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LOCAL OPERATORS BETWEEN SPACES OF ULTRADIFFERENTIABLE FUNCTIONS AND ULTRADISTRIBUTIONS

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In this article we characterize certain ultradifferential operators by the condition of being local. First, we examine the continuity properties of local linear operators on spaces of ultradifferentiable functions in the sense of Beurling and of Roumieu. Next, a structure theorem for vector-valued ultradistributions with support at the origin is proved. This result leads to a representation theorem for continuous local operators from spaces of ultradifferentiable functions into various spaces of ultradistributions. In combination with the continuity results we thus obtain in many cases the desired characterization.

0. Introduction

The starting-point of this note is the abstract characterization of differential operators due to Peetre [11]. Given an open subset Ω of $\mathbb{R}^n$, his result says, in particular, that a linear operator $T:\mathcal{D}(\Omega) \to \mathcal{C}(\Omega)$ is local in the sense of

$$\operatorname{supp}(T\varphi) \subset \operatorname{supp}\varphi \qquad \text{for all } \varphi \in \mathcal{D}(\Omega)$$

if and only if there exists a unique representation

$$T\varphi = \sum_{\beta \in \mathbb{N}_0^n} a_\beta D^\beta \varphi \qquad \text{for all } \varphi \in \mathcal{D}(\Omega)$$

with some locally finite family $\{a_\beta : \beta \in \mathbb{N}_0^n\} \subset \mathcal{C}(\Omega)$. The present article is devoted to a similar characterization concerning operators between spaces of ultradifferentiable functions and ultradistributions in the sense of Komatsu

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[6]; see section 1 for the basic definitions.

Our first object is to investigate the continuity properties of local linear operators within the theory of ultradistributions. In section 2 it is shown that completely satisfying results are available for those local operators whose range does not contain any non-trivial distribution with finite support. In the general case, there is a fundamental difference between local operators on spaces of ultradifferentiable functions in the sense of Beurling on the one side and of Roumieu on the other: In the Beurling case these mappings have to be "almost" continuous similar to the classical situation [11], whereas "highly" discontinuous local operators can be constructed in the Roumieu case. On the other hand, local operators on spaces of ultradistributions with compact support behave nicely in the Roumieu case, but not necessarily in the Beurling case. All positive results of this section are consequences of the abstract automatic continuity theory for generalized local operators from [1].

We then proceed to the representation of continuous local operators on spaces of ultradifferentiable functions. The main idea will be to associate with a given local operator a certain vector-valued ultradistribution with support at the origin and then to apply a suitable structure theorem. In the scalar-valued case, ultradistributions with support at the origin have been characterized independently by Droste [4] and Komatsu [7]. In section 3 we extend their result to the case of ultradistributions with values in an arbitrary complete locally convex space by using topological tensor products and sequence spaces. The final section contains the main representation of continuous local operators as ultradifferential operators, which is similar to the case of ordinary distributions [11] and of ω-ultradifferentiable functions [3]. The representation is more involved than the classical situation, since the infinite series defining ultradifferential operators cause some convergence problem. As it turns out, however, the

desired convergence actually holds for a wide and natural class of range spaces.

1. Preliminaries

In this paper, we shall use the standard notations and fundamental results on ultradifferentiable functions and ultradistributions from Komatsu [6;7]. For convenience, let us briefly recall some basic definitions. Let Ω denote an arbitrary open subset of $\mathbb{R}^n$ and consider a sequence $(M_p)_{p=0,1,\dots}$ of positive real numbers. In this situation, for every compact $K\subset\Omega$ and real $h>0$ let

$$p_{K,h}(f) := \sup_{\alpha\in\mathbb{N}_0^n}\ \sup_{t\in K} \frac{|D^\alpha f(t)|}{h^{|\alpha|}M_{|\alpha|}} \quad (\leq\infty)$$

for all $f\in\mathcal{C}^\infty(\Omega)$. Then the spaces of ultradifferentiable functions of Beurling, resp. Roumieu type are defined to be

$$\mathcal{E}^{(M_p)}(\Omega) := \{f\in\mathcal{C}^\infty(\Omega) : \text{For all compact } K\subset\Omega \text{ and all real } h>0 \text{ we have } p_{K,h}(f)<\infty\},$$

$$\mathcal{E}^{\{M_p\}}(\Omega) := \{f\in\mathcal{C}^\infty(\Omega) : \text{For all compact } K\subset\Omega \text{ there is some } h>0 \text{ such that } p_{K,h}(f)<\infty\}.$$

Both spaces are endowed with the natural locally convex topologies which are suggested by the definitions [6,p.44]. We are mainly interested in ultradifferentiable functions with compact support: For compact $K\subset\Omega$ and real $h>0$ consider the Banach space

$$\mathcal{D}_K^{\{M_p\},h} := \{f\in\mathcal{C}^\infty(\mathbb{R}^n) : \operatorname{supp} f\subset K \text{ and } p_{K,h}(f)<\infty\}$$

with respect to the norm $p_{K,h}$ and then the locally convex spaces given by

$$\mathcal{D}^{(M_p)}(\Omega) := \operatorname*{ind}_{K\subset\subset\Omega}\ \operatorname*{proj}_{h\to 0+} \mathcal{D}_K^{\{M_p\},h},$$

$$\mathcal{D}^{\{M_p\}}(\Omega) := \operatorname*{ind}_{K\subset\subset\Omega}\ \operatorname*{ind}_{h\to\infty} \mathcal{D}_K^{\{M_p\},h}.$$

The corresponding dual spaces are the spaces of ultradistributions of Beurling, resp. Roumieu type and will always be equipped with the strong topology. Clearly, whenever $\mathcal{F}_1(\Omega)$ and $\mathcal{F}_2(\Omega)$ are spaces of ultradifferentiable functions or ultradistributions on Ω, a linear operator $T:\mathcal{F}_1(\Omega) \to \mathcal{F}_2(\Omega)$ is called <u>local</u> if

$$\operatorname{supp}(Tf) \subset \operatorname{supp} f \qquad \text{for all } f \in \mathcal{F}_1(\Omega) .$$

In order to apply the theory of ultradistributions, we have to impose certain natural restrictions on the defining sequence $(M_p)_{p=0,1,\dots}$. The following conditions will be used in this note:

(M.0) $M_0 = 1$;

(M.1) $M_p^2 \leq M_{p-1} M_{p+1}$ for all $p \in \mathbb{N}$ (logarithmic convexity) ;

(M.2) $\exists\, A,H>0$ such that $M_p \leq A H^p \min_{0 \leq q \leq p} M_q M_{p-q}$ for all $p \in \mathbb{N}_0$ (stability under ultradifferential operators) ;

(M.3) $\exists\, A>0$ such that $\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq A\, p \frac{M_p}{M_{p+1}}$ for all $p \in \mathbb{N}$ (strong non-quasi-analyticity) ;

(M.3)' $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$ (non-quasi-analyticity) .

Standard examples are the Gevrey sequences $M_p = (p!)^s$, p^{ps}, or $\Gamma(1+ps)$ for any real $s>1$, which satisfy all of these conditions. Let us point out that the rather mild conditions (M.1) and (M.3)' will suffice for the automatic continuity theory in section 2. This section is mainly based on our paper [1]. Therefore, we shall constantly use some special terminology from [1] within the proofs of the next section.

2. Continuity properties of local operators

Throughout this section, let Ω be an open subset of $\mathbb{R}^n$ and consider a sequence $(M_p)_{p=0,1,\dots}$ of positive real num-

bers such that conditions (M.1) and (M.3)' are satisfied. In the following, the asterisk * stands for the Beurling case (M_p) or for the Roumieu case $\{M_p\}$. We first attack the problem of continuity for local operators between spaces of functions.

2.1.THEOREM. Every local linear operator $T : \mathcal{D}^*(\Omega) \to L^1_{loc}(\Omega)$ is necessarily continuous.

Proof. (a) Given an arbitrary compact subset $\widetilde{\Omega}$ of Ω, we endow the space $X:=\{f\in\mathcal{D}^*(\Omega) : \operatorname{supp} f \subset\widetilde{\Omega}\}$ with its canonical topology and consider the Banach space $Y:=L^1(\widetilde{\Omega})$. Then we have to prove the continuity of the local linear operator $\widetilde{T}:X\to Y$ given by $\widetilde{T}f := (Tf)|\widetilde{\Omega}$ for all $f\in X$.

(b) First assume that $* = (M_p)$. In this case, the canonical topology of X is complete metrizable. In order to apply the general automatic continuity result [1,Cor.3.9], we consider on X and on Y the support capacities E_s given by $E_s(F) :=\{f\in X$, resp. $Y : \operatorname{supp} f\subset F\}$ for all closed $F\subset\widetilde{\Omega}$. Since X admits appropriate partitions of unity according to [6, Lemma 5.1], one easily verifies that the support capacity E_s for X is 2-spectral in the sense of [1,Def.1.1]. Hence the continuity of $\widetilde{T}:X\to Y$ follows immediately from [1, Cor.3.9].

(c) It remains to prove the assertion in the Roumieu case $* = \{M_p\}$, which is a little bit more involved. For each real h>o let

$$X_h := \mathcal{D}^{\{M_p\},h}_{\widetilde{\Omega}} \quad \text{and} \quad T_h:=\widetilde{T}|X_h:X_h\to Y .$$

Again, T_h is local with respect to the support capacities E_s on the Banach spaces X_h and Y in the sense of [1,Def. 1.3]. From [1,Remark 3.8] we conclude that the separating space $\mathfrak{S}(T_h)$ is contained in $E_s(\Lambda_2(T_h))$ for all h>o. Consequently, each of the operators T_h and hence $\widetilde{T}$ will be continuous, if $\Lambda_2(T_h)$ is seen to be finite for every h>o. Unfortunately, this is not a direct consequence of the basic result [1,Th.3.5(b)], since the support capacity on X_h is not 2-spectral [1,Def.1.1]. However, it follows from [1,

Th.3.5(a)] that $\Lambda_1(T_h)$ is finite for every h>o. It therefore suffices to show that

$$\Lambda_2(T_h) \subset \Lambda_1(T_{h+1}) \qquad \text{for all } h>o .$$

The proof of this inclusion will be similar to a corresponding argument in the proof of [1,Th.3.5].

