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SINE - LAPLACE EQUATION, SINH - LAPLACE EQUATION AND HARMONIC MAPS

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The relationship between harmonic maps from R^2 to S^2, H^2, $S^{1,1}(+1)$, $S^{1,1}(-1)$ and the $\mp$ sinh - Laplace, $\mp$ sine - Laplace equation is found respectively. Existence theorems of some boundary value problems for the above harmonic maps are obtained. In the cases of H^2, $S^{1,1}(+1)$, $S^{1,1}(-1)$ the results are global.

1. Introduction

The relationship between the harmonic map and the famous sine - Gordon equation was pointed out by K. Pohlmeyer in [1]. In fact, from the harmonic map of the Minkowski plane $R^{1,1}$ to a sphere $S^2 \subset R^3$ one can construct a Lax pair and its integrability condition is the sine - Gordon equation. Conversely, the solution of the sine - Gordon equation can be used to construct a harmonic map from $R^{1,1}$ to S^2 in general by solving a system of completely integrable linear partial differential equations. Moreover, Gu developed this idea, established the global existence theorem for the Cauchy problems of the harmonic map from $R^{1,1}$ to any n - dimensio-

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nal complete Riemannian manifold M_n and gave a geometrical interpretation of the single soliton solutions of the sine-Gordon equation in [2].

Recently [3] Gu found the relationship between the sinh-Gordon equation and the harmonic map of $R^{1,1}$ to the sphere $S^{1,1}$ with indefinite metric in a 3 dimensional Minkowski space $R^{2,1}$, furthermore he proved an existence theorem of global solutions of the harmonic map with suitable initial conditions. It should be mentioned that the sinh-Gordon equation has been geometrically interpretated through time-like surfaces of positive constant curvature in $R^{2,1}$ by S.S. Chern in [4].

Thus, the following problem arises naturally: Whether there exists similar relationship between the nonlinear elliptic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = \mp \sinh u \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = \mp \sin u \qquad \begin{matrix}(3)\\(4)\end{matrix}$$

and the harmonic maps. We call equations (1) - (4) $\mp$sinh-Laplace and $\mp$sine-Laplace equations respectively.

Let S^2, H^2, $S^{1,1}(+1)$ and $S^{1,1}(-1)$ be the sphere, hyperbolic plane, the "spheres" with indefinite metric of constant curvature +1 and -1 respectively. In this paper we shall show the intimate relationship between equation (1) - (4) and the harmonic maps from R^2 to S^2, H^2 and the indefinite $S^{1,1}(1)$, $S^{1,1}(-1)$ respectively. From the harmonic maps, we can obtain the Lax pairs with one of the equations (1) - (4) as their integrability conditions respectively, and from each solution of (1) - (4) by using the solution of the Lax pairs, we can construct the corresponding harmonic maps.

Further, we consider certain boundary value problems for harmonic maps from R^2 to the above mentioned 2 dimensional manifolds of constant curvature. Owing to the known properties of non-linear elliptic equations, we obtained existence theorems of the solutions. In the cases of H^2, $S^{1,1}(+1)$ and $S^{1,1}(-1)$, the results are global.

2. The case of S^2

Consider a unit sphere S^2 in 3 dimensional Euclidean space R^3 and choose the parameter (t,x) such that the metric of S^2 can be written in the form

$$ds^2 = \cosh^2\frac{\alpha}{2}dt^2 + \sinh^2\frac{\alpha}{2}dx^2 \qquad (\alpha \neq 0) \tag{5}$$

locally. Considering the formula of curvature

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} \tag{6}$$

we obtain

$$1 = \frac{\dfrac{\partial^2\alpha}{\partial t^2} + \dfrac{\partial^2\alpha}{\partial x^2}}{-\sinh\alpha} \tag{7}$$

this is the negative sinh-Laplace equation (1). Consequently, if the metric of $S^2 \subset R^3$ is locally written in the form (5), then α must be the solution of the negative sinh-Laplace equation (1). Evidently, the converse is also true, if $\alpha \neq 0$.

