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CANCELLATION LAW FOR HOMOTOPY EQUIVALENT REPRESENTATIONS OF GROUPS OF ODD ORDER

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We prove that if V and W are real G-modules and G is a group of odd order, then V and W are stably G-homotopy equivalent if and only if they are G-homotopy equivalent.

Let G be a compact Lie group. We will say after Kawakubo [4] that the cancellation law holds for real G-modules when the following is true: "if $S(V\oplus U)$ and $S(W\oplus U)$ are G-homotopy equivalent for some G-module U, then S(V) and S(W) are G-homotopy equivalent". It is proved in [4] that the cancellation law holds for an arbitrary compact abelian topological group G.

In this paper we give a proof that the cancellation law holds for every compact Lie group such that G/G_0 is a group of odd order (here G_0 denotes the identity component of G).

All the finite groups considered in this paper are assumed to be of odd order.

We shall use notation $V \overset{s}{\underset{G}{\sim}} W$ for the stable G-homotopy equivalence and $V \overset{u}{\underset{G}{\sim}} W$ for the G-homotopy equivalence.

Let us begin with the following simple observation.

LEMMA 1. If the cancellation law holds for groups of order less than |G| and for such G-modules that for every $I \neq H \trianglelefteq G$ $V^H = W^H = 0$, then it holds for G.

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PROOF. Assume that $V \overset{s}{\sim} W$. We shall prove that $V \overset{u}{\sim} W$ by the induction on the dimension of V.

1° Let $\dim V = 1$. It is obvious that if $V \overset{s}{\sim} W$, then $\dim V^H = \dim W^H$ for every subgroup $H \subseteq G$. It follows that $\dim W = \dim V = 1$ and V and W have the same kernel. The isomorphism class of a one-dimensional representation is uniquely determined by its kernel, whence V and W are isomorphic, whence $V \overset{u}{\sim} W$.

2° Assume that the cancellation holds for G-modules of dimension less than $\dim V$. By the assumption it is enough to consider the case when $V^H, W^H \neq 0$ for some $I \neq H \trianglelefteq G$. Of course if $f: S(V \oplus U) \to S(W \oplus U)$ is a stable G-homotopy equivalence, then $f^H: S(V^H \oplus U^H) \to S(W^H \oplus U^H)$ is a stable G/H-homotopy equivalence. $|G/H| < |G|$, whence by the assumption of the lemma there exists an unstable G/H-homotopy equivalence

$$g: S(W^H) \to S(V^H).$$

Let us consider the G-homotopy equivalence

$$(1) \qquad f * g: S(V \oplus U \oplus W^H) \to S(W \oplus U \oplus V^H).$$

If V and W are decomposed as $V = V^H \oplus V_H$, $W = W^H \oplus W_H$ then (1) may be written as

$$(2) \qquad f * g: S(V_H \oplus U \oplus V^H \oplus W^H) \to S(W_H \oplus U \oplus V^H \oplus W^H).$$

It follows that $f * g$ is a stable G-homotopy equivalence of V_H and W_H. By the inductive assumption $V_H \overset{u}{\sim} W_H$. Combining this with $V^H \overset{u}{\sim} W^H$ we obtain $V \overset{u}{\sim} W$, which completes the proof.

We shall prove our main theorem by the induction on the order of G. It is obvious that the cancellation holds for trivial group. In view of Lemma 1 to make the inductive step it is enough to prove the following:

if $V \overset{s}{\sim} W$ and $V^H = W^H = 0$ for $I \neq H \trianglelefteq G$, then $V \overset{u}{\sim} W$.

In particular we need only consider such representations that $V^G = W^G = 0$. It is a standard fact that such representations of groups of odd order admit a complex structure (Curtis-Reiner [1], p. 223). It will be convenient to assume for the rest of this paper that the considered representations are complex. But we shall still consider the unoriented homotopy equivalence. The cancellation law for oriented homotopy equivalence is proved in tom Dieck [2].

The essential fact we need in our proof is the theorem about the existence of equivariant maps and congruences between the degrees on fixed points subsets. The construction and the proof can be found in [3], section 8.4.

THEOREM 2. Let V and W be complex representations of G, such that for every subgroup $H \subseteq G$ $\dim V^H = \dim W^H$. Then there exists an equivariant map $f\colon S(V) \longrightarrow S(W)$. If H is an isotropy subgroup of the action of G on $S(V)$, then $\deg f^H (\mathrm{mod}\, |N(H)/H|)$ is determined by $\{\deg f^K\}_{H \subset K}$. For every integer k there exists a map f_k such that $\deg f_k^H = \deg f^H + k\,|N(H)/H|$ and for every subgroup H' which is not subconjugate to H $\deg f_k^{H'} = \deg f^{H'}$.

