem 4.1

Assume that M is a simple or compact Riemannian manifold of class C^1 and that N is a complete Riemannian manifold of class C^3 which satisfies (ASS. Q_0, f_0, r_0). Let $u \in \mathcal{B}_{R_0}(Q_0)$ for $R_0 < r_0$, be a local minimum of the energy functional E(u). Then u is a constant. For the case $r_0 = \infty$ any bounded solution is constant.

Here a Riemannian manifold M is said to be simple if it is described by coordinates x on $\mathbb{R}^m$ and by metric

$$d\sigma^2 = h_{\alpha\beta}(x)dx^\alpha dx^\beta$$

for which there exist positive constants λ and μ such that

(4.1) $$\lambda|\xi|^2 \leqq h_{\alpha\beta}(x)\xi^\alpha\xi^\beta \leqq \mu|\xi|^2$$

for all $x,\xi \in \mathbb{R}^m$. In the following we use these coordinates $\{x\}$ for M when M is simple.

Proof

If M is compact, the assertion of the theorem is an immediate consequence of the maximum principle for $|u(x)|^2$. (see Lemma 2.2)

If M is simple the assertion of the theorem follows from proof of Theorem 3.1 and Theorem 3.2 : Since M is simple, Theorem 3.1 is valid with $x_0 = Q_0$ and $R_0 = \infty$. Moreover from the last part of the proof of Theorem 3.2, for any $R > 0$ there exists $k \in \{0,1,2,\ldots, \varepsilon_0^{-1}\sup|u|^2+1\}$ such that

$$\rho^{-n+2}\int_{B(0,\rho)} |Du|^2 dx \leqq \varepsilon_0 \qquad \text{for } \rho = 2^{-k}R$$

therefore we can take a monotone increasing sequence $\{R_i\}$, $R_i \to \infty$ such that

$$R_i^{-n+2}\int_{B(0,R_i)} |Du|^2 dx \leqq \varepsilon_0 \qquad \text{for any } R_i$$

Then by Theorem 3.1 for $\rho < R_i$

$$\rho^{-n+2-2\alpha} \int_{B(0,\rho)} |Du|^2 dx \leqq Cr_i^{-n+2-2\alpha} \int_{B(0,R_i)} |Du|^2 dx$$

$$\leqq CR_i^{-n+2-2\alpha} \varepsilon_0 \quad \text{for every } R_i$$

therefore, letting $R_i \to \infty$, we get

$$\rho^{-n+2-2\alpha} \int_{B(0,\rho)} |Du|^2 dx = 0 \quad \text{for all } \rho > 0$$

i.e. $Du \equiv 0$ and this means that u is a constant map.

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