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STABILITY OF EINSTEIN - HERMITIAN VECTOR BUNDLES

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Einstein - Hermitian vector bundles are defined by a certain curvature condition. We prove that over a compact Kähler manifold a bundle satisfying this condition is semistable in the sense of Mumford - Takemoto and a direct sum of stable Einstein - Hermitian subbundles.

1. Introduction

In his paper [6] Kobayashi introduced the notion of an Einstein - Hermitian vector bundle, i.e. a bundle satisfying a certain curvature condition (Einstein condition). This condition is (over a compact complex manifold) sufficient for the bundle to be semistable in the sense of Bogomolov [1], as Kobayashi proved in [6]. In his recent paper [4] Kobayashi announces the following

THEOREM. *An Einstein - Hermitian vector bundle over a compact Kähler manifold is semistable in the sense of Mumford - Takemoto and a direct sum of stable Einstein - Hermitian subbundles.*

The purpose of this note is to give a proof of this Theorem *). Therefore we at first recall the definition and some properties of Einstein - Hermitian bundles and state a result on the first Chern forms of an Einstein - Hermitian bundle and a subbundle (Proposition 1), which is proved by purely differential geometric methods.

Next we define a differential geometric version of the (semi-)stability in the sense of Mumford - Takemoto and prove the first part of the Theorem: An Einstein - Hermitian vector bundle is semistable (Proposition 2). The main tool of the proof is a vanishing theorem of Kobayashi for sections in Einstein - Hermitian bundles.

If an Einstein - Hermitian bundle E is not stable, we get by Proposition 1 a splitting of E outside an analytic subset of the base manifold, which we extend to a global splitting by sheaf theoretic arguments. This leads to a splitting of E into stable subbundles (Proposition 4).

Although stable bundles and Einstein - Hermitian bundles share many common properties, it is not known until now if every stable bundle satisfies the Einstein condition.

2.Einstein - Hermitian vector bundles

We start with some general facts on Einstein - Hermitian bundles; details can be found in Kobayashi's papers [4], [5],[6] or in [7],[8].

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*) I communicated this proof to Prof.Kobayashi, and he wrote me that his proof is in essential parts different from mine. He kindly suggested a publication of my proof because his will not be published "probably for another year or so"

Let M always be a n - dimensional complex manifold with Kähler metric g and associated 2 - form ω.

Let $\mathcal{E}$ denote the sheaf of differentiable functions and $\mathcal{E}^{1,1}$ the sheaf of differentiable (1,1) - forms on M. Then we define a morphism

$$\tilde{g} : \mathcal{E}^{1,1} \longrightarrow \mathcal{E}$$

as follows:

With respect to local coordinates $z_1,\dots,z_n$, a (1,1) - form σ is given by

$$\sigma = \sum_{i,j=1}^{n} \sigma_{ij} dz_i \wedge d\bar{z}_j ,$$

the metric g determines a matrix (g_{ij}), and we set

$$(g^{ij}) = (g_{ij})^{-1} .$$

We define

$$\tilde{g}(\sigma) = \sum_{i,j} g^{ji} \sigma_{ij} .$$

With

$$\omega = \frac{i}{2} \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j$$

one gets

$$\text{(2.1)} \qquad \sigma \wedge \omega^{n-1} = \frac{2}{in} \tilde{g}(\sigma) \omega^n .$$

Now let E be a holomorphic vector bundle of rank r over M and h a Hermitian metric in E with associated curvature operator D^2. With respect to a local frame, D^2 is given by a matrix $\Omega = (\Omega_{\alpha\beta})$ of (1,1) - forms.

DEFINITION. *The triple (E,h,g) is called an Einstein - Hermitian vector bundle (h is then called an Einstein - Hermitian metric) with factor φ, if*

$$(2.2) \qquad \tilde{g}(\Omega) = \varphi I_r \qquad \text{(Einstein condition)}$$

with a differentiable real function φ, where I_r denotes the $r \times r$ unit matrix.

By definition, every line bundle is an Einstein - Hermitian bundle.

If h is an Einstein metric with factor φ and f a positive function on M, then fh is an Einstein metric with factor

$$\varphi' = \varphi - \Delta \log f \quad ,$$

which becomes a constant by a suitable choice of f if M is compact. So we always can assume that the factor of an Einstein - Hermitian bundle over a compact Kähler manifold is a constant.