Let l be the radius vectors of the points on S^2, equation (5) can be written as

$$\begin{aligned} l_t &= \cosh\frac{\alpha}{2}m , \\ l_x &= \sinh\frac{\alpha}{2}n , \end{aligned} \tag{8}$$

where m, n are unit vectors and l, m, n are mutually orthogonal. For definiteness, we assume that det(l,m,n) = 1.

Hence we can write

$$\begin{aligned} l_t &= \cosh\frac{\alpha}{2} m & l_x &= \sinh\frac{\alpha}{2} n \\ m_t &= -\cosh\frac{\alpha}{2} l - \sigma n & m_x &= -\lambda n \\ n_t &= \sigma m & n_x &= -\sinh\frac{\alpha}{2} l + \lambda m \end{aligned} \tag{9}$$

From the integrability condition $l_{xt} = l_{tx}$, we have

$$\lambda = -\frac{1}{2}\alpha_t \quad , \quad \sigma = \frac{1}{2}\alpha_x \tag{10}$$

Substituting (10) into (9) and calculating $m_{tx} = m_{xt}$, $n_{tx} = n_{xt}$, we can verify easily that these integrability conditions of (9), (10) are the negative sinh - Laplace equation

$$\alpha_{xx} + \alpha_{tt} = -\sinh\alpha \tag{1'}$$

Consequently, equation (9) and (10) constitute a Lax pair of the group SO(3) with negative sinh - Laplace equation as its integrability condition. On the otherhand, from (8) we have

$$l_t^2 - l_x^2 = 1 \quad , \quad l_t \cdot l_x = 0 \tag{11}$$

By differentiation, it is easily seen that $l(t,x)$ satisfies

$$l_{tt} + l_{xx} + (l_t^2 + l_x^2)l = 0 \tag{12}$$

This is exactly the equation of harmonic maps from R^2 to S^2. Consequently, $l(t,x)$ is a harmonic map from R^2 to S^2. In this paper, a normalized harmonic map is defined as a harmonic map $l(t,x)$ which satisfies the condition (11).

Moreover, let $\alpha(t,x)$ be a given solution to the negative sinh - Laplace equation on a simply connected region

$\Omega \subset R^2$. By solving the Lax pair with the initial condition $(t,x) = (t_o,x_o)$, $(l,m,n) = (l_o,m_o,n_o)$, where $(t_o,x_o) \in \Omega$, $(l_o,m_o,n_o) \in SO(3)$, and extracting $l(t,x)$ from the solution, we obtain a normalized harmonic map $l(t,x)$ from Ω to the sphere. Thus, we can construct a normalized harmonic map from a solution of the negative sinh - Laplace equation.

Now there arises another question. Whether all of the harmonic maps $l(t,x)$ from R^2 to S^2 can be obtained through solving $\alpha(t,x)$ from the negative sinh - Laplace equation. In the following we prove that in general a harmonic map from a region $\Omega \subset R^2$ to S^2 can always be constructed through a normalized harmonic map together with a conformal map.

Let $l(t,x)$ be a harmonic map from $\Omega \subset R^2$ to S^2. Then $l(t,x)$ satisfies equation (12) and the equation $l^2 = 1$. Since (12) is a system of elliptic equations with analytic coefficients, the regular solution $l(t,x)$ must be analytic. Hence the vector - valued function $l(t,x)$ can be complexified to $l(\tau,\zeta)$ where τ, ζ are complex variables and $l(\tau,\zeta)$ is a complex analytic function defined in a region Ω_c $(\supset \Omega)$ in $\mathbb{C}^2$, $l(\tau,\zeta)$ still satisfies the equation

$$l_{\tau\tau} + l_{\zeta\zeta} + (l_\tau^2 + l_\zeta^2)l = 0 \tag{13}$$

Introduce

$$\xi = \frac{\tau + i\zeta}{2} \quad , \quad \eta = \frac{\tau - i\zeta}{2} \tag{14}$$

as two new independent variables, then τ, ζ take real values if and only if ξ and η are conjugate. Using (14), we can rewrite (13) into