We want to apply Theorem 2 to prove a theorem in which the existence of equivariant homotopy equivalences for subgroups of G implies the existence of a G-homotopy equivalence. Apparently some restrictions should be made in such a theorem on the kind of groups considered and some coherence assumptions for the degrees of the involved H-homotopy equivalences for $H \subset G$. Before formulating the theorem we will prove two lemmas which enable us to show that for groups of odd order the coherence assumptions may be stated in a simple form.

LEMMA 3. Let E be the regular complex representation of a group G of odd order. Assume that $2E \subset V$. Then if f_1, f_2 are self G-homotopy equivalences of $S(V)$ and for some subgroup $H \subseteq G$

$$(3) \qquad \deg f_1^H = \deg f_2^H,$$

then for every subgroup $K \subseteq G$

$$(4) \qquad \deg f_1^K = \deg f_2^K.$$

PROOF. By the Theorem 10.1 in Rubinsztein [5] there is a ring homomorphism

$$(5) \qquad [S(V), S(V)]_G \approx A(G),$$

where $A(G)$ is the Burnside ring of G. By Proposition 1.5.1 in tom Dieck [3]

$$(6) \qquad A(G)^* = \{\pm 1\}.$$

Let $V = V' \oplus C$, where C is the trivial representation of G. Let $g: S(C) \longrightarrow S(C)$ be any map of degree -1. Obviously g is a G-homotopy equivalence. Then $g_1 := id_{S(V)}$ and $g_2 := id_{S(V')} * g$ are G-homotopy equivalences of $S(V)$ and for every $H \subseteq G$

$$(7) \qquad \deg g_1^H = - \deg g_2^H.$$

It follows that $[g_1] \neq [g_2]$ in $[S(V), S(V)]_G$. A class of a G-homotopy equivalence is an invertible element in $[S(V), S(V)]_G$, whence by (5), (6) and the fact that $[g_1] \neq [g_2]$ we obtain that $[g_1]$ and $[g_2]$ are the only classes of self G-homotopy equivalences of $S(V)$. From this and (7) it follows that if f_1 and f_2 satisfy (3), then they belong to the same class in $[S(V), S(V)]_G$, whence they satisfy (4), which completes the proof.

What we need directly in the announced theorem is the following.

LEMMA 4. Let F be a group of odd order. If f_1, f_2 are F-homotopy equivalences from $S(V)$ to $S(W)$ and either

(9) $$V^F = W^F = 0$$

<u>or</u>

(10) $$\deg f_1 = \deg f_2,$$

<u>then for every subgroup</u> $H \subsetneq F$

(11) $$\deg f_1^H = \deg f_2^H.$$

PROOF. Let $g\colon S(W) \longrightarrow S(V)$ be an F-homotopy equivalence. Then $g_i := gf_i$ are self F-homotopy equivalences of $S(V)$ and $\deg g_i^H = \deg f_i^H \cdot \deg g^H$, whence

(12) $$\deg g_1^H = \deg g_2^H \Longleftrightarrow \deg f_1^H = \deg f_2^H.$$

It is thus enough to prove that

(13) $$\deg g_1^H = \deg g_2^H.$$

If E is the regular representation of F, then $g_i * \mathrm{id}\colon S(V \oplus 2E) \longrightarrow S(V \oplus 2E)$ are self F-homotopy equivalences of $S(V \oplus 2E)$. If $V^F = W^F = 0$, then

(14) $$\deg(g_1 * \mathrm{id})^F = \deg(g_2 * \mathrm{id})^F = 1.$$

If $\deg f_1 = \deg f_2$, then

(15) $$\deg g_1 * \mathrm{id} = \deg g_2 * \mathrm{id}.$$

Now it is enough to apply Lemma 3 to $g_i * \mathrm{id}$ in both cases to obtain

(16) $$\deg g_1^H = \deg(g_1 * \mathrm{id})^H = \deg(g_2 * \mathrm{id})^H = \deg g_2^H$$

which yields (13). This completes the proof.

With Lemma 4 we are ready to prove a theorem about the existence of G-homotopy equivalence.