If E is an Einstein - Hermitian bundle with factor φ, then the dual bundle E* is an Einstein - Hermitian bundle with factor $-\varphi$; the symmetric powers S^kE and the exterior powers Λ^kE are Einstein - Hermitian bundles with factor $k\varphi$ (with the induced metrics). If E' is another bundle satisfying the Einstein condition with factor φ', then $E \otimes E'$ is an Einstein - Hermitian bundle with factor $\varphi + \varphi'$.

3. First Chern form and stability

Let $\mathfrak{F}$ be a coherent sheaf of rank r over M. Then

$$\det \mathfrak{F} = (\Lambda^r \mathfrak{F})^{**}$$

is a line bundle. The first Chern form of $\mathcal{F}$ is defined by

$$c_1(\mathcal{F}) = c_1(\det\mathcal{F}) = \frac{i}{2\pi}\Omega ,$$

where Ω is the curvature form of a Hermitian metric h in $\det\mathcal{F}$. The cohomology class of $c_1(\mathcal{F})$ in $H^2(M,\mathbb{C})$ is independent of the choice of the metric h.

If E is a holomorphic vector bundle of rank r with Hermitian metric h, then one has

$$c_1(E) = \frac{i}{2\pi}\operatorname{trace}(\Omega) ,$$

where Ω is the matrix of the curvature operator associated to h with respect to a local frame. A simple calculation shows

$$(3.1) \qquad c_1(\Lambda^p E) = \binom{r-1}{p-1} c_1(E)$$

for $1 \leq p \leq r$.

<u>DEFINITION. Let $\mathcal{F}$ be a coherent sheaf over</u> M.

i) <u>We define</u>

$$d(\mathcal{F}) = c_1(\mathcal{F}) \wedge \omega^{n-1} .$$

ii) <u>If</u> M <u>is compact we define</u>

$$\deg(\mathcal{F}) = \int_M d(\mathcal{F})$$

<u>and</u>

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\operatorname{rk}(\mathcal{F})} .$$

Both $\deg(\mathcal{F})$ and $\mu(\mathcal{F})$ depend on the choice of g, but not on the choice of the metric in $\det\mathcal{F}$.

If M is compact and (E,h,g) an Einstein - Hermitian vector bundle with constant factor φ, then one has

$$(3.2) \qquad \varphi = \frac{\pi n}{\mathrm{vol}(M)} \mu(E)$$

(see [4] 3.3), where $\mathrm{vol}(M) = \int_M \omega^n$.

Our first result is

PROPOSITION 1. Let (E,h,g) be an Einstein - Hermitian vector bundle of rank r with constant factor φ over M and

$$(*) \qquad 0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

an exact sequence of vector bundles.

i) One has

$$\frac{d(F)}{rk(F)} \leq \frac{d(E)}{rk(E)} \quad ;$$

if equality holds, the sequence (*) splits and F and G are (equipped with the induced metrics) Einstein - Hermitian bundles with factor φ.

ii) If M is compact one has

$$\mu(F) \leq \mu(E)$$

with the same consequences as in i) for the case of equality.

Proof.

We have only to prove i), because ii) then follows by integration.

Since ω^n is a positive 2n - form it suffices to show

$$\frac{2\pi}{is}\tilde{g}(c_1(F)) \leq \frac{2\pi}{ir}\tilde{g}(c_1(E))$$

(see (2.1)) with $s = rk(F)$.

For this we choose a unitary local frame $e_1,\dots,e_r$ for E such that $e_1,\dots,e_s$ is a local frame for F. Let $\Omega = (\Omega_{\alpha\beta})$, Ω^F, Ω^G be the matrices representing the curvature operators in E, F, G, and $A = (a_{\alpha\beta})$ the matrix of the second fundamental form of F in E with respect to this frame. Then one has

$$(3.3) \qquad \Omega = \begin{pmatrix} \Omega^F + A \wedge {}^t\bar{A} & * \\ * & \Omega^G + {}^t\bar{A} \wedge A \end{pmatrix}$$

([3] p.78, [8] proof of Lemma 1.6) and

$$\frac{2\pi}{i} c_1(E) = \sum_{\alpha=1}^{r} \Omega_{\alpha\alpha},$$

$$\frac{2\pi}{i} c_1(F) = \sum_{\alpha=1}^{s} \Omega_{\alpha\alpha} - \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r-s} a_{\alpha\beta}\wedge\bar{a}_{\alpha\beta}.$$