(d) Given an arbitrary h>o, we claim that every continuity point of the first type for T_{h+1} is also a continuity point of the second type for T_h in the sense of [1,Def.3.4]. Consider some $t\in\widetilde{\Omega}$ and an open neighborhood U of t in Ω such that $\widetilde{T}$ is continuous on the space $E_s(\overline{U}) := \{f\in X_{h+1} : \operatorname{supp} f \subset \overline{U}\}$. Further, let V be an open neighborhood of t in Ω such that $\overline{V}\subset U$. According to a property of ultradifferentiable functions which will be derived in the subsequent independent lemma, we obtain a pair of functions $\varphi_1,\varphi_2\in\mathcal{C}^\infty(\mathbb{R}^n)$ with the properties $\operatorname{supp}\varphi_1\subset U$, $\operatorname{supp}\varphi_2\cap\overline{V}=\emptyset$, and $\varphi_1+\varphi_2\equiv 1$ on some neighborhood of the compact set $\widetilde{\Omega}$ such that for j=1,2 the definition

$$A_{h,j}f := \varphi_j f \qquad \text{for all } f\in X_h$$

yields a continuous linear mapping $A_{h,j}:X_h\to X_{h+1}$. Since T_{h+1} is local and $\operatorname{supp}\varphi_2\cap\overline{V}=\emptyset$, we obtain

$$p_{\widetilde{\Omega}\setminus V}T_h f = p_{\widetilde{\Omega}\setminus V}T_{h+1}(A_{h,1}f + A_{h,2}f) = p_{\widetilde{\Omega}\setminus V}T_{h+1}A_{h,1}f$$

for all $f\in X_h$, where $p_{\widetilde{\Omega}\setminus V}:Y\to Y/E_s(\widetilde{\Omega}\setminus V)$ denotes the canonical quotient mapping. Now observe that T_{h+1} is continuous on $E_s(\overline{U})$ and that the range of $A_{h,1}$ is contained in $E_s(\overline{U})$. We conclude that $p_{\widetilde{\Omega}\setminus V}T_h$ is continuous on X_h so that t turns out to be a continuity point of the second type for T_h. This completes the proof of the theorem modulo the following auxiliary result.

2.2.LEMMA. As before, assume that the sequence $(M_p)_{p=o,1,\dots}$ satisfies (M.1) and (M.3)'. Further, let $K\subset\mathbb{R}^n$ be compact and $U_1,\dots,U_m\subset\mathbb{R}^n$ be open such that $K\subset U_1\cup\dots\cup U_m$. Then there exist functions $\varphi_1,\dots,\varphi_m\in\mathcal{C}^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi_j\subset U_j$ and $p_{K,1}(\varphi_j)<\infty$ for j=1,...,m such that $\varphi_1+\dots+\varphi_m\equiv 1$ holds on some neighborhood of K. In this situation, for every real h>o and j=1,...,m we have

$$A_{h,j}f := \varphi_j f \in \mathcal{D}_K^{\{M_p\},h+1} \quad \underline{\text{for all}} \quad f \in \mathcal{D}_K^{\{M_p\},h}$$

and obtain, by this definition, a continuous mapping

$$A_{h,j}: \mathcal{D}_K^{\{M_p\},h} \to \mathcal{D}_K^{\{M_p\},h+1} .$$

Proof. First, from [6,Lemma 4.3] we obtain a sequence of real $N_p>0$ for all $p\in\mathbb{N}_0$ with the properties (M.1) and (M.3)' such that for every $L>0$ there exists some $C>0$ satisfying $N_p \leqq C L^p M_p$ for all $p\in\mathbb{N}_0$. Then, in view of the non-quasi-analyticity, [6,Lemma 5.1] supplies us with a set of functions

$$\varphi_j \in \mathcal{D}^{\{N_p\}}(U_j) \quad \text{for } j=1,\dots,m$$

such that $\varphi_1+\dots+\varphi_m \equiv 1$ holds on some neighborhood of K. Now fix a real $k>0$ such that

$$q_{K,k}(\varphi_j) := \sup_{\alpha\in\mathbb{N}_0^n} \sup_{t\in K} \frac{|D^\alpha \varphi_j(t)|}{k^{|\alpha|} N_{|\alpha|}} < \infty$$

for $j=1,\dots,m$ and choose some $C>0$ such that $k^p N_p \leqq C M_p$ for all $p\in\mathbb{N}_0$. It follows that

$$p_{K,1}(\varphi_j) \leqq C\, q_{K,k}(\varphi_j) < \infty \quad \text{for } j=1,\dots,m ,$$

which proves the first part of the lemma. Finally, for each $j=1,\dots,m$ we have by [6,Prop.2.7]

$$p_{K,h+1}(\varphi_j f) \leqq M_0\, p_{K,1}(\varphi_j)\, p_{K,h}(f) \quad \text{for all } f\in\mathcal{D}_K^{\{M_p\},h} ,$$

from which the remaining assertions are immediate.

We note that theorem 2.1 remains valid for local operators on the space $\mathcal{D}^\omega(\Omega)$ of all test functions on Ω with compact support in the sense of Björck [2,p.358], where the defining mapping $\omega:\mathbb{R}^n\to\mathbb{R}$ is only subject to the conditions (α) and (β) from [2,p.353 and 355]. Indeed, in this case [2,Th.1.3.7] supplies us with suitable partitions of unity so that the proof can be carried out exactly as in the Beurling case. Also, the classical situation $\mathcal{D}(\Omega)$ can be treated in a similar way [1,Example 3.11].

Frequently it is possible to improve the continuity assertion in 2.1 by means of a suitable closed graph theorem. We restrict ourselves to the following typical application.

2.3.COROLLARY. Consider another sequence $(N_p)_{p=0,1,\ldots}$ of positive real numbers with the properties (M.1) and (M.3)' and let # stand for either (N_p) or $\{N_p\}$. Then every local linear operator $T: \mathcal{D}^*(\Omega) \to \mathcal{D}^\#(\Omega)$ is continuous with respect to the natural topologies on both spaces. The same is true for every local linear operator $T: \mathcal{E}^*(\Omega) \to \mathcal{E}^\#(\Omega)$.

Proof. (a) In the first case, we note that the space $\mathcal{D}^\#(\Omega)$ belongs to the Raikov class $\mathcal{D}$ because of [13,Prop. 2.3 and 2.5]. It therefore suffices to show that the map $T: \mathcal{D}^*(\Omega) \to \mathcal{D}^\#(\Omega)$ has a closed graph. But this is immediate after theorem 2.1.

(b) In order to prove the second assertion, let $(K_j)_{j\in\mathbb{N}}$ be an increasing sequence of compact sets in Ω such that $\overset{\circ}{K}_j \uparrow \Omega$ for $j \to \infty$. Given any $j\in\mathbb{N}$, we choose by [6,Lemma 5.1] some $\varphi_j \in \mathcal{D}^*(\Omega)$ with $\varphi_j \equiv 1$ on a neighborhood of K_j and define $A_j\psi := \varphi_j\psi$ for all $\psi\in\mathcal{E}^*(\Omega)$. Then $A_j: \mathcal{E}^*(\Omega) \to \mathcal{D}^*(\Omega)$ is continuous by [6,Th.2.8] and $T|\mathcal{D}^*(\Omega): \mathcal{D}^*(\Omega) \to \mathcal{E}^\#(\Omega)$ is continuous by what we proved in (a). Thus $T_j\psi := T(\varphi_j\psi)$ for all $\psi\in\mathcal{E}^*(\Omega)$ defines a continuous linear mapping $T_j: \mathcal{E}^*(\Omega) \to \mathcal{E}^\#(\Omega)$. Now let q denote an arbitrary continuous seminorm on $\mathcal{E}^\#(\Omega)$. Then there exists some $j\in\mathbb{N}$ such that $q(f) = 0$ for all $f\in\mathcal{E}^\#(\Omega)$ with $K_j \cap \operatorname{supp} f = \emptyset$. Since T is local, we obtain $q(T\psi - T_j\psi) = 0$ for all $\psi\in\mathcal{E}^*(\Omega)$. We conclude that $q\circ T = q\circ T_j$ holds on $\mathcal{E}^*(\Omega)$, which proves the continuity of the mapping $T: \mathcal{E}^*(\Omega) \to \mathcal{E}^\#(\Omega)$.

We next turn to the more complicated situation of local operators within the context of ultradistributions. In certain cases we obtain nice continuity properties which are analogous to the classical situation [11].

2.4.THEOREM. Consider a local linear operator

$$T: \mathcal{D}^{(N_p)}(\Omega) \to \mathcal{E}^{*\prime}(\Omega) \quad \text{or} \quad T: \mathcal{E}^{\{N_p\}\prime}(\Omega) \to \mathcal{E}^{*\prime}(\Omega) ,$$

where the sequence $(N_p)_{p=0,1,\dots}$ satisfies (M.1) and (M.3)'. Then there exists an at most countable subset Λ of Ω without any cluster point in Ω such that the restriction

$$T|\mathcal{D}^{(N_p)}(\Omega\setminus\Lambda) \text{ , resp. } T|\mathcal{E}^{\{N_p\}}(\Omega\setminus\Lambda)$$

is continuous.

Proof. (a) We first consider the case $*=(M_p)$. As before, let $(K_j)_{j\in\mathbb{N}}$ be an increasing sequence of compact sets in Ω such that $\overset{\circ}{K}_j \uparrow \Omega$ for $j\to\infty$. Given any $j\in\mathbb{N}$, we define

$$Y := \mathcal{E}^{(M_p)}(\Omega) \quad \text{and}$$

$$X_j := \{f\in\mathcal{D}^{(N_p)}(\Omega) \text{ , resp. } \mathcal{D}^{\{N_p\}}(\Omega) : \operatorname{supp} f\subset K_j\} .$$

Then Y is the dual of an (F)-space and hence countably boundedly generated in the sense of [1,Section 2]. It follows from [6,Th.2.6,5.4, and 5.8] that in both cases the space X_j is complete metrizable with respect to the natural subspace topology. According to the standard characterization of the ultradistributions with compact support [6,Th.5.9], it makes sense to consider the restriction $T_j := T|X_j : X_j \to Y$ also in the second case. Now, in view of partitions of unity, we may apply [1,Th.3.6] to the local operator T_j with respect to the usual support capacities. Hence, for every $j\in\mathbb{N}$ we obtain some finite subset Λ_j of K_j such that T_j is continuous on $\{f\in X_j : \operatorname{supp} f\subset K\}$ for all compact subsets K of $K_j\setminus\Lambda_j$. Moreover, we may assume $\Lambda_{j+1}\cap\overset{\circ}{K}_j\subset\Lambda_j$ for all $j\in\mathbb{N}$ so that

$$\Lambda := \bigcup_{j=1}^{\infty}\Lambda_j$$

is at most countable and has no cluster point in Ω. In the first case, we immediately arrive at the continuity of the mapping

$$T|\mathcal{D}^{(N_p)}(\Omega\setminus\Lambda) : \mathcal{D}^{(N_p)}(\Omega\setminus\Lambda) \to \mathcal{E}^{(M_p)}(\Omega) .$$

In the second case, we note that $\mathcal{E}^{\{N_p\}}(\Omega\setminus\Lambda)$ is bornological by [6,Th.5.12] and fix an arbitrary bounded subset B of this space. Then [6,Prop.5.11] supplies us with some

compact $K \subset \Omega \setminus \Lambda$ such that B can be regarded as a bounded set in $\mathcal{D}^{\{N_p\},}(\Omega)$ with $\operatorname{supp} f \subset K$ for all $f \in B$. Taking any $j \in \mathbb{N}$ such that $K \subset K_j$, we conclude that $T(B) = T_j(B)$ is bounded in Y. This proves the continuity of the mapping

$$T | \mathcal{E}^{\{N_p\},}(\Omega \setminus \Lambda) : \mathcal{E}^{\{N_p\},}(\Omega \setminus \Lambda) \to \mathcal{E}^{(M_p),}(\Omega) ,$$

where, of course, $\mathcal{E}^{\{N_p\},}(\Omega \setminus \Lambda)$ is canonically identified with a linear subspace of $\mathcal{E}^{\{N_p\},}(\Omega)$.