<u>THEOREM 5. Let</u> G <u>be a group of odd order, non-isomorphic to a cyclic p-group. Let</u> V <u>and</u> W <u>be complex</u> G-<u>modules such that</u> $V^H = W^H = 0$ <u>for every</u> $I \neq H \trianglelefteq G$. <u>If there exists a set</u> $\{f_H\}_{H \subset G, H \neq G}$ <u>of</u> H-<u>homotopy equivalences</u> $f_H\colon S(V) \longrightarrow S(W)$ <u>such that</u>

(17) $$\deg f_{H_1} = \deg f_{H_2}$$

<u>for every</u> $H_1, H_2 \subset G$, <u>then there exists a</u> G-<u>homotopy equivalence</u> $f\colon S(V) \to S(W)$ <u>such that for every</u> $H \subset G$

(18) $$\deg f_H^H = \deg f^H$$

(here the lower "H" is an index and the upper "H" is the symbol for restriction to the fixed points of the action of H).

PROOF. By the assumption f_H is an H-homotopy equivalence, in particular $\dim V^H = \dim W^H$. This equality holds also for G, because $V^G = W^G = 0$.
Let G be a p-group. We adopt the notation as in tom Dieck [2] . From the above equalities it follows that

(19) $$X := r(V-W) \in RO_0(G).$$

(r denotes the restriction of scalars).

All the irreducible factors in V and W are imprimitive, because they are faithful and G is not a cyclic group. It follows that X represents a class in $iO'(G)$. Moreover $\mathrm{res}_H X \in RO_h(H)$ for every proper subgroup $H \subset G$, whence by Theorem 2 in [2] $X \in RO_1(H)$. It follows that $\mathrm{res}\, X = 0$, where res is the restriction map $\mathrm{res}\colon iO'(G) \to \prod_H iO(H)$. By the Proposition 3.2 in [2] this map is injective, whence $X \in RO_1(G)$. By the construction of G-homotopy equivalences given in [3] section 9.5 there exists a G-homotopy equivalence $f\colon S(V) \to S(W)$.

Let H_1 be a normal subgroup of index p in G. By the assumption $V^H = W^H = 0$, whence by Lemma 4 applied to H_1

(20) $$\deg f_{H_1}^K = \deg f^K$$

for every $K \subset H_1$, in particular

(21) $$\deg f_{H_1} = \deg f.$$

Let H_2 be an arbitrary subgroup of G. Combining (21) with (17) we obtain

(22) $$\deg f_{H_2} = \deg f.$$

Applying the second part of Lemma 4 to $F = H_2$, $f_1 = f$, $f_2 = f_{H_2}$ we obtain

(23) $$\deg f_{H_2}^{H_2} = \deg f^{H_2}$$

which completes the proof for p-groups.

Assume now that G is not a p-group. By Theorem 2 there exists a G-map $f_o: S(V) \longrightarrow S(W)$. Let H be a maximal element among those isotropy subgroups for which

(24) $$\deg f_o^H \neq \deg f_H^H.$$

We shall show that the same inequality holds for all subgroups conjugate to H. Every element $g \in G$ acts on V and W as a unitary map. We shall denote by g_V, g_W, g_V^H, g_W^H the restrictions of these maps to $S(V)$, $S(W)$, $S(V)^H$ and $S(W)^H$ respectively. Let $K = gHg^{-1}$. Then

$$(g_V^{-1})^K : S(V)^K = g(S(V)^H) \longrightarrow S(V)^H$$

and

$$g_W^H : S(W)^H \longrightarrow g(S(W)^H) = S(W)^K .$$

For f_o^K we have a formula

$$f_o^K = g_W^H \circ f_o^H \circ (g_V^{-1})^K ,$$

whence

(25) $$\deg f_o^K = \deg g_W^H \cdot \deg f_o^H \cdot \deg (g_V^{-1})^K$$

Let us consider a K-homotopy equivalence φ_K given by the formula $\varphi_K = g_W \circ f_H \circ g_V^{-1}$. Then

$$\varphi_K^K = g_W^H \circ f_H^H \circ (g_V^{-1})^K .$$

We obtain the following formulas for $\deg \varphi_K$ and $\deg \varphi_K^K$

(26) $$\deg \varphi_K^K = \deg g_W^H \cdot \deg f_H^H \cdot \deg (g_V^{-1})^K$$

(27) $$\deg \varphi_K = \deg g_W \cdot \deg f_H \cdot \deg g_V^{-1} = \deg f_H = \deg f_K$$

The second equality in (27) is obtained from the fact that g_W and g_V^{-1} are odd-order self-homeomorphisms of $S(W)$ and $S(V)$ respectively, whence $\deg g_W = \deg g_V^{-1} = 1$. The third is a consequence of the assumption (17).