The Einstein condition (2.2) gives

$$\tilde{g}(\Omega_{\alpha\alpha}) = \varphi$$

and therefore

$$\frac{2\pi}{ir}\tilde{g}(c_1(E)) = \varphi,$$

$$\frac{2\pi}{is}\tilde{g}(c_1(F)) = \varphi - \frac{1}{s}\sum_{\alpha=1}^{s} \sum_{\beta=1}^{r-s} \tilde{g}(a_{\alpha\beta}\wedge\bar{a}_{\alpha\beta}).$$

From the fact that A is a matrix of (1,0) - forms ([3] p.78) one gets by a simple calculation

$$\tilde{g}(a_{\alpha\beta} \wedge \bar{a}_{\alpha\beta}) \geqslant 0 ,$$

which proves the desired inequality. Equality occurs if and only if $A = 0$, but in this case the sequence (*) splits ([2] p.422, [8] Corollar 1.5), and from (3.3) it is easily seen that F and G satisfy the Einstein condition with factor φ. Thus the proof is complete.

DEFINITION. Let M be compact. A vector bundle E over M is called stable (semistable), if for every coherent subsheaf $\mathcal{F}$ of E with $0 < rk(\mathcal{F}) < rk(E)$ one has

$$\mu(\mathcal{F}) < \mu(E) \qquad (\mu(\mathcal{F}) \leqslant \mu(E)).$$

Remarks
i) From some arguments in [9] Chapter II, §1 it follows, that for the (semi-)stability condition of a vector bundle one has to consider only reflexive subsheaves (a coherent sheaf $\mathcal{F}$ is called reflexive, if the canonical morphism $\mathcal{F} \longrightarrow \mathcal{F}^{**}$ is an isomorphism).
ii) If M is projective algebraic and g induced by an ample divisor H, then our (semi-)stability coincides with the H-(semi-)stability of Mumford - Takemoto.

4. Proof of the Theorem

Now let M always be compact.

The second part of Proposition 1 means in particular, that an Einstein - Hermitian vector bundle over M satisfies the semistability condition with respect to subbundles. We strengthen now this statement and prove the first part of the Theorem.

PROPOSITION 2. *If* (E,h,g) *is an Einstein - Hermitian vector bundle with constant factor* φ *over* M, *then* E *is semistable.*

For the proof we need

PROPOSITION 3 (Kobayashi [6]). *Let* E *be as in Proposition* 2. *If* $\varphi < 0$, *then* E *has no nontrivial global holomorphic sections. If* $\varphi = 0$, *then every global holomorphic section is parallel.*

(see also [8] Satz 3.3)

Proof of Proposition 2

Let $\mathcal{F}$ be a reflexive subsheaf of E with

$$0 < s = \mathrm{rk}(\mathcal{F}) < r = \mathrm{rk}(E) \quad .$$

The inclusion

$$\Phi : \mathcal{F} \hookrightarrow E$$

induces a morphism

$$\det\Phi : \det\mathcal{F} \longrightarrow (\Lambda^s E)^{**} \cong \Lambda^s E$$

which is a monomorphism of sheaves because $\mathcal{F}$ is torsion-free. By tensoring with $(\det\mathcal{F})^*$ one gets a nontrivial global holomorphic section

$$f : \mathcal{O}_M \hookrightarrow \Lambda^s E \otimes (\det\mathcal{F})^* \, .$$

From (3.2) we know

$$\varphi = \frac{\pi n}{\mathrm{vol}(M)} \, \mu(E) ,$$

and $\det\mathcal{F}$ (as a line bundle) admits an Einstein-Hermitian metric with factor

$$\psi = \frac{\pi n}{\mathrm{vol}(M)}\,\mu(\det\mathcal{F}) = \frac{\pi n s}{\mathrm{vol}(M)}\,\mu(\mathcal{F}) \ .$$

The bundle $\Lambda^s E \otimes (\det\mathcal{F})^*$ then is Einstein-Hermitian with factor $s\varphi - \psi$, and from the existence of the section f one gets by Proposition 3

$$s\varphi - \psi \geq 0$$

or equivalently

$$\mu(\mathcal{F}) \leq \mu(E)$$

as asserted.