$$l_{\xi\eta} + (l_\xi \cdot l_\eta)l = 0 \tag{15}$$

Since $l^2 = 1$, we obtain

$$(l_\xi^2)_\eta = 0 \quad , \quad (l_\eta^2)_\xi = 0 \tag{16}$$

and hence

$$l_\xi^2 = k(\xi) \quad , \quad l_\eta^2 = h(\eta) \tag{17}$$

where $k(\xi)$ and $h(\eta)$ are all analytic functions. We assume

$$\begin{aligned} l_\xi^2 &= (l_t - il_x)^2 = l_t^2 - l_x^2 - 2il_t \cdot l_x \neq 0 \\ l_\eta^2 &= (l_t + il_x)^2 = l_t^2 - l_x^2 + 2il_t \cdot l_x \neq 0 \end{aligned} \tag{18}$$

in the region Ω. Then we can solve the differential equations

$$\frac{\partial \xi'}{\partial \xi} = 2k^{\frac{1}{2}}(\xi) \quad , \quad \frac{\partial \eta'}{\partial \eta} = 2h^{\frac{1}{2}}(\eta) \tag{19}$$

and obtain the solutions

$$\xi' = \beta(\xi) \quad , \quad \eta' = \overline{\beta}(\eta) \tag{20}$$

in an open set Ω_1 $(\Omega \subset \Omega_1 \subset \Omega_c)$ such that ξ' and η' are conjugate when ξ and η are conjugate. Denote

$$\xi' = t' + ix' \quad , \quad \eta' = t' - ix'$$

then

$$t' + ix' = \beta(t + ix) \tag{21}$$

is a conformal mapping T from the region Ω to some region Ω'. Let $l(t',x')$ be the composition of T^{-1} and $l(t,x)$ (i.e. $l \circ T^{-1}$), then we have $l_{\xi'}^2 = 1/4$ or

$$l_{t'}^2 - l_{x'}^2 = 1 \quad , \quad l_{t'} \cdot l_{x'} = 0 \tag{22}$$

on Ω'. Hence $l(t',x')$ is a normalized harmonic map and can be determined from a solution of the negative sinh - Laplace equation. Consequently, when the condition (18) is satisfied, a harmonic map from Ω to S^2 can be constructed by solving the negative sinh - Laplace equation, and solving the Lax pair (9), (10), and then combining the solution with a conformal map.

THEOREM 1. Let $\alpha(t,x)$ be a given solution to the negative sinh - Laplace equation (1) on a simply connected region $\Omega \subset R^2$, we can determine a normalized harmonic map $l(t,x)$ from Ω to S^2 through solving the Lax pair (9), (10). Conversely, any harmonic map from $\Omega \subset R^2$ to S^2 satisfying (18) equals to the product of a conformal map of region Ω and a normalized harmonic map constructed by solving the negative sinh - Laplace equation and Lax pair (9), (10).

Remark. As is known, in 2 dimensional case, the product of a conformal map and a harmonic map is also a harmonic map.

Remark. If $l_\xi^2 = l_\eta^2 = 0$ identically, l is a conformal map.

3. Other cases

Let l, m be vectors in space $R^{2,1}$, their scalar product is defined by

$$l \cdot m = l_1 m_1 + l_2 m_2 - l_3 m_3 \tag{23}$$

$S^{1,1}(+1)$ is the locus of points in $R^{2,1}$ satisfying the equation

$$l^2 = 1 \tag{24}$$

If the metric of $S^{1,1}(+1)$ is written in the following form locally

$$ds^2 = \cos^2\frac{\alpha}{2}\,dt^2 - \sin^2\frac{\alpha}{2}\,dx^2 \qquad (\alpha \neq 0)\,, \tag{25}$$

calculating the condition $K = 1$, we have

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial^2\alpha}{\partial x^2} = -\sin\alpha \tag*{(3)'}$$

this is just the negative sine - Laplace equation (3). Consequently, in order that the metric of $S^{1,1}(+1) \subset R^{2,1}$ can be written in the form (25), the necessary and sufficient condition is that α must be the solution of the negative sine -

Laplace equation (3) and $\alpha \neq 0$.