The equality (27) enables us to apply Lemma 4. We obtain

(28) $$\deg \varphi_K^K = \deg f_K^K$$

Combining (28) and (26) we obtain

(29) $$\deg f_K^K = \deg g_W^H \cdot \deg f_H^H \cdot \deg g_V^{-1\ K}.$$

Using (25) and (24) we get

(30) $$\deg f_o^K \neq \deg g_W^H \cdot \deg f_H^H \cdot \deg (g_V^{-1})^K.$$

(29) and (30) yield

(31) $$\deg f_K^K \neq \deg f_o^K .$$

If H is not the trivial subgroup, then $N(H) \neq G$, because V^H and W in this case and H is assumed to be an isotropy subgroup of the action of G on $S(V)$.
Let $H \subset K \leq N(H)$. By Lemma 4 applied to K-homotopy equivalences f_K and $f_{N(H)}$

$$\deg f_{N(H)}^K = \deg f_K^K = \deg f_o^K .$$

(By the assumption (24) $\deg f_K^K = \deg f_o^K$ for every isotropy subgroup of the action of G on $S(V)$ containing H as a proper subgroup. To obtain the same for arbitrary K we use the fact, that there exists an isotropy subgroup K', $K \subset K'$ such that $S(V)^K = S(V)^{K'}$, this is an easy consequence of the fact, that for representations the intersection of isotropy subgroups is an isotropy subgroup. For the proof see Rubinsztein [5], Remark 8.2.)

Applying Theorem 2 to the group $N(H)$ we obtain

$$\deg f_0^H \equiv \deg f_{N(H)}^H \pmod{|N(H)/H|}.$$

By the other part of Theorem 2 applied to G there exists a G-map $f_1 : S(V) \longrightarrow S(W)$ such that $\deg f_1^M = \deg f_0^M$ for every subgroup M which is not subconjugate to H and $\deg f_1^H = \deg f_{N(H)}^H = \deg f_H^H$.

After a finite number of such steps we obtain a G-map $f_n : S(V) \longrightarrow S(W)$ such that $\deg f_n^H = \deg f_H^H$ for every proper subgroup H, except possibly for the trivial one. In particular if p divides the order of G, then for a Sylow subgroup G_p and arbitrary subgroup $H \subset G_p$ we have by Lemma 4 $\deg f_H^H = \deg f_{G_p}^H$, which implies $\deg f_n^H = \deg f_{G_p}^H$ for nontrivial subgroups of G_p.

By Theorem 2 applied to G_p we get

$$\deg f_n \equiv \deg f_{G_p} \pmod{|G_p|} \tag{32}$$

This is true for every prime p dividing the order of G. If I denotes the trivial subgroup of G, then by the assumption (17)

$$\deg f_{G_p} = \deg f_I \tag{33}$$

Combining (32) and (33) we obtain

$$\deg f_n \equiv \deg f_I \pmod{|G_p|} \tag{34}$$

It follows that

$$\deg f_n \equiv \deg f_I (\mathrm{mod}\, |G|).$$

By Theorem 2 there exists a G-map $f\colon S(V) \longrightarrow S(W)$ such that for every $H \subsetneq G$

$$\deg f^H = \deg f_H^H = \pm 1.$$

By Proposition 8.2.4 and Remark 8.2.5 in [3] f is a G-homotopy equivalence. This completes the proof of Theorem 5.

Now we are ready to prove our main theorem.

THEOREM 6. The cancellation law holds for groups of odd order.

PROOF. We shall prove the theorem by induction on the order of G. To start the induction let us recall that

I. The cancellation holds for abelian groups (Kawakubo [4], Theorem 2.5).

To make the inductive step it is enough to consider stably homotopy equivalent complex representations V and W such that $V^H = W^H = 0$ for every $I \neq H \trianglelefteq G$ (Lemma 1).

Let us assume that the cancellation holds for every group of order less than $|G|$. We shall consider several cases.

II. Let G be a semidirect product of Z_p and F, so that

$$G = Z_p \tilde{\times} F, \quad Z_p \triangleleft G, \quad F \cap Z_p = I$$

Assume moreover that the action of $F = G/Z_p$ on Z_p is effective.