(b) In the remaining case $*=\{M_p\}$, the preceding arguments require only minor supplements. We first employ [6,Th.7.3] to conclude that

$$\mathcal{E}^{\{M_p\},}(\Omega) \text{ is continuously embedded in } \mathcal{E}^{(M_p),}(\Omega) .$$

Hence part (a) supplies us with a suitable subset Λ of Ω such that the restriction

$$T | \mathcal{D}^{(N_p)}(\Omega \setminus \Lambda) : \mathcal{D}^{(N_p)}(\Omega \setminus \Lambda) \to \mathcal{E}^{\{M_p\},}(\Omega) , \text{ resp.}$$

$$T | \mathcal{E}^{\{N_p\},}(\Omega \setminus \Lambda) : \mathcal{E}^{\{N_p\},}(\Omega \setminus \Lambda) \to \mathcal{E}^{\{M_p\},}(\Omega)$$

is continuous with respect to the $\mathcal{E}^{(M_p),}(\Omega)$ - topology and in particular closed with respect to the natural strong topology on the range space. We now observe that the domain spaces are ultrabornological by [6,Th.2.6 and 5.12] and that the range space endowed with its natural topology belongs to the class $\mathcal{D}$ of Raikov [13]. In fact, this follows easily from [6,Th.5.4,5.8, and Prop.5.11] by means of the standard permanence properties of the Raikov class $\mathcal{D}$ [13,Prop.2.3 and 2.4]. We conclude that T has the desired continuity property.

Again, the preceding theorem carries over to the case of ultradifferentiable functions in the sense of Björck [2] and may be often combined with closed graph theorems. Also, the result can be easily transferred to the non-compact situation.

2.5.COROLLARY. Let $\mathcal{F}(\Omega)$ be one of the spaces $\mathcal{E}^{(N_p)}(\Omega)$

or $\mathcal{D}^{\{N_p\}}(\Omega)$, where $(N_p)_{p=0,1,\dots}$ satisfies (M.1) and (M.3)', and consider a local linear operator T from $\mathcal{F}(\Omega)$ into $\mathcal{D}^{*\prime}(\Omega)$. Then there exists a discrete subset Λ of Ω such that T is continuous on

$$\mathcal{F}(\Omega,\Lambda) := \{f\in\mathcal{F}(\Omega) : f \equiv 0 \text{ on a neighborhood of } \Lambda\} ,$$

where this space is endowed with the inductive limit topology with respect to the spaces $\mathcal{F}_F(\Omega):=\{f\in\mathcal{F}(\Omega):\operatorname{supp} f\subset F\}$ for all relatively closed $F\subset\Omega$ with $F\cap\Lambda=\emptyset$.

Proof. From theorem 2.4 we obtain an appropriate $\Lambda\subset\Omega$ such that the restriction $T|\mathcal{F}_L(\Omega): \mathcal{F}_L(\Omega)\to\mathcal{D}^{*\prime}(\Omega)$ is continuous for all compact $L\subset\Omega\setminus\Lambda$. Now consider an arbitrary closed subset F of Ω with $F\cap\Lambda=\emptyset$ and fix a continuous seminorm q on $\mathcal{D}^{*\prime}(\Omega)$. Then there exists a compact $K\subset\Omega$ such that $q(f)=0$ for all $f\in\mathcal{D}^{*\prime}(\Omega)$ satisfying $K\cap\operatorname{supp} f=\emptyset$. From [6,Lemma 5.1] we obtain some $\varphi\in\mathcal{D}^{(N_p)}(\Omega)$, $\mathcal{D}^{\{N_p\}}(\Omega)$ respectively such that $\varphi\equiv 1$ on a neighborhood of K. Because of [6,Th.2.8 and 6.2], the multiplication operator $A: \mathcal{F}_F(\Omega)\to\mathcal{F}_{F\cap\operatorname{supp}\varphi}(\Omega)$ given by $A\psi:=\varphi\psi$ for all $\psi\in\mathcal{F}_F(\Omega)$ is continuous. It follows that $S\psi:=T(\varphi\psi)$ for all $\psi\in\mathcal{F}_F(\Omega)$ defines a continuous linear mapping $S: \mathcal{F}_F(\Omega)\to\mathcal{D}^{*\prime}(\Omega)$. On the other hand we have $q\circ T = q\circ S$ on $\mathcal{F}_F(\Omega)$, since T is local. We conclude that T is continuous on $\mathcal{F}_F(\Omega)$. The assertion follows.

We close this section with some counterexamples to show that theorem 2.4 is best possible. The following result seems to be new even in the classical situation.

2.6.EXAMPLE. Let $\mathcal{F}(\Omega)$ be $=\mathcal{D}^{\{N_p\}}(\Omega), \mathcal{E}^{(N_p)}(\Omega)$, or $\mathcal{E}'(\Omega)$, where the sequence $(N_p)_{p=0,1,\dots}$ satisfies (M.1) and (M.3)'. Then there exists a local linear operator $T: \mathcal{F}(\Omega)\to\mathcal{D}^{*\prime}(\Omega)$ such that T is discontinuous on $\mathcal{F}(U)$ for all nonempty open subsets U of Ω. Moreover, the range of T can be chosen to be contained in $\mathcal{C}'(\Omega)$.

Proof. (a) We may of course assume $\Omega=\mathbb{R}^n$ and restrict

ourselves to give the detailed construction of T only for the case $\mathcal{F} := \mathcal{D}^{\{N_p\}}(\mathbb{R}^n)$, which is sufficiently typical. For compact $K \subset \mathbb{R}^n$ and $m \in \mathbb{N}$, we consider the usual seminorm

$$p_{K,m}(f) := \sup \{ \frac{|D^\alpha f(t)|}{m^{|\alpha|} N_{|\alpha|}} : \alpha \in \mathbb{N}_o^n \text{ and } t \in K \} \quad (\leqq \infty)$$

for all $f \in C^\infty(\mathbb{R}^n)$ and write for simplicity $\mathcal{F}_K^m := \mathcal{D}_K^{\{N_p\},m}$. Given any $t \in \mathbb{R}^n$ and $m \in \mathbb{N}$, we say that an $f \in \mathcal{F}$ is locally of order $\leqq m$ at the point t, if there exists some compact neighborhood K of t such that $p_{K,m}(f) < \infty$. Let $\mathcal{F}(t,m)$ denote the linear space of all $f \in \mathcal{F}$, which are locally of order $\leqq m$ at t.

(b) We claim that for all choices of $t \in \mathbb{R}^n$, $m \in \mathbb{N}$, and $\varepsilon > 0$ there exists some $f \in \mathcal{F}^{m+2}_{\nabla(t,\varepsilon)}$ with $f \notin \mathcal{F}(t,m)$, where $\nabla(t,\varepsilon)$ is the closed ball in $\mathbb{R}^n$ centered at t with radius ε. Assuming this assertion to be false, a simple translation argument ensures that $\mathcal{F}^{m+2}_{\nabla(s,\varepsilon)} \subset \mathcal{F}(s,m)$ holds for an arbitrary point $s \in \mathbb{R}^n$. This implies that $\mathcal{F}^{m+2}_K \subset \mathcal{F}(s,m)$ for every $s \in K := \nabla(t,\varepsilon/2)$. Consequently, for each $f \in \mathcal{F}^{m+2}_K$ there exists a finite covering of K consisting of bounded open sets $U_1, \ldots, U_k$ such that $p_{\overline{U}_j,m}(f) < \infty$ for $j = 1, \ldots, k$. We now choose a set of functions $\varphi_1, \ldots, \varphi_k \in C^\infty(\mathbb{R}^n)$ according to lemma 2.2 and obtain by [6,Prop.2.7] the estimates

$$p_{K,m+1}(\varphi_j f) = p_{\overline{U}_j,m+1}(\varphi_j f) \leqq N_o p_{K,1}(\varphi_j) p_{\overline{U}_j,m}(f) < \infty$$

for $j = 1, \ldots, k$ and hence $p_{K,m+1}(f) < \infty$. It follows that the inclusion mapping between the Banach spaces $\mathcal{F}^{m+1}_K$ and $\mathcal{F}^{m+2}_K$ is bijective and therefore a topological isomorphism by the open mapping theorem. On the other hand, this mapping is known to be compact [6,Prop.2.2], which forces $\mathcal{F}^{m+1}_K$ to be of finite dimension. But this is clearly impossible, for instance by virtue of lemma 2.2.

(c) We now turn to the construction of T. Let $(t_m)_{m \in \mathbb{N}}$ be a sequence of distinct points $t_m \in \mathbb{R}^n$ which are dense in $\mathbb{R}^n$. In view of part (b) and Zorn's lemma, for every $m \in \mathbb{N}$ there exists a linear functional F_m on $\mathcal{F}$, which vanishes on $\mathcal{F}(t_m,m)$ and satisfies $F_m(f_m) = 1$ for some $f_m \in \mathcal{F}^{m+2}_{\nabla(t_m,1/m)}$.