Hence $l = l(t,x)$ satisfies

$$l_t = \cos\frac{\alpha}{2} m \qquad l_x = \sin\frac{\alpha}{2} n \tag{26}$$

Here $m^2 = 1$, $n^2 = -1$, $m \cdot n = 0$, and $\det(l,m,n) > 0$. The Lax pair of the group SO(2,1) takes the following form

$$\begin{aligned} l_t &= \cos\frac{\alpha}{2} m & l_x &= \sin\frac{\alpha}{2} n \\ m_t &= -\cos\frac{\alpha}{2} l - \frac{\alpha_x}{2} n & m_x &= \frac{\alpha_t}{2} n \\ n_t &= -\frac{\alpha_x}{2} m & n_x &= \sin\frac{\alpha}{2} l + \frac{\alpha_t}{2} m \end{aligned} \tag{27}$$

Calculating $m_{xt} = m_{tx}$, $n_{xt} = n_{tx}$ we obtain the negative sine - Laplace equation again. Similar to the section 2 we have

<u>THEOREM 2. For any solution of the negative sine - Laplace equation (3') in a simply connected region Ω, we can always construct a normalized harmonic map $l(t,x)$ from $\Omega \subset R^2$ to $S^{1,1}(+1)$ through solving the Lax pair (26), (27). Conversely, any harmonic map $l(t,x)$ from $\Omega \subset R^2$ to $S^{1,1}(+1)$ satisfying (18) equals to the product of a conformal map of region Ω and a normalized harmonic map constructed by solving the negative sine - Laplace equation and Lax pair (27).</u>

We have the similar results on the sinh - Laplace equation and sine - Laplace equation, corresponding to the harmonic map from $\Omega \subset R^2$ to $H^2 \subset R^{2,1}$ and to $S^{1,1}(-1) \subset R^{1,2}$ respectively. For simplicity, we write down here the metrics and the Lax pairs only.

If we write the metric of H^2 in the form

$$ds^2 = \cosh^2\frac{\alpha}{2} dt^2 + \sinh^2\frac{\alpha}{2} dx^2 \tag{28}$$

we will obtain the sinh - Laplace equation

$$\alpha_{tt} + \alpha_{xx} = \sinh \alpha$$

Here H^2 is considered as the upper component of

$$l^2 = -1 \tag{29}$$

in $R^{2,1}$. The corresponding Lax pair is

$$\begin{aligned} l_t &= \cosh\frac{\alpha}{2} m & l_x &= \sinh\frac{\alpha}{2} n \\ m_t &= \cosh\frac{\alpha}{2} l - \frac{1}{2}\alpha_x n & m_x &= \frac{1}{2}\alpha_t n \\ n_t &= \frac{1}{2}\alpha_x m & n_x &= \sinh\frac{\alpha}{2} l - \frac{1}{2}\alpha_t m \end{aligned} \tag{30}$$

with $l^2 = -1$, $m^2 = n^2 = 1$, $l \cdot m = l \cdot n = m \cdot n = 0$ and $\det(l,m,n) = 1$.

If we write the metric of $S^{1,1}(-1)$ in the form

$$ds^2 = \cos^2\frac{\alpha}{2} dt^2 - \sin^2\frac{\alpha}{2} dx^2 \tag{31}$$

Here $S^{1,1}(-1)$ is the sphere

$$l^2 = -1 \tag{32}$$

in the space $R^{1,2}$ with $l^2 = -l_1^2 - l_2^2 + l_3^2$, we will obtain the sine - Laplace equation

$$\alpha_{tt} + \alpha_{xx} = \sin \alpha$$

The related Lax pair is

$$\begin{aligned} l_t &= \cos\frac{\alpha}{2} m & l_x &= \sin\frac{\alpha}{2} n \\ m_t &= \cos\frac{\alpha}{2} l - \frac{1}{2}\alpha_x n & m_x &= \frac{1}{2}\alpha_t n \\ n_t &= -\frac{1}{2}\alpha_x m & n_x &= -\sin\frac{\alpha}{2} l + \frac{1}{2}\alpha_t m \end{aligned} \tag{33}$$

with $l^2 = -1$, $m^2 = 1$, $n^2 = -1$, $l \cdot m = l \cdot n = m \cdot n = 0$, $\det(l,m,n) = 1$.