There exists a Z_p-homotopy equivalence f_{Z_p}. By Lemma 4 $\deg f_{Z_p}$ is uniquely determined. To prove that

a G-homotopy equivalence exists we need to show that for every proper subgroup $H \subset G$ there exists an H-homotopy equivalence f_H such that $\deg f_H = \deg f_{Z_p}$ and apply Theorem 5. First consider the case when $H+Z_p$ (the subgroup generated by the union $H \cup Z_p$) is proper. Then by the inductive assumption there exists an $H+Z_p$-homotopy equivalence f_{H+Z_p} which is also a Z_p-homotopy equivalence, whence $\deg f_{H+Z_p}$ is uniquely determined and $\deg f_{H+Z_p} = \deg f_{Z_p}$. The same map is also an H-homotopy equivalence and it may be taken for f_H.
Assume now that H is a proper subgroup and $H+Z_p = G$. It follows that $Z_p \not\subset H$, whence $H \cap Z_p = I$ and $G = Z_p \tilde{\times} H$. By the inductive assumption there exists an H-homotopy equivalence f_H. It is well-known that every irreducible faithful representation of G is induced from Z_p. It follows that $\mathrm{res}_H V$ and $\mathrm{res}_H W$ have trivial factors, so f_H may be choosen to have the degree 1 or -1 as required.

III. Let G be a semidirect product of a p-torus $P = Z_p \times \dots \times Z_p \neq Z_p$ and F. Assume moreover that P is a minimal normal subgroup in G and that the action of F on P is effective. To proceed with the proof we will need some preparations. For $K \subset G$ denote by $V(K)$ (resp. $W(K)$) the intersection of all subrepresentations containing V^K (W^K resp.).

<u>LEMMA 7. Let V and W be stably G-homotopy equivalent representations of G. Suppose that</u>

$$V(K) = \mathrm{ind}_{N(K)}^{G} V^K, \quad W(K) = \mathrm{ind}_{N(K)}^{G} W^K \tag{35}$$

<u>Then $V(K)^{\perp}$ and $W(K)^{\perp}$ are stably G-homotopy equivalent.</u>

PROOF. V^K and W^K are stably $N(K)$-homotopy equivalent. Let U be a representation of $N(K)$ such that there exists an $N(K)$-homotopy equivalence

$$f\colon S(V^K \oplus U) \longrightarrow S(W^K \oplus U).$$

It is proved in tom Dieck [3], p. 251 that f induces a G-homotopy equivalence of induced representations

$$f'\colon S(\mathrm{ind}^G_{N(K)}(V^K \oplus U)) \longrightarrow S(\mathrm{ind}^G_{N(K)}(W^K \oplus U)).$$

Combining this with (35) we obtain $V(K) \overset{s}{\underset{G}{\sim}} W(K)$, whence $V(K)^{\perp} \overset{s}{\underset{G}{\sim}} W(K)^{\perp}$.

We would like to apply Lemma 7 to reduce case III to the situation when F is not isomorphic to Z_q and for every F such that $G = P \rtimes F$ there exists $K \triangleleft F$ such that $V^K = W^K = 0$. Then we could easily apply Theorem 5. To do this we are going to choose suitable subgroups in the considered semidirect product to apply Lemma 7.

Let P' be a subgroup of P such that $P/P' \approx Z_p$. P' is not a trivial group, because by the assumption $P \neq Z_p$. Denote by $F(P')$ the subgroup in G consisting of those elements $x \in N(P')$ for which the induced action on P/P' is trivial. Obviously $P \subseteq F(P')$, whence

$$F(P') = P \rtimes F'$$

where F' is a suitable subgroup in F.

We can also define $F(P')$ as the kernel of the action of $N(P')$ on P/P'. From this it is obvious that

$$F(P') \trianglelefteq N(P').$$

It follows that

(39) $$K(P') := [F(P'), F(P')] + P' \trianglelefteq N(P').$$

Let us observe that

$$F(P')/(P'+[F',F']) = (P \times F')/(P'+[F',F']) = (P/P') \times (F'/[F',F'])$$

and the right side is an abelian group, whence $[F(P'),F(P')] \cap P \subseteq P'$. It follows that $K(P') \cap P = P'$ and finally

(40) $$N(K(P')) = N(P').$$

It is also easy to see that

(41) $$K(P') = [F',F'] + P'.$$

LEMMA 8. Let $K \leq F \leq N$ and $K \trianglelefteq N$. Let X be an F-module. Then

(42) $$\mathrm{res}_K\, \mathrm{ind}_F^N X = \bigoplus_{s \in N/F} X_s,$$

where $\rho_{X_s}(k) = \rho_X(s^{-1}ks)$, for $k \in K$.

PROOF. This is a straightforward consequence of the double coset formula for the restriction of the induced representation (see Serre [6], Proposition 22). It is enough to observe that the double coset KsF is equal to the left coset sF and that for every $s \in N$ $sFs^{-1} \cap K = K$

We want to show now that we can apply Lemma 7 to $K = K(P')$.