We then define

$$T(f) := \sum_{m=1}^{\infty} F_m(f)\,\delta_{t_m} \qquad \text{for all } f\in\mathcal{F}.$$

Since each $f\in\mathcal{F}$ is globally of finite order, the above series is in fact a finite sum for any given $f\in\mathcal{F}$. Moreover, it is immediate that the mapping $T\colon \mathcal{F}\to\mathcal{C}'(\mathbb{R}^n)$ is linear and local. Consider now an open and nonempty subset U of $\mathbb{R}^n$. From [6,Th.7.3] we know that $\mathcal{E}^{(N_p)}(U)$ is dense in $\mathcal{E}^{\{N_p\}}(U)$. Using [6,Lemma 5.1 and Th.2.8] it is easy to deduce that $\mathcal{D}^{(N_p)}(U)$ is dense in $\mathcal{D}^{\{N_p\}}(U)$ as well. Since $\mathcal{D}^{(N_p)}(U)$ is certainly contained in $\mathcal{F}(t_m,m)$ for all $m\in\mathbb{N}$, we conclude that T vanishes on a dense subspace of $\mathcal{D}^{\{N_p\}}(U)$. On the other hand, we have $\nabla(t_k,1/k)\subset U$ for some suitable $k\in\mathbb{N}$ and obtain from [6,Lemma 5.1] a $\varphi\in\mathcal{D}^*(\mathbb{R}^n)$ such that $\varphi(t_k)=1$ as well as $\varphi(t_m)=0$ for all $m=1,\dots,k+1$ with $m\neq k$. Then the identity

$$\langle T(f_k),\varphi\rangle = \sum_{m=1}^{k+1} F_m(f_k)\varphi(t_m) = F_k(f_k) = 1$$

shows that the restriction $T|\mathcal{D}^{\{N_p\}}(U)\colon \mathcal{D}^{\{N_p\}}(U)\to\mathcal{D}^{*\prime}(\mathbb{R}^n)$ cannot be continuous.

(d) In the classical situation $\mathcal{F}=\mathcal{E}'(\mathbb{R}^n)$, an $f\in\mathcal{E}'(\mathbb{R}^n)$ is said to be locally of order $\leq m$ at a given point $t\in\mathbb{R}^n$, if there exists a compact neighborhood K of t and a constant $c>0$ such that

$$|\langle f,\varphi\rangle| \leq c \sup\{|D^\alpha\varphi(s)| : s\in K,\ |\alpha|\leq m\}$$

holds for all $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi\subset K$. In this case, assertion (b) can be easily verified by means of appropriate derivatives of the Dirac measure at t, and the above construction can be carried out even much simplier. The remaining case $\mathcal{F}=\mathcal{E}^{(N_p)\prime}(\mathbb{R}^n)$ can be treated along the same lines.

2.7.EXAMPLE. Let $\mathcal{F}(\Omega)$ stand for $\mathcal{D}^{(N_p)}(\Omega)$ or $\mathcal{E}^{\{N_p\}\prime}(\Omega)$, where $(N_p)_{p=0,1,\dots}$ satisfies (M.1) and (M.3)', and con-

sider a countable $\Lambda \subset \Omega$ without any cluster point in Ω. Then there exists a local linear operator $T: \mathcal{F}(\Omega) \to \mathcal{D}^{*\prime}(\Omega)$ such that T is discontinuous on $\mathcal{F}(U)$ for all open $U \subset \Omega$ with $U \cap \Lambda \neq \emptyset$ and $T = 0$ on $\mathcal{F}(\Omega \setminus \Lambda)$. Moreover, the range of T can be chosen to be contained in $C'(\Omega)$.

Proof. The result is known for the case $\mathcal{D}(\Omega)$, but the classical argument [11,p.119] has to be modified to work also in the context of distributions. Given any $\lambda \in \Lambda$, let V_λ be the linear space consisting of all germs at the point λ of functions $f \in \mathcal{E}^{(N_p)}(\Omega)$, resp. distributions $f \in \mathcal{D}^{\{N_p\}}(\Omega)$ and let $[f]_\lambda$ denote the corresponding germ of f at λ. Then there exists a linear functional F_λ on V_λ such that $F_\lambda([p]_\lambda)=0$ for all polynomials p in n variables and $F_\lambda([e_\lambda]_\lambda)=1$, where $e_\lambda(t) := \exp(t_1-\lambda_1+\ldots+t_n-\lambda_n)$ for all $t \in \Omega$. Of course, p and e_λ are considered as ultradistributions in the second case. Now let

$$T(f) := \sum_{\lambda \in \Lambda} F_\lambda([f]_\lambda)\, \delta_\lambda \qquad \text{for all } f \in \mathcal{F}(\Omega) .$$

It is clear that T is well-defined, linear, and local with $T = 0$ on $\mathcal{F}(\Omega \setminus \Lambda)$. Finally, let $U \subset \Omega$ be open with $U \cap \Lambda \neq \emptyset$. From [6,Lemma 5.1] we obtain some $\varphi \in \mathcal{D}^*(\Omega)$ such that $\varphi(\lambda)=1$ for exactly one $\lambda \in U \cap \Lambda$ and $\varphi(\mu)=0$ for all $\mu \in \Lambda$ with $\lambda \neq \mu$. Then $\langle T(f), \varphi \rangle = 0$ whenever $f \in \mathcal{F}(U)$ coincides with a polynomial in n variables on a neighborhood of λ. On the other hand we have $\langle T(f), \varphi \rangle = 1$ if $f \in \mathcal{F}(U)$ is equal to e_λ on some neighborhood of λ . Hence the assertion follows easily by means of [6,Th.7.3].

3. Vector-valued ultradistributions supported by the origin

We start with some preliminaries on sequence spaces. Given a Banach space X with norm $\|\cdot\|$, an $n \in \mathbb{N}$ and a sequence $C = (C_p)_{p=0,1,\ldots}$ of positive real numbers, let $l^1_n(C,X)$, resp. $l^\infty_n(C,X)$ denote the space of all

$$x = (x_\beta)_{\beta \in \mathbb{N}_0^n} \in \prod_{\beta \in \mathbb{N}_0^n} X$$

such that

$$\|x\|_{C,1} := \sum_{\beta\in\mathbb{N}_o^n} C_{|\beta|}\|x_\beta\| < \infty \text{ , resp.}$$

$$\|x\|_{C,\infty} := \sup_{\beta\in\mathbb{N}_o^n} C_{|\beta|}\|x_\beta\| < \infty .$$

Clearly, these spaces are Banach spaces with respect to the corresponding norms. In the scalar-valued case $X=\mathbb{R}$ or $\mathbb{C}$, we write $l_n^1(C)$ and $l_n^\infty(C)$ for simplicity. The proof of the following lemma is straightforward and therefore omitted.

3.1.LEMMA. Let $C^{-1}:=(C_p^{-1})_{p=o,1,\dots}$. Then the Banach space $\mathcal{L}(l_n^1(C),X)$ is isometrically isomorphic to $l_n^\infty(C^{-1},X)$ by means of the isomorphism $\Psi\colon \mathcal{L}(l_n^1(C),X)\to l_n^\infty(C^{-1},X)$ given by

$$\Psi(T) := (Te_\beta)_{\beta\in\mathbb{N}_o^n} \qquad \text{for all } T\in\mathcal{L}(l_n^1(C),X) ,$$

where $e_\beta:=(\delta_{\alpha\beta})_{\alpha\in\mathbb{N}_o^n}$ for each $\beta\in\mathbb{N}_o^n$.

For the remainder of this section, let Ω be an open set in $\mathbb{R}^n$ containing the origin. We also assume that the sequence $(M_p)_{p=o,1,\dots}$ of positive real numbers satisfies all the conditions (M.0), (M.1), (M.2), and (M.3). In the following, * stands for either (M_p) or $\{M_p\}$. Given a compact subset K of Ω, we define

$$\mathcal{D}_K^{*\prime}(\Omega) := \{f\in\mathcal{D}^{*\prime}(\Omega) : \operatorname{supp} f\subset K\}$$

and endow this space with the strong topology coming from $\mathcal{D}_b^{*\prime}(\Omega)$.

3.2.LEMMA. $\mathcal{D}_K^{(M_p),\prime}(\Omega)$ is a (DFN)-space and $\mathcal{D}_K^{\{M_p\},\prime}(\Omega)$ is an (FN)-space; in particular, both spaces are ultrabornological.

Proof. According to [6,Lemma 5.1] there exists a $\varphi\in\mathcal{D}^*(\Omega)$ with $\varphi\equiv 1$ on some neighborhood of K. Let $A\colon \mathcal{E}^*(\Omega)\to\mathcal{D}^*(\Omega)$ denote the continuous linear mapping given by $A\psi:=\varphi\psi$ for all $\psi\in\mathcal{E}^*(\Omega)$. Then the transpose $A'\colon\mathcal{D}^{*\prime}(\Omega)\to\mathcal{E}^{*\prime}(\Omega)$ is con-

tinuous with respect to the strong topologies and satisfies $A'=id$ on $\mathcal{D}_K^{*'}(\Omega)$. We conclude that the strong topologies of $\mathcal{D}^{*'}(\Omega)$ and $\mathcal{E}^{*'}(\Omega)$ coincide on the subspace $\mathcal{D}_K^{*'}(\Omega)$. Because of [6,Th.5.8], $\mathcal{D}_K^{*'}(\Omega)$ is a complete subspace of $\mathcal{D}^{*'}(\Omega)$ and hence also of $\mathcal{E}^{*'}(\Omega)$. By [6,Th.5.4 and 5.12] we know that $\mathcal{D}^{\{M_p\}'}(\Omega)$ is an (FN)-space and that $\mathcal{E}^{(M_p)'}(\Omega)$ is a (DFN)-space. Hence the main assertions follow from well-known permanence properties of these spaces [12,Prop.5.1.1 and 5.1.7]. The last statement can be verified by means of [12,Th.4.4.12] and [8,p.4oo] .

According to the structure theorem for scalar-valued ultradistributions with support at the origin [4,Satz 2.1; 7,Th.3.1], an ultradistribution $f\in\mathcal{D}^{*'}(\Omega)$ satisfies $\operatorname{supp} f\subset\{o\}$ if and only if it has a representation

$$(1)\qquad f=\sum_{\beta\in\mathbb{N}_o^n} a_\beta D^\beta\delta_o \ ,$$

where the scalar sequence $(a_\beta)_{\beta\in\mathbb{N}_o^n}$ fulfills

$(2^{(M_p)})$ $\exists\ L,C>o$ such that $|a_\beta|\leq C L^{|\beta|} M_{|\beta|}^{-1}$ for all $\beta\in\mathbb{N}_o^n$

in the Beurling case $*=(M_p)$ and

$(2^{\{M_p\}})$ $\forall\ L>o\ \exists C>o$ such that $|a_\beta|\leq CL^{|\beta|}M_{|\beta|}^{-1}$ for all $\beta\in\mathbb{N}_o^n$

in the Roumieu case $*=\{M_p\}$, respectively. Moreover, the representation (1) is unique and converges in $\mathcal{D}_b^{*'}(\Omega)$. Conversely, if a family of scalars a_β satisfies condition (2^*), then the series in (1) converges in $\mathcal{D}_b^{*'}(\Omega)$ and represents an ultradistribution $f\in\mathcal{D}_{\{o\}}^{*'}(\Omega)$.