4. A boundary value problem for normalized harmonic maps

From now on we will consider a boundary value problem for normalized harmonic maps. By using the properties of the elliptic equations (1) - (4), some existence theorems can be obtained.

We can pose the following boundary value problem: Let Ω be a simply connected region with C^2 boundary and $v = (v_1, v_2)$ be a field of unit vectors on the boundary $\partial\Omega$. We wish to find the normalized harmonic map from Ω to S^2 (resp. $S^{1,1}(+1)$, H^2, $S^{1,1}(-1)$) such that l_v^2 (the square length of the v - direction derivative of l) along $\partial\Omega$ equals to a given function σ.

For the S^2 case, from equation (8) we have

$$l_v = l_t v_1 + l_x v_2 = v_1 \cosh\frac{\alpha}{2} m + v_2 \sinh\frac{\alpha}{2} n \tag{34}$$

The boundary condition $l_v^2 = \sigma$ means that

$$v_1^2 \cosh^2\frac{\alpha}{2} + v_2^2 \sinh^2\frac{\alpha}{2} = \sigma \tag{35}$$

It follows that

$$\cosh^2\frac{\alpha}{2} = \sigma + v_2^2 \tag{36}$$

Suppose that

$$\sigma + v_2^2 > 1 \tag{37}$$

we can get two boundary values α defined by

$$\alpha = \pm 2\cosh^{-1}\sqrt{\sigma + v_2^2} \tag{38}$$

for the negative sinh - Laplace equation. Using the known result of the nonlinear elliptic equations [5], it follows that the Dirichlet problem for the negative sinh - Laplace equation admits solutions if Ω is small enough (or the variation of α is small enough). With no loss of generality, we can take + sign in (38), since the negative sign can be compensated by a symmetry. Hence the harmonic map is uniquely determined except an orthogonal transformation in R^3.

For the case $S^{1,1}(+1)$ and $S^{1,1}(-1)$, the condition (37) should be replaced by

$$o < \sigma + v_2^2 < 1 \qquad (40)$$

and for the case of H^2, we still have the condition (37).

It is interesting to point out that the Dirichlet problem for positive and negative sine - Laplace equation and positive sinh - Laplace equation are solvable globally [5] since the right hand sides satisfy

$$|\sin\alpha| \le 1 \quad \text{or} \quad \frac{d}{d\alpha}\sinh\alpha = \cosh\alpha > 0$$

Hence for the cases H^2, $S^{1,1}(+1)$, $S^{1,1}(-1)$ the region Ω can be arbitrary large.

Thus we have

THEOREM 3. _If (37) is satisfied for the cases of S^2 and H^2, or (40) is satisfied for the cases of $S^{1,1}(+1)$ and $S^{1,1}(-1)$. There exists a normalized harmonic map from a simply-connected region Ω to S^2 (or H^2, $S^{1,1}(+1)$, $S^{1,1}(-1)$) such that at the boundary $l_v^2 = \sigma$ hold. In the S^2 case, Ω should be small enough. In the cases of H^2, $S^{1,1}(+1)$ and $S^{1,1}(-1)$, the results are global. The maps are determined except for a rotation in R^3, $R^{2,1}$ or $R^{1,2}$ respectively._

The results of this section can be listed as follows:

Target manifold	boundary condition	equation for α	normalized harmonic map
$S^2 \subset R^3$	$l_v^2 = \sigma$ with $\sigma + v_2^2 > 1$	$\Delta\alpha = -\mathrm{sh}\alpha$	exist for small Ω
$H^2 \subset R^{2,1}$	$l_v^2 = \sigma$ with $\sigma + v_2^2 > 1$	$\Delta\alpha = \mathrm{sh}\alpha$	exist globally
$S^{1,1}(+1) \subset R^{2,1}$	$l_v^2 = \sigma$ with $0 < \sigma + v_2^2 < 1$	$\Delta\alpha = -\sin\alpha$	exist globally
$S^{1,1}(-1) \subset R^{1,2}$	$l_v^2 = \sigma$ with $0 < \sigma + v_2^2 < 1$	$\Delta\alpha = \sin\alpha$	exist globally.

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