LEMMA 9. Let X be an $F(P')$-module such that $K(P') \leq \ker \rho_X$ and $X^P = 0$. Then

(43) $$(\mathrm{ind}_{F(P')}^G X)^{K(P')} = \mathrm{ind}_{F(P')}^{N(P')} X.$$

PROOF. It is enough to apply Lemma 8. to $K(P') \leq F(P') \leq N(P')$ to obtain

(44) $$\operatorname{res}_{K(P')} \operatorname{ind}_{F(P')}^{N(P')} X = \bigoplus_{s \in N(P')/F(P')} X_s,$$

where $\varrho_{X_s}(k) = \varrho_X(s^{-1}ks) = id_X$, for $k \in K(P')$ (because $s^{-1}ks = k' \in K(P')$ and $K(P') \subseteq \ker \varrho_X$, whence $\varrho_X(k') = id_X$). Thus $\operatorname{res}_{K(P')} \operatorname{ind}_{F(P')}^{N(P')} X$ is a trivial $K(P')$-module, so

(45) $$\operatorname{ind}_{F(P')}^{N(P')} X \subseteq (\operatorname{ind}_{F(P')}^{G} X)^{K(P')}.$$

Applying Lemma 8. to $P \subseteq F(P') \subseteq G$, $P \trianglelefteq G$ we obtain

$$\operatorname{res}_P \operatorname{ind}_{F(P')}^{G} X = \bigoplus_{s \in G/F(P')} X_s,$$

where $\varrho_{X_s}(g) = \varrho_X(s^{-1}gs)$, for $g \in P$. Obviously for $s \notin N(P')$ $X_s^{P'} = 0$, because $P' \neq sP's^{-1} \subseteq \ker \varrho_{X_s}$ and $X_s^P = 0$. It follows that

(46) $$\dim (\operatorname{ind}_{F(P')}^{G} X)^{P'} = |N(P')/F(P')| \dim X = \dim \operatorname{ind}_{F(P')}^{N(P')} X.$$

Combining this with the obvious inequality

(47) $$\dim (\operatorname{ind}_{F(P')}^{G} X)^{K(P')} \leq \dim (\operatorname{ind}_{F(P')}^{G} X)^{P'}$$

and the inclusion (45) we obtain (43).

We can return now to the inductive step in case III. This will be split into two subcases.

IIIa. Suppose that there exists a subgroup $P' \subseteq P$ such that $P/P' \approx Z_p$ and $V^{K(P')}, W^{K(P')} \neq 0$. Let X_1 be an irreducible $F(P')$ submodule of $V^{K(P')}$ ($K(P')$ is normal in $F(P')$, so $V^{K(P')}$ is an $F(P')$-module). P is normal, whence $X_1^P = 0$ and $\ker \varrho_{X_1} \cap P = P'$. It follows that $F(P')$ is exactly the isotropy subgroup of $\mathrm{res}_P X_1$ with respect to the action of G on the isomorphism classes of irreducible representations of P ($\mathrm{res}_P X_1$ is irreducible because X_1 factorizes to a representation of an abelian group and as an irreducible representation is therefore one-dimensional).

Now it is enough to apply the Mackey's irreducibility criterion to see that $\mathrm{ind}_{F(P')}^G X_1$ is irreducible (for the details see the description of the irreducible representations of a semidirect product by an abelian group given in Serre [6], Proposition 25).

By the Frobenius reciprocity

$$(48) \qquad \langle \mathrm{ind}_{F(P')}^G X_1, V\rangle_G = \langle X_1, \mathrm{res}_{F(P')} V\rangle_{F(P')} .$$

By the assumption the right side is not zero, whence the irreducible representation $\mathrm{ind}_{F(P')}^G X_1$ is contained in V. Every irreducible factor in $V(K(P'))$ and $W(K(P'))$ is of that form, whence

$$(49) \qquad V(K(P')) = \mathrm{ind}_{F(P')}^G X, \quad W(K(P')) = \mathrm{ind}_{F(P')}^G Z,$$

where X and Z satisfy the conditions

$$(50) \qquad K(P') \subseteq \ker \varrho_X, \; K(P') \subseteq \ker \varrho_Z, \; X^P = Z^P = 0.$$

Applying Lemma 9 to X and Z we obtain

$$(51) \qquad \begin{aligned} (\mathrm{ind}_{F(P')}^G X)^{K(P')} &= \mathrm{ind}_{F(P')}^{N(P')} X \\ (\mathrm{ind}_{F(P')}^G X)^{K(P')} &= \mathrm{ind}_{F(P')}^{N(P')} X \end{aligned}$$