Now let λ^* denote the linear space of all scalar sequences $(a_\beta)_{\beta\in\mathbb{N}_o^n}$ satisfying condition (2^*), and for each $k\in\mathbb{N}$ let

$$A^{(k)}:=(k^{-p}M_p)_{p\in\mathbb{N}_o} \quad\text{and}\quad B^{(k)}:=(k^pM_p)_{p\in\mathbb{N}_o}$$

so that

$$\lambda^{(M_p)}=\bigcup_{k=1}^{\infty} l_n^\infty(A^{(k)}) \quad\text{and}\quad \lambda^{\{M_p\}}=\bigcap_{k=1}^{\infty} l_n^\infty(B^{(k)}) \ .$$

Let λ^* be equipped with the corresponding inductive, resp. projective locally convex limit topology of Banach spaces. Then we have:

3.3.LEMMA. The mapping $\Phi^*: \lambda^* \to \mathcal{D}'^*_{\{o\}}(\Omega)$ given by

$$\Phi^*\Big((a_\beta)_{\beta\in\mathbb{N}_o^n}\Big) := \sum_{\beta\in\mathbb{N}_o^n} a_\beta D^\beta\delta_o \qquad \forall\ (a_\beta)_{\beta\in\mathbb{N}_o^n}\in\lambda^*$$

is a topological linear isomorphism.

Proof. (a) We first consider the Beurling case $*=(M_p)$. If B_k denotes the unit ball of the Banach space $l^\infty_n(A^{(k)})$, then for each $k\in\mathbb{N}$ the set

$$\Phi^{(M_p)}(B_k) = \{ \sum_{\beta\in\mathbb{N}_o^n} a_\beta D^\beta\delta_o : \sup_{\beta\in\mathbb{N}_o^n} |a_\beta| M_{|\beta|} k^{-|\beta|} \leq 1 \}$$

is bounded in $\mathcal{D}^{(M_p)'}_b(\Omega)$ by the first structure theorem of Komatsu [6,Th.8.1]. From this we get the continuity of

$$\Phi^{(M_p)} \quad \text{on} \quad \lambda^{(M_p)} = \operatorname{ind}_{k\to\infty} l^\infty_n(A^{(k)}) .$$

Since $\Phi^{(M_p)}$ is known to be bijective, it follows that $\Phi^{(M_p),-1}$ has a closed graph. Since $\mathcal{D}^{(M_p)'}_{\{o\}}(\Omega)$ is ultrabornological by 3.2 and since $\lambda^{(M_p)}$ is the countable inductive limit of Banach spaces, a standard version of the closed graph theorem [13,Prop.2.3 and 2.4] confirms the continuity of $\Phi^{(M_p),-1}$. This settles the Beurling case.

(b) In the Roumieu case $*=\{M_p\}$ we observe that $\lambda^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}'}_{\{o\}}(\Omega)$ are both (F)-spaces. In view of the classical open mapping theorem, it therefore suffices to prove the continuity of the linear isomorphism $\Phi^{\{M_p\}}$. To this end, let B denote an arbitrary bounded set in $\lambda^{\{M_p\}}$. Hence for every $k\in\mathbb{N}$ there exists some $c>0$ such that

$$|a_\beta| \leq c\, M^{-1}_{|\beta|}\, k^{-|\beta|} \qquad \text{for all } \beta\in\mathbb{N}_o^n \text{ and } a\in B.$$

Consequently $\Phi^{\{M_p\}}(B)$ is bounded in $\mathcal{D}^{\{M_p\}'}_b(\Omega)$ by [6,Th. 8.7]. This proves the continuity of Φ^* on λ^* for $*=\{M_p\}$.

In the following, let E be a complete locally convex topological vector space. As in the case of usual distributions [15], any continuous linear mapping $f: \mathcal{D}^*(\Omega) \to E$ is called an E-valued ultradistribution on Ω of class (M_p), resp. $\{M_p\}$. Since $\mathcal{D}^*(\Omega)$ is complete and barrelled [6,Th. 2.6] and since $\mathcal{D}_b^{*\prime}(\Omega)$ is complete and nuclear [6,Th.5.4], we have canonical topological isomorphisms

$$\mathcal{L}_b(\mathcal{D}^*(\Omega),E) \cong \mathcal{D}_b^{*\prime}(\Omega) \hat{\otimes} E \cong \mathcal{L}_\varepsilon(E'_\tau,\mathcal{D}_b^{*\prime}(\Omega))$$

by [16,Prop.5o.5 and 5o.4]. Given a closed linear subspace $\mathcal{H}$ of $\mathcal{D}_b^{*\prime}(\Omega)$, let $\mathcal{H}(E)$ consist of all $f \in \mathcal{L}(\mathcal{D}^*(\Omega),E)$ such that $e' \circ f \in \mathcal{H}$ for every $e' \in E'$. Then $\mathcal{H}(E)$ is a closed linear subspace of $\mathcal{L}_b(\mathcal{D}^*(\Omega),E)$, and we obtain the following canonical isomorphisms

$$\mathcal{H}(E) \cong \mathcal{H} \hat{\otimes} E \cong \mathcal{L}_\varepsilon(E'_\tau,\mathcal{H}) .$$

In view of lemma 3.2, this remark applies, in particular, to the space $\mathcal{H}=\mathcal{D}_{\{o\}}^{*\prime}(\Omega)$. An obvious combination with lemma 3.3 leads therefore to the following representation of vector-valued ultradistributions with support at the origin.

3.4.LEMMA. The canonical mapping $\theta^*: \lambda^* \otimes E \to \mathcal{D}_{\{o\}}^{*\prime}(\Omega,E)$ given by

$$\theta^*\Big(\sum_{j=1}^{k} a_j \otimes e_j\Big)(\varphi) := \sum_{\beta \in \mathbb{N}_o^n} \sum_{j=1}^{k} a_{j,\beta} D^\beta \varphi(o)\, e_j ,$$

where $a_j \in \lambda^*$, $e_j \in E$, $\varphi \in \mathcal{D}^*(\Omega)$ and the series on the right-hand side converges in $\mathcal{L}_b(\mathcal{D}^*(\Omega),E)$, extends to a topological linear isomorphism θ^* from $\lambda^* \hat{\otimes} E$ onto $\mathcal{D}_{\{o\}}^{*\prime}(\Omega,E)$.

Now, let $\mathcal{U}$ denote a neighborhood base at o consisting of absolutely convex sets in E and consider the space

$$\lambda^*\{E\} := \{x=(x_\beta)_{\beta \in \mathbb{N}_o^n} : x_\beta \in E,\ (q_U(x_\beta))_{\beta \in \mathbb{N}_o^n} \in \lambda^*\ \forall\ U \in \mathcal{U}\} ,$$

where q_U denotes the seminorm associated with $U \in \mathcal{U}$. Further, for each $x \in \lambda^*\{E\}$ and $m \in \mathbb{N}$, let $x^{(m)} \in \lambda^*\{E\}$ be given by

$$x_\beta^{(m)} := x_\beta \quad \text{for } \beta \in \mathbb{N}_o^n \text{ with } |\beta| < m,\ x_\beta^{(m)} := o \text{ otherwise.}$$

We also consider the injection $\rho^*\colon \lambda^* \otimes E \to \lambda^*\{E\}$ given by

$$\rho^*\Big(\sum_{j=1}^{k} a_j \otimes e_j\Big) := \Big(\sum_{j=1}^{k} a_{j,\beta} e_j\Big)_{\beta\in\mathbb{N}_o^n} ,$$

where $a_j \in \lambda^*$ and $e_j \in E$ for $j=1,\dots,k$. In order to identify $\lambda^*\{E\}$ with the completion $\lambda^* \hat{\otimes} E$, we have to introduce a suitable topology on $\lambda^*\{E\}$. This is easy in the Roumieu case, since it can be immediately deduced from [5,§22; Satz 4.2] and [8,§3o;Prop.8.1] that the space $\lambda^{\{M_p\}}$ is a perfect sequence space with its normal topology. Consequently, $\lambda^{\{M_p\}}\{E\}$ carries a natural topology and then admits the following representation according to a result of Pietsch, cf. [9,p.196-198].

3.5.LEMMA. In the Roumieu case $*=\{M_p\}$, ρ^* extends to a topological linear isomorphism from the space $\lambda^* \hat{\otimes} E$ onto $\lambda^*\{E\}$, and $x^{(m)} \to x$ as $m \to \infty$ for all $x \in \lambda^*\{E\}$.

In order to handle the Beurling case, we observe that the complete space E is the projective limit of the Banach spaces E_U, where $U \in \mathcal{U}$ and E_U is the completion of $E/q_U^{-1}(o)$ with respect to the norm induced by q_U [14,Ch.II;Prop. 5.4]. For all $U,V \in \mathcal{U}$ with $U \subset V$ let $\pi_U\colon E \to E_U$ and $\pi_{V,U}\colon E_U \to E_V$ denote the canonical continuous linear maps and define

$$\tilde{\pi}_U(x) := \big(\pi_U(x_\beta)\big)_{\beta\in\mathbb{N}_o^n} \qquad \forall\, x=(x_\beta)_{\beta\in\mathbb{N}_o^n} \in \prod_{\beta\in\mathbb{N}_o^n} E ,$$

$$\tilde{\pi}_{V,U}(x) := \big(\pi_{V,U}(x_\beta)\big)_{\beta\in\mathbb{N}_o^n} \quad \forall\, x=(x_\beta)_{\beta\in\mathbb{N}_o^n} \in \prod_{\beta\in\mathbb{N}_o^n} E_U .$$

Then we have

$$\lambda^{(M_p)}\{E\} = \bigcap_{U\in\mathcal{U}} \bigcup_{k\in\mathbb{N}} \{x \in \prod_{\beta\in\mathbb{N}_o^n} E : \tilde{\pi}_U(x) \in l_n^\infty(A^{(k)},E_U)\} .$$

Further, $\tilde{\pi}_{V,U}$ maps $l_n^\infty(A^{(k)},E_U)$ continuously into $l_n^\infty(A^{(k)},E_V)$ for each $k \in \mathbb{N}$ so that we obtain a continuous linear mapping

$$\tilde{\pi}_{V,U}\colon \operatorname*{ind}_{k\to\infty} l_n^\infty(A^{(k)},E_U) \to \operatorname*{ind}_{k\to\infty} l_n^\infty(A^{(k)},E_V) .$$

We thus arrive at a projective spectrum

$$\{ \operatorname{ind}_{k\to\infty} l^\infty_n(A^{(k)},E_U) , \tilde{\pi} \}_{U\in\mathcal{U}}$$

and endow $\lambda^{(M_p)}\{E\}$ with the corresponding natural topology

$$\lambda^{(M_p)}\{E\} = \operatorname{proj}_{U\in\mathcal{U}} \operatorname{ind}_{k\to\infty} l^\infty_n(A^{(k)},E_U) .$$

3.6.LEMMA. The assertions of lemma 3.5 remain valid in the Beurling case $*=(M_p)$.