Now let $\mathcal{F}$ be as in the proof above with $\mu(\mathcal{F}) = \mu(E)$. Then one has

$$s\varphi - \psi = 0 \ ,$$

and by Proposition 3 the section f is parallel, in particular has no zeroes, i.e. $\det\Phi$ is a monomorphism of bundles. Thus Φ must be a monomorphism of bundles outside the singularity set S of $\mathcal{F}$ (S is the set of points where $\mathcal{F}$ is not locally free), and over $\tilde{M} = M \setminus S$ one gets an exact sequence of vector bundles

$$(*) \qquad 0 \longrightarrow \mathcal{F}|\tilde{M} \longrightarrow E|\tilde{M} \longrightarrow \mathcal{G}|\tilde{M} \longrightarrow 0$$

with $\mathcal{G} = E/\mathcal{F}$. By (3.1) and Proposition 1, the equality $\mu(\mathcal{F}) = \mu(E)$ implies

$$\frac{d(\mathfrak{F}|\tilde{M})}{s} = \frac{d(E|\tilde{M})}{r}$$

and therefore (again by Proposition 1) the sequence (*) splits.

Now $\mathfrak{F}$ and E are normal sheaves ([9] Chapter II Lemma 1.1.12; a sheaf $\mathfrak{F}$ is called normal, if for every open subset $U \subset M$ and every analytic subset $A \subset U$ with $\operatorname{codim} A \geqslant 2$ the restriction map $\mathfrak{F}(U) \longrightarrow \mathfrak{F}(U \setminus A)$ is an isomorphism), and because $\mathfrak{F}$ is reflexive we have $\operatorname{codim} S \geqslant 3$ ([9] Chapter II Lemma 1.1.10). Therefore the splitting of (*) implies the global splitting of the sequence

$$0 \longrightarrow \mathfrak{F} \longrightarrow E \longrightarrow \mathfrak{G} \longrightarrow 0 ,$$

because the morphism

$$\tilde{\Psi} : E|\tilde{M} \longrightarrow \mathfrak{F}|\tilde{M}$$

with

$$\tilde{\Psi} \circ \Phi|\tilde{M} = \mathrm{id}_{\mathfrak{F}|\tilde{M}}$$

extends canonically to a morphism

$$\Psi : E \longrightarrow \mathfrak{F}$$

with

$$\Psi \circ \Phi = \mathrm{id}_{\mathfrak{F}} .$$

As an immediate consequence we have the second part of the Theorem:

PROPOSITION 4. Let (E,h,g) be an Einstein-Hermitian vector bundle with constant factor φ over M. Then E is a direct sum of stable Einstein-Hermitian subbundles with factor φ.

Proof

If E is not stable, there exists a reflexive subsheaf $\mathfrak{F}$ of E with $\mu(\mathfrak{F}) = \mu(E)$. As we have seen, the sequence

$$0 \longrightarrow \mathfrak{F} \longrightarrow E \longrightarrow \mathfrak{G} \longrightarrow 0$$

splits; in particular, $\mathfrak{F}$ and $\mathfrak{G}$ are bundles and $\mathfrak{F}$ is a subbundle of E. From $\mu(\mathfrak{F}) = \mu(E)$ one gets by Proposition 1, that $\mathfrak{F}$ and $\mathfrak{G}$ satisfy the Einstein condition with factor φ, and a simple induction on the rank of E completes the proof.

References

[1] Bogomolov,F.A.: Holomorphic tensors and vector bundles on projective varieties. Math.USSR Izvestija 13, 499 - 555 (1979)

[2] Griffiths,P.A.: The extension problem in complex analysis II. Amer.J.Math. 88, 366 - 446 (1966)

[3] Griffiths,P.A., Harris,J.: Principles of algebraic geometry. New York: Wiley 1978

[4] Kobayashi,S.: Curvature and stability of vector bundles. Proc.Japan Acad. 58 A 4, 158 - 162 (1982)

[5] Kobayashi,S.: Einstein-Hermitian vector bundles and stability. to appear in Proc.Symp. Global Riemannian Geometry, Durham, England (1982)

[6] Kobayashi,S.: First Chern class and holomorphic tensor fields. Nagoya Math.J. 77, 5 - 11 (1980)

[7] Lübke,M.: Chernklassen von Hermite - Einstein - Vektorbündeln. Math.Ann. 260, 133 - 141 (1982)

[8] Lübke,M.: Hermite - Einstein - Vektorbündel. Dissertation, Bayreuth 1982.

[9] Okonek,C., Schneider,M., Spindler,H.: Vector bundles on complex projective spaces. Boston - Basel - Stuttgart: Birkhäuser 1980

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