Combining this with (49) we obtain

$$(52)\qquad \begin{aligned} V^{K(P')} &= V(K(P'))^{K(P')} = \mathrm{ind}_{F(P')}^{N(P')} X \\ W^{K(P')} &= W(K(P'))^{K(P')} = \mathrm{ind}_{F(P')}^{N(P')} Z. \end{aligned}$$

By the transitivity of the induction we have

$$(53)\qquad \begin{aligned} V(K(P')) &= \mathrm{ind}_{N(P')}^{G}\, \mathrm{ind}_{F(P')}^{N(P')} X \\ W(K(P')) &= \mathrm{ind}_{N(P')}^{G}\, \mathrm{ind}_{F(P')}^{N(P')} Z \end{aligned}$$

This together with (52) and (40) yields

$$(54)\qquad \begin{aligned} V(K(P')) &= \mathrm{ind}_{N(K(P'))}^{G}\, V^{K(P')} \\ W(K(P')) &= \mathrm{ind}_{N(K(P'))}^{G}\, W^{K(P')}. \end{aligned}$$

Applying Lemma 7 to $K = K(P')$ we obtain

$$(55)\qquad V(K(P'))^{\perp} \underset{G}{\overset{s}{\sim}} W(K(P'))^{\perp}.$$

We shall show now that $V(K(P')) \underset{G}{\overset{u}{\sim}} W(K(P'))$. For this let us observe that $N(P') \neq G$ because P was assumed to be a minimal normal subgroup of G and $P' \neq I$. By (39), (40) $K(P') \trianglelefteq N(P') = N(K(P'))$, whence $V^{K(P')} \underset{N(P')}{\overset{s}{\sim}} W^{K(P')}$, which in view of (52) may be rewritten as

$$(56)\qquad \mathrm{ind}_{F(P')}^{N(P')} X \underset{N(P')}{\overset{s}{\sim}} \mathrm{ind}_{F(P')}^{N(P')} Z.$$

By the inductive assumption the cancellation law holds for $N(P')$ so we can write $\underset{N(P')}{\overset{u}{\sim}}$ in the above formula. The $N(P')$-homotopy equivalence of $\mathrm{ind}_{F(P')}^{N(P')} X$ and $\mathrm{ind}_{F(P')}^{N(P')} Z$ induces a G-homotopy equivalence of induced G-modules, this combined with (53) yields

$$(57)\qquad V(K(P')) \underset{G}{\overset{u}{\sim}} W(K(P')).$$

From (55) and (57) it follows that it is enough to consider such V and W that $V^{K(P')} = W^{K(P')} = 0$. This may be done for every P', whence III. is reduced to the following second subcase:

IIIb. Suppose that for every $P' \subset P$ such that $P/P' \approx Z_p$ $V^{K(P')} = W^{K(P')} = 0$.

If $\dim V \neq 0$, then for a suitable P' $V^{P'} \neq 0$, whence $K(P') \neq P'$. It follows that $F(P') \neq P$, in particular $N(P') \neq P$. On the other hand $N(P') \neq G$, because P is a minimal normal subgroup. It follows that the sequence $P \subset N(P') \subset G$ is strictly increasing, whence the order of F is not a prime number. By the famous Feit and Thompson theorem F is solvable. Let $K \trianglelefteq F$ be a normal subgroup of index q, where q is a prime number. We shall show that

$$(58) \qquad V^K = W^K = 0$$

By the description of the irreducible representations of a semidirect product every irreducible representation of G is of the form $\mathrm{ind}^G_{F(P')} X = \mathrm{ind}^G_{P \rtimes F'} X$, where $\mathrm{res}_{F'} X$ is an irreducible representation of $F' = F(P') \cap F$ and $\mathrm{res}_P X$ is an isotypic representation of P such that $\ker \mathrm{res}_P X = P'$ (we consider faithful representations only). Applying the double coset formula we obtain

$$(59) \qquad \mathrm{res}_F \mathrm{ind}^G_{F(P')} X = \mathrm{ind}^F_{F'} \mathrm{res}_{F'} X$$

Applying the same formula once more to $\mathrm{ind}^F_{F'} \mathrm{res}_{F'} X$ we obtain the following formula for $\mathrm{res}_K \mathrm{ind}^G_{F(P')} X$.