Proof. It follows from [5,§22;Prop.4.3] that the space

$$Y := \operatorname{proj}_{k\to\infty} l^1_n(A^{(k)},-1)$$

is the projective limit of a strict and compact (even nuclear) projective spectrum. According to [5,§26;Satz 2.4], $\lambda^{(M_p)}$ can therefore be identified with Y'_b . Since Y is an (FN)-space, we know that Y is barrelled and that Y'_b is complete and nuclear. We thus obtain a canonical topological isomorphism

$$\lambda^{(M_p)} \hat{\otimes} E \cong \mathcal{L}_b(Y,E)$$

by [16,Prop.5o.5]. Since E is the projective limit of the spaces E_U for $U\in\mathcal{U}$, we may employ [1o,Satz 1 and 2] and 3.1 to arrive at the following canonical topological isomorphisms

$$\begin{aligned}\lambda^{(M_p)} \hat{\otimes} E &\cong \operatorname{proj}_{U\in\mathcal{U}} \mathcal{L}_b(Y,E_U)\\ &\cong \operatorname{proj}_{U\in\mathcal{U}} \operatorname{ind}_{k\to\infty} \mathcal{L}_b(l^1_n(A^{(k)},-1),E_U)\\ &\cong \operatorname{proj}_{U\in\mathcal{U}} \operatorname{ind}_{k\to\infty} l^\infty_n(A^{(k)},E_U) = \lambda^{(M_p)}\{E\} .\end{aligned}$$

It remains to check that for every $x\in\lambda^{(M_p)}\{E\}$ we have $x^{(m)}\to x$ for $m\to\infty$. This amounts to show that $\tilde{\pi}_U(x^{(m)}-x)\to o$ as $m\to\infty$, where $U\in\mathcal{U}$ is arbitrarily chosen. Taking any $k\in\mathbb{N}$ such that $\tilde{\pi}_U(x)\in l^\infty_n(A^{(k)},E_U)$, we obtain

$$\begin{aligned}\| \tilde{\pi}_U(x^{(m)}-x) \|_{A^{(k+1)},\infty} &= \sup_{\beta\in\mathbb{N}_o^n} q_U(x^{(m)}_\beta-x_\beta) M_{|\beta|}^{(k+1)^{-|\beta|}}\\ &\leq \left(\frac{k}{k+1}\right)^m \| \tilde{\pi}_U(x) \|_{A^{(k)},\infty} \to o \quad \text{for } m\to\infty .\end{aligned}$$

This proves the desired convergence.

We are now able to prove the following structure theorem for vector-valued ultradistributions with support at the origin.

3.7.THEOREM. Let $\mathcal{Q}$ denote the family of all continuous seminorms on a given complete locally convex topological vector space E. Then every E-valued ultradistribution $f\in\mathcal{L}(\mathcal{D}^*(\Omega),E)$ with $\operatorname{supp} f\subset\{o\}$ has a unique representation of the form

$$(3)\qquad f=\sum_{\beta\in\mathbb{N}_o^n} a_\beta\otimes D^\beta\delta_o\ ,$$

where the E-valued sequence $(a_\beta)_{\beta\in\mathbb{N}_o^n}$ satisfies

$$(4^{(M_p)})\quad \forall\, q\in\mathcal{Q}\ \exists\, L,C>o \text{ such that } q(a_\beta)\leq CL^{|\beta|}M_{|\beta|}^{-1}\quad \forall\,\beta\in\mathbb{N}_o^n$$

in the Beurling case $*=(M_p)$, resp.

$$(4^{\{M_p\}})\quad \forall\, q\in\mathcal{Q},L>o\ \exists\, C>o \text{ such that } q(a_\beta)\leq CL^{|\beta|}M_{|\beta|}^{-1}\quad \forall\,\beta\in\mathbb{N}_o^n$$

in the Roumieu case $*=\{M_p\}$. Conversely, whenever $a_\beta\in E$ are given such that condition (4^*) is fulfilled, then the series in (3) converges in $\mathcal{L}_b(\mathcal{D}^*(\Omega),E)$ to an E-valued ultradistribution $f\in\mathcal{L}(\mathcal{D}^*(\Omega),E)$ with $\operatorname{supp} f\subset\{o\}$.

Proof. By 3.4,3.5, and 3.6 we know that the mapping $\Theta^*\rho^{*-1}$ is a topological linear isomorphism from $\lambda^*\{E\}$ onto $\mathcal{D}'^*_{\{o\}}(\Omega,E)$. Given an arbitrary $a\in\lambda^*\{E\}$, it is clear that the identity

$$\Theta^*\rho^{*-1}(a^{(m)})=\sum_{|\beta|<m} a_\beta\otimes D^\beta\delta_o$$

holds for all $m\in\mathbb{N}$, since Θ^* and ρ^* are the canonical mappings. Because $a^{(m)}$ converges to a in the topology of $\lambda^*\{E\}$, we obtain by continuity

$$\Theta^*\rho^{*-1}(a)=\sum_{\beta\in\mathbb{N}_o^n} a_\beta\otimes D^\beta\delta_o\ ,$$

where the series on the right-hand side converges in $\mathcal{L}_b(\mathcal{D}^*(\Omega),E)$. This completes the proof of the theorem.

4. Representation of local operators

Throughout this section, let Ω denote an arbitrary open subset of $\mathbb{R}^n$ and consider a sequence $(M_p)_{p=0,1,\ldots}$ of positive real numbers such that (M.0), (M.1), (M.2), and (M.3) are fulfilled. As usual, * stands for either (M_p) or $\{M_p\}$. We shall need two technical results concerning the functions $v_\pm\colon \mathbb{R}^n \times \Omega \to \mathbb{R}^n \times \Omega$ given by $v_\pm(x,y) := (x \pm y, y)$ for all $(x,y) \in \mathbb{R}^n \times \Omega$. These functions are bijective and in fact inverse to each other. Furthermore, it is easily seen that the definition $V_\pm \varphi := \varphi \circ v_\pm$ for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n \times \Omega)$ yields a pair of continuous linear mappings from $\mathcal{D}^*(\mathbb{R}^n \times \Omega)$ into itself, which are again inverse to each other. In particular, we know that the transposed mappings $V'_\pm : \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega) \to \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$ given by $V'_\pm f := f \circ V_\pm$ for all $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$ are continuous with respect to the strong topology.

4.1.LEMMA. $\operatorname{supp}(V'_\pm f) = v_\pm(\operatorname{supp} f)$ for all $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$.

Proof. The assertions follow immediately from the identities $\operatorname{supp}(\varphi \circ v_\pm) = v_\mp(\operatorname{supp} \varphi)$ for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n \times \Omega)$.

4.2.LEMMA. Let $\mathcal{F}$ be a linear subspace of $\mathcal{D}^{*\prime}(\Omega)$ and suppose that $\mathcal{F}$ is equipped with a complete locally convex topology such that the following conditions are satisfied:

(a) $\mathcal{F}$ is continuously embedded in $\mathcal{D}^{*\prime}_b(\Omega)$.

(b) The multiplication $(\varphi, f) \mapsto \varphi f$ defines a continuous bilinear mapping from $\mathcal{E}^*(\mathbb{R}^n) \times \mathcal{F}$ into $\mathcal{F}$.

Then the operators $V'_\pm$ map $\mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}$ continuously into itself, where $\mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}$ is canonically embedded in the space $\mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$.

Proof. We write $\mathcal{D}^* := \mathcal{D}^*(\mathbb{R}^n)$, $\mathcal{D}^{*\prime} := \mathcal{D}^{*\prime}(\mathbb{R}^n)$, $\mathcal{E}^* := \mathcal{E}^*(\mathbb{R}^n)$ for shortness and observe that there are canonical topological isomorphisms $\mathcal{L}(\mathcal{D}^*, \mathcal{E}^*) \cong \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{E}^*$ as well as $\mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega) \cong \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{D}^{*\prime}(\Omega)$ according to [16,Prop.5o.5] and [7,Th.2.3]. We now have to introduce some auxiliary

mappings. First, let $A_{\pm}: \mathcal{D}^{*\prime} \to \mathcal{L}(\mathcal{D}^*, \mathcal{E}^*) \cong \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{E}^*$ be given by

$$\langle A_{\pm} g, \varphi \rangle := \langle g, \varphi(\cdot \pm y) \rangle \quad \forall\ g \in \mathcal{D}^{*\prime},\ \varphi \in \mathcal{D}^*,\ y \in \mathbb{R}^n .$$

Then $A_{\pm}$ are obviously related to the usual convolution in the theory of ultradistributions and hence continuous by [6,Th.6.1o]. Next, we employ assumption (b) on $\mathcal{F}$ to obtain a continuous linear mapping $B: \mathcal{E}^* \hat{\otimes} \mathcal{F} \to \mathcal{F}$ such that

$$B(\varphi \otimes f) = \varphi f \quad \forall\ \varphi \in \mathcal{E}^* \text{ and } f \in \mathcal{F} .$$

Finally, let $J: \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{F} \to \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{D}^{*\prime}(\Omega) \cong \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$ denote the canonical mapping and consider the compositions

$$C_{\pm} := J \circ (I_{\mathcal{D}^{*\prime}} \hat{\otimes} B) \circ (A_{\pm} \hat{\otimes} I_{\mathcal{F}}) ,$$

where I is the identity operator and $(\mathcal{D}^{*\prime} \hat{\otimes} \mathcal{E}^*) \hat{\otimes} \mathcal{F}$ is canonically identified with $\mathcal{D}^{*\prime} \hat{\otimes} (\mathcal{E}^* \hat{\otimes} \mathcal{F})$. Then $C_{\pm}$ are continuous linear mappings from $\mathcal{D}^{*\prime} \hat{\otimes} \mathcal{F}$ into $\mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$. Now, in order to prove the lemma, it certainly suffices to show that $V'_{\pm} | \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{F} = C_{\pm}$. To this end, let $f \in \mathcal{F}$ and $g \in \mathcal{D}^{*\prime}$ be arbitrarily given. Further, let $H \subset \mathbb{R}^n$ and $L \subset \Omega$ be compact subsets and consider a pair of functions $\psi_1 \in \mathcal{D}^*_H$ and $\psi_2 \in \mathcal{D}^*_L$. We then define a continuous linear functional M on $\mathcal{L}(\mathcal{D}^*, \mathcal{E}^*) \cong \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{E}^*$ by the formula

$$M(S) := \langle f, S(\psi_1)\psi_2 \rangle \qquad \text{for all } S \in \mathcal{L}(\mathcal{D}^*, \mathcal{E}^*) .$$

In the particular case $S = k \otimes \varphi$ with $k \in \mathcal{D}^{*\prime}$ and $\varphi \in \mathcal{E}^*$ we have

$$\begin{aligned} M(k \otimes \varphi) &= \langle f, \langle k, \psi_1 \rangle \varphi \psi_2 \rangle \\ &= \langle k \otimes (\varphi f), \psi_1 \otimes \psi_2 \rangle \\ &= \langle J \circ (I_{\mathcal{D}^{*\prime}} \hat{\otimes} B)(k \otimes \varphi \otimes f), \psi_1 \otimes \psi_2 \rangle . \end{aligned}$$

This implies by continuity

$$M(S) = \langle J \circ (I_{\mathcal{D}^{*\prime}} \hat{\otimes} B)(S \otimes f), \psi_1 \otimes \psi_2 \rangle \quad \forall\ S \in \mathcal{D}^{*\prime} \hat{\otimes} \mathcal{E}^* .$$

By means of this identity we obtain

$$\langle V'_{\pm}(g \otimes f), \psi_1 \otimes \psi_2 \rangle = \langle g \otimes f, V_{\pm}(\psi_1 \otimes \psi_2) \rangle$$

$$= \langle g \otimes f, (x,y) \mapsto \psi_1(x \pm y)\psi_2(y) \rangle$$

$$= \langle f, y \mapsto \langle g, \psi_1(\cdot \pm y) \rangle \psi_2(y) \rangle$$

$$= \langle f, \langle A_\pm g, \psi_1 \rangle \psi_2 \rangle$$

$$= M(A_\pm g)$$

$$= \langle J \circ (I_{\mathcal{D}^{*\prime}} \hat{\otimes} B) \circ (A_\pm \hat{\otimes} I_{\mathcal{F}})(g \otimes f), \psi_1 \otimes \psi_2 \rangle$$

$$= \langle C_\pm (g \otimes f), \psi_1 \otimes \psi_2 \rangle .$$

In view of [7,Th.2.1] we conclude that

$$V'_\pm(g \otimes f) = C_\pm(g \otimes f) \qquad \text{for all } g \in \mathcal{D}^{*\prime},\ f \in \mathcal{F}$$

and hence by continuity $V'_\pm = C_\pm$ on $\mathcal{D}^{*\prime} \hat{\otimes} \mathcal{F}$. This completes the proof of lemma 4.2.

We now introduce a certain class of spaces which will be important for the representation of local operators. Let us call a linear subspace $\mathcal{F}(\Omega)$ of $\mathcal{D}^{*\prime}(\Omega)$ <u>admissible</u>, if $\mathcal{F}(\Omega)$ is endowed with a complete locally convex topology such that the following five conditions are fulfilled:

(i) $\mathcal{F}(\Omega)$ is continuously embedded in $\mathcal{D}^{*\prime}(\Omega)$.

(ii) For each compact subset K of Ω the space $\mathcal{F}_K := \{f \in \mathcal{F}(\Omega) : \operatorname{supp} f \subset K\}$ is closed in $\mathcal{F}(\Omega)$.

(iii) For each compact $K \subset \Omega$ the multiplication $(\varphi, f) \mapsto \varphi f$ defines a continuous bilinear mapping from $\mathcal{E}^*(\mathbb{R}^n) \times \mathcal{F}_K$ into $\mathcal{F}_K$; moreover, for every $\varphi \in \mathcal{D}^*(\Omega)$ the multiplication $f \mapsto \varphi f$ is a continuous linear mapping on $\mathcal{F}(\Omega)$.

(iv) Given any $f \in \mathcal{D}^{*\prime}(\Omega)$, we have $f \in \mathcal{F}(\Omega)$ if and only if for each compact $K \subset \Omega$ there is some $f_K \in \mathcal{F}(\Omega)$ such that $f = f_K$ on $\mathcal{D}^*_K$.

(v) For every continuous seminorm q on $\mathcal{F}(\Omega)$ there exists some compact $K \subset \Omega$ such that $q(f) = 0$ for all $f \in \mathcal{F}(\Omega)$ with $K \cap \operatorname{supp} f = \emptyset$.

Let us list some natural examples concerning admissible subspaces $\mathcal{F}(\Omega)$ of $\mathcal{D}^{*\prime}(\Omega)$, namely

$$C^k(\Omega), L^p_{loc}(\Omega), H^{p,k}_{loc}(\Omega), \mathcal{E}^*(\Omega), \mathcal{D}'(\Omega), \mathcal{D}^{\#\prime}(\Omega), \mathcal{D}^{*\prime}(\Omega) .$$

Here $k=0,1,\dots,\infty$ and $1 \leq p \leq \infty$; and # stands for either (N_p) or $\{N_p\}$, where the sequence $(N_p)_{p=0,1,\dots}$ is supposed to satisfy the same conditions as $(M_p)_{p=0,1,\dots}$ and $(M_p) \prec (N_p)$ in the sense of [6,Def.3.9]. Indeed, in these situations the preceding conditions (i) - (v) can be easily deduced from standard facts on distributions and ultradistributions; we refer in particular to [6,Th.2.8,6.2,6.4] and [16,Th.41.1]. We now turn to the main characterization theorem for continuous local operators.

4.3.THEOREM. Let $\mathcal{F}(\Omega)$ be an admissible subspace of $\mathcal{D}^{*\prime}(\Omega)$ and let $\mathcal{Q}$ denote the family of all continuous seminorms on $\mathcal{F}(\Omega)$. Then every continuous local linear operator $T: \mathcal{D}^*(\Omega) \to \mathcal{F}(\Omega)$ admits a unique representation of the form

$$(5) \qquad T\varphi = \sum_{\beta \in \mathbb{N}_0^n} f_\beta \cdot D^\beta \varphi \qquad \text{for all } \varphi \in \mathcal{D}^*(\Omega) ,$$

where $f_\beta \in \mathcal{F}(\Omega)$ for all $\beta \in \mathbb{N}_0^n$ and the following estimates hold in the Beurling case $*=(M_p)$:

$$(6^{(M_p)}) \quad \forall\, q \in \mathcal{Q}\ \exists\, L,C>0 \text{ such that } q(f_\beta) \leq C L^{|\beta|} M_{|\beta|}^{-1} \quad \forall\, \beta \in \mathbb{N}_0^n ,$$

resp. in the Roumieu case $*=\{M_p\}$:

$$(6^{\{M_p\}}) \quad \forall\, q \in \mathcal{Q}, L>0\ \exists\, C>0 \text{ such that } q(f_\beta) \leq C L^{|\beta|} M_{|\beta|}^{-1} \quad \forall\, \beta \in \mathbb{N}_0^n .$$

Conversely, whenever $f_\beta \in \mathcal{F}(\Omega)$ are given for all $\beta \in \mathbb{N}_0^n$ such that condition (6^*) is fulfilled, then the series in (5) converges in $\mathcal{L}_b(\mathcal{D}^*(\Omega), \mathcal{F}(\Omega))$ and defines a continuous local linear operator $T: \mathcal{D}^*(\Omega) \to \mathcal{F}(\Omega)$.

Proof. (a) We first observe that the uniqueness part of the theorem is obvious. Indeed, given $f_\beta \in \mathcal{F}(\Omega) \subset \mathcal{D}^{*\prime}(\Omega)$ for all $\beta \in \mathbb{N}_0^n$ such that

$$\sum_{\beta \in \mathbb{N}_0^n} f_\beta \cdot D^\beta \varphi = 0 \qquad \text{for all } \varphi \in \mathcal{D}^*(\Omega) ,$$

we successively insert functions of the form $\varphi = pg$, where p is a suitable monomial in n variables and $g \in \mathcal{D}^*(\Omega)$ is

identical to 1 on a given compact subset of Ω. Then we arrive at $f_\beta = o$ for every $\beta \in \mathbb{N}_o^n$.

(b) Next we consider a sequence of $f_\beta \in \mathcal{F}(\Omega)$ for all $\beta \in \mathbb{N}_o^n$ such that condition (6*) is satisfied and fix an arbitrary bounded subset B of $\mathcal{D}^*(\Omega)$. According to [6,Th.2.6] this set B is contained and bounded in $\mathcal{D}_H^*$ for some suitable compact $H \subset \Omega$. By [6,Lemma 5.1] there exists a $g \in \mathcal{D}^*(\Omega)$ such that $g \equiv 1$ on some neighborhood of H. From property (iii) of the admissible space $\mathcal{F}(\Omega)$ we conclude that (6*) remains valid with f_β replaced by $g_\beta := (-1)^{|\beta|} g f_\beta \in \mathcal{F}_K$ for all $\beta \in \mathbb{N}_o^n$, where $K := \operatorname{supp} g$. Hence the easier part of theorem 3.7 confirms that the series

$$S := \sum_{\beta \in \mathbb{N}_o^n} (D^\beta \delta_o) \otimes g_\beta$$

converges in the space $\mathcal{L}_b(\mathcal{D}^*(\mathbb{R}^n), \mathcal{F}_K)$, which is by [16, Prop.5o.5] canonically isomorphic with $\mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}_K$. In view of lemma 4.2 we arrive at the convergence of

$$V_+'(S) = \sum_{\beta \in \mathbb{N}_o^n} V_+'(D^\beta \delta_o \otimes g_\beta)$$

in the same topology as before. Now, for all $\beta \in \mathbb{N}_o^n$, $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ and $\psi \in \mathcal{D}^*(\Omega)$ we have the following identities

$$\begin{aligned} \langle V_+'(D^\beta \delta_o \otimes g_\beta), \varphi \otimes \psi \rangle &= \langle D^\beta \delta_o \otimes g_\beta, (x,y) \mapsto \varphi(x+y)\psi(y) \rangle \\ &= \langle g_\beta, y \mapsto (-1)^{|\beta|} D^\beta \varphi(y)\psi(y) \rangle \\ &= \langle g f_\beta \cdot D^\beta \varphi, \psi \rangle = \langle f_\beta \cdot D^\beta \varphi, g\psi \rangle . \end{aligned}$$

Consequently the series given by

$$\sum_{\beta \in \mathbb{N}_o^n} g f_\beta \cdot D^\beta \varphi \qquad \text{for all } \varphi \in \mathcal{D}^*(\mathbb{R}^n)$$

converges in $\mathcal{L}_b(\mathcal{D}^*(\mathbb{R}^n), \mathcal{F}(\Omega))$ and is equal to

$$\sum_{\beta \in \mathbb{N}_o^n} f_\beta \cdot D^\beta \varphi \qquad \text{for all } \varphi \in \mathcal{D}_H^* .$$

In particular it follows that the last series converges uniformly on B in $\mathcal{F}(\Omega)$. This proves the second part of the theorem, since the operator given by formula (5) is obviously local.