$$(60) \qquad \mathrm{res}_K \mathrm{ind}^G_{F(P')} X = \bigoplus_{s \in K \backslash F/F'} \mathrm{ind}^K_{K_s} X_s = \bigoplus_{s \in F/F'} \mathrm{ind}^K_{K_s} X_s,$$

where $K_s = sF's^{-1} \cap K = s(F' \cap K)s^{-1}$ and $\varrho_{X_s}(k) = \varrho_X(s^{-1}ks)$ for $k \in K_s$. It is obvious that X_s contains a trivial factor if and only if $X_1 = \mathrm{res}_{F' \cap K} X$ contains a trivial factor. On the other hand by the Frobenius reciprocity

(61) $$\langle \mathrm{ind}_{K_s}^{K} X_s, 1_K \rangle_K = \langle X_s, 1_{K_s} \rangle_{K_s},$$

and if $\mathrm{res}_K \, \mathrm{ind}_{F(P')}^{G} X$ contains a trivial factor, then the left side is not zero for some s, whence $\mathrm{res}_{F' \cap K} X$ contains a trivial factor. Obviously $F' \cap K \trianglelefteq F'$ so $X^{F' \cap K}$ is an F'-module, whence by the irreducibility of X $X = X^{F' \cap K}$ and X factorizes to an F'/F' K-module. The last group is abelian, whence $[F', F'] \subseteq \ker \varrho_X$. By the assumption also $P' \subset \ker \varrho_X$, whence $K(P') = [F', F'] + P' \subseteq \subseteq \ker \varrho_X$, which is a contradiction

With (58) proved we are ready to apply Theorem 5. Let H be a proper subgroup of G. If $H+P \neq G$, then by the inductive assumption there exists an $H+P$-homotopy equivalence f_{H+P}, which is in particular H- and P-homotopy equivalence. By Lemma 4

(62) $$\deg f_{H+P} = \deg f_P$$

for every P-homotopy equivalence f_P, because $V^P = W^P = 0$

If $H+P = G$, then $P \cap H \trianglelefteq H$ and $P \cap H \trianglelefteq P$, whence $P \cap H \trianglelefteq G$. Since P is a minimal normal subgroup of G this means that $P \cap H = I$ and G is a semidirect product of P and H. It follows that there exists a subgroup $K \trianglelefteq H$ of index q in H, and by (58) $V^K = W^K = 0$. Obviously $P+K$ is a proper subgroup in G, and by the inductive assumption there exist equivariant homotopy equivalences f_H and f_{P+K}. Using the fact that f_H is in particular a K-homotopy equivalence and that f_{P+K} is also P- and K-homotopy equivalence and applying Lemma 4 first to K, then to P we obtain

(63) $$\deg f_H = \deg f_{P+K} = \deg f_P.$$

(62) and (63) enable us to apply Theorem 5. It follows that $V \underset{G}{\overset{u}{\sim}} W$.

It remains to consider the last case.

IV. Suppose that G is not an abelian group and not a semidirect product by a p-torus with an effective action. G is solvable, whence a minimal normal subgroup of G is a p-torus $P = Z_p \times \ldots \times Z_p$.

Let H be a proper subgroup of G. If $P+H \neq G$, then by the inductive assumption we have an f_{P+H}, and as in IIIb

(64) $$\deg f_{P+H} = \deg f_P$$

so that f_{P+H} may be taken for f_H.

Let $P+H = G$. Then G is a semidirect product of P and H. Suppose that $G = P \times H$. Then the order of H is not a prime number, because in this case G would be abelian. It follows that there exists a non-trivial proper normal subgroup $K \triangleleft H$.

If G is not a direct product of H and P, then K:= the kernel of the action of H on P is a non-trivial proper normal subgroup of H. In both cases K is contained in the kernel of the action of H on P, hence K is normal in the whole G. By the inductive assumption there exist equivariant homotopy equivalences f_H, f_K, f_{K+P}, f_P. $V^K = W^K = V^P = W^P = 0$, whence applying Lemma 4 to K and P we obtain

(65) $$\deg f_H = \deg f_K = \deg f_{K+P} = \deg f_P.$$

Applyig Theorem 5 we obtain $V \underset{G}{\overset{u}{\sim}} W$, which completes the inductive proof of Theorem 6.

By Theorem 3.1 in Kawakubo [4] the cancellation law holds for a compact Lie group G if and only if the cancellation law holds for G/G_0. This together with Theorem 6 yields the following.

<u>THEOREM 10. Let G be a compact Lie group. If</u> G/G_0 <u>is a group of odd order, then the cancellation law holds for</u> G.

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