(c) Finally, let $T \in \mathcal{L}(\mathcal{D}^*(\Omega), \mathcal{F}(\Omega))$ be an arbitrary local operator and consider a compact subset H of Ω. Again [6, Lemma 5.1] supplies us with some $g \in \mathcal{D}^*(\Omega)$ such that $g \equiv 1$ on a neighborhood of H. We put $K := \operatorname{supp} g$ and define a local linear operator $T_g : \mathcal{D}^*(\mathbb{R}^n) \to \mathcal{F}_K$ by

$$T_g(\varphi) := T(g\varphi) \qquad \text{for all } \varphi \in \mathcal{D}^*(\mathbb{R}^n) .$$

Then T_g is continuous by [6,Th.2.8] and satisfies $T_g = T$ on $\mathcal{D}^*_H$. We next observe that

$$\mathcal{L}(\mathcal{D}^*(\mathbb{R}^n), \mathcal{F}_K) \cong \mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}_K \subset \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$$

canonically, according to [16,Prop.5o.5] and [7,Th.2.3]. Hence there exists an ultradistribution $S_g \in \mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}_K \subset \mathcal{D}^{*\prime}(\mathbb{R}^n \times \Omega)$ such that

$$\langle S_g, \varphi \otimes \psi \rangle = \langle T(g\varphi), \psi \rangle \qquad \text{for all } \varphi \in \mathcal{D}^*(\mathbb{R}^n), \psi \in \mathcal{D}^*(\Omega).$$

Since T is local, we have the inclusion

$$\operatorname{supp} S_g \subset D_K := \{(x,x) \in \mathbb{R}^n \times \Omega : x \in K\} .$$

Indeed, if $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ and $\psi \in \mathcal{D}^*(\Omega)$ satisfy $\operatorname{supp}(\varphi \otimes \psi) \cap D_K = \emptyset$, it follows from $(K \cap \operatorname{supp} \varphi) \cap \operatorname{supp} \psi = \emptyset$ and $\operatorname{supp} T(g\varphi) \subset \operatorname{supp}(g\varphi) \subset K \cap \operatorname{supp} \varphi$ that $\langle S_g, \varphi \otimes \psi \rangle = \langle T(g\varphi), \psi \rangle = 0$. This implies by the usual compactness argument and [7,Th.2.1] that $\langle S_g, \rho \rangle = 0$ holds for all $\rho \in \mathcal{D}^*(\mathbb{R}^n \times \Omega)$ with $\operatorname{supp} \rho \cap D_K = \emptyset$. Consequently $\operatorname{supp} S_g \subset D_K$. From 4.1 and 4.2 we obtain

$$V'_-(S_g) \in \mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}_K \quad \text{and} \quad \operatorname{supp} V'_-(S_g) \subset \{0\} \times K .$$

Now let $R_g \in \mathcal{L}(\mathcal{D}^*(\mathbb{R}^n), \mathcal{F}_K) \cong \mathcal{D}^{*\prime}(\mathbb{R}^n) \hat{\otimes} \mathcal{F}_K$ be given by

$$\langle R_g \varphi, \psi \rangle = \langle V'_-(S_g), \varphi \otimes \psi \rangle \qquad \text{for all } \varphi \in \mathcal{D}^*(\mathbb{R}^n), \psi \in \mathcal{D}^*(\Omega).$$

Then it is clear that R_g is supported by the origin. Hence the representation theorem 3.7 supplies us with a sequence of $f_{g,\beta} \in \mathcal{F}_K$ for all $\beta \in \mathbb{N}_0^n$, which satisfies condition (6*) as well as

$$\langle V'_-(S_g), \varphi \otimes \psi \rangle = \sum_{\beta \in \mathbb{N}_0^n} (-1)^{|\beta|} D^\beta \varphi(0) \langle f_{g,\beta}, \psi \rangle$$

for all $\varphi\in\mathcal{D}^*(\mathbb{R}^n)$ and $\psi\in\mathcal{D}^*(\Omega)$. In virtue of [7,Th.2.1] we conclude that

$$\langle V'_-(S_g),\rho\rangle = \sum_{\beta\in\mathbb{N}_o^n} \langle D^\beta\delta_o \otimes f_{g,\beta},\rho\rangle$$

for all $\rho\in\mathcal{D}^*(\mathbb{R}^n\times\Omega)$ and hence

$$\begin{aligned}\langle T(g\varphi),\psi\rangle &= \langle V'_-(S_g),V_+(\varphi\otimes\psi)\rangle \\ &= \sum_{\beta\in\mathbb{N}_o^n} \langle D^\beta\delta_o \otimes f_{g,\beta},(x,y)\mapsto\varphi(x+y)\psi(y)\rangle \\ &= \sum_{\beta\in\mathbb{N}_o^n} \langle(-1)^{|\beta|}f_{g,\beta}\cdot D^\beta\varphi,\psi\rangle\end{aligned}$$

for all $\varphi\in\mathcal{D}^*(\mathbb{R}^n)$ and all $\psi\in\mathcal{D}^*(\Omega)$. This implies

$$T\varphi = \sum_{\beta\in\mathbb{N}_o^n} (-1)^{|\beta|}f_{g,\beta}\cdot D^\beta\varphi \qquad \text{for all } \varphi\in\mathcal{D}^*_H.$$

Now, for every $\beta\in\mathbb{N}_o^n$ let $f_\beta\in\mathcal{D}^{*\prime}(\Omega)$ be given by

$$f_\beta|\mathcal{D}^*_H := (-1)^{|\beta|}f_{g,\beta}|\mathcal{D}^*_H\ ,$$

where $H\subset\Omega$ is compact and $g\in\mathcal{D}^*(\Omega)$ is identical to 1 on a neighborhood of H. It is immediate after the first part of the proof that $f_\beta\in\mathcal{D}^{*\prime}(\Omega)$ is well-defined. Moreover, by property (iv) of the admissible space $\mathcal{F}(\Omega)$ we have $f_\beta\in\mathcal{F}(\Omega)$ for all $\beta\in\mathbb{N}_o^n$. Finally, it can be easily deduced from property (v) that for every continuous seminorm q on $\mathcal{F}(\Omega)$ there is a suitable $g\in\mathcal{D}^*(\Omega)$ such that $q(f_\beta) = q(f_{g,\beta})$ holds for every $\beta\in\mathbb{N}_o^n$. We conclude that conditions (5) and (6*) are satisfied by the above sequence. This finishes the proof of the theorem.

4.4.COROLLARY. Let $(N_p)_{p=0,1,\dots}$ be a sequence with the same properties as $(M_p)_{p=0,1,\dots}$ and assume that $(M_p)\prec(N_p)$ in the sense of [6,Def.3.9]. Further, let # stand for *, (N_p) or $\{N_p\}$, and let $\mathcal{Q}$ denote the family of all continuous seminorms on $\mathcal{E}^\#(\Omega)$. Then every continuous local linear operator $T\colon \mathcal{E}^{\#\prime}(\Omega)\to\mathcal{D}^{*\prime}(\Omega)$ has a unique representation

$$(7)\qquad T\varphi = \sum_{\beta\in\mathbb{N}_o^n} D^\beta(f_\beta\cdot\varphi) \qquad \text{for all } \varphi\in\mathcal{E}^{\#\prime}(\Omega)\ ,$$

where $f_\beta\in\mathcal{E}^\#(\Omega)$ for all $\beta\in\mathbb{N}_o^n$ and

$(8^{(M_p)})$ $\forall q \in \mathcal{Q}\ \exists L, C>0$ such that $q(f_\beta) \leq C L^{|\beta|} M_{|\beta|}^{-1} \quad \forall \beta \in \mathbb{N}_o^n$

in the Beurling case $*=(M_p)$, resp.

$(8^{\{M_p\}})$ $\forall q \in \mathcal{Q}, L>0\ \exists C>0$ such that $q(f_\beta) \leq C L^{|\beta|} M_{|\beta|}^{-1}\ \forall \beta \in \mathbb{N}_o^n$

in the Roumieu case $*=\{M_p\}$. Conversely, whenever $f_\beta \in \mathcal{E}^\#(\Omega)$ satisfy condition (8*), then the series in (7) converges in $\mathcal{L}_b(\mathcal{E}^{\#'}(\Omega), \mathcal{D}^{*'}(\Omega))$ and defines a local linear operator $T \in \mathcal{L}(\mathcal{E}^{\#'}(\Omega), \mathcal{D}^{*'}(\Omega))$.

Proof. First note that the spaces $\mathcal{D}^*(\Omega)$ and $\mathcal{E}^\#(\Omega)$ are reflexive by [6,Th.2.6 and 5.12] and that $\mathcal{E}^\#(\Omega)$ is an admissible subspace of $\mathcal{D}^{*'}(\Omega)$. Next observe that for every local $T \in \mathcal{L}(\mathcal{E}^{\#'}(\Omega), \mathcal{D}^{*'}(\Omega))$ the adjoint $T': \mathcal{D}^*(\Omega) \to \mathcal{E}^\#(\Omega)$ is local as well. Hence the assertion can be easily derived from theorem 4.3.

We finally note that the continuity hypothesis in the preceding results is to some extent redundant by the results of section 2:

4.5.REMARK. Let $\mathcal{F}(\Omega)$ be an admissible subspace of $\mathcal{D}^{*'}(\Omega)$ and suppose that the closed graph theorem is valid for linear mappings from $\mathcal{D}^*(\Omega)$ to $\mathcal{F}(\Omega)$. Then we have:

(a) In the Beurling case, every local linear operator $T: \mathcal{D}^{(M_p)}(\Omega) \to \mathcal{F}(\Omega)$ is continuous on $\mathcal{D}^{(M_p)}(\Omega \setminus \Lambda)$ for some discrete subset Λ of Ω so that theorem 4.3 applies to the restriction $T|\mathcal{D}^{(M_p)}(\Omega \setminus \Lambda)$; cf. 2.4.

(b) In the Roumieu case, every local linear operator $T: \mathcal{E}^{\{M_p\},}(\Omega) \to \mathcal{D}^{*'}(\Omega)$ is continuous on $\mathcal{E}^{\{M_p\},}(\Omega \setminus \Lambda)$ for a discrete subset Λ of Ω and corollary 4.4 applies to the restriction $T|\mathcal{E}^{\{M_p\},}(\Omega \setminus \Lambda)$; cf. 2.4.

(c) If $\mathcal{F}(\Omega)$ is contained in $L^1_{loc}(\Omega)$, then the continuity hypothesis in theorem 4.3 can be omitted; cf. 2.1.

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