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ON THE SYMMETRIC ALGEBRA OF AN IDEAL [1]

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The symmetric algebra of an ideal I may be compared to the Rees algebra via the canonical epimorphism $\alpha:\mathrm{Sym}(I)\to \mathcal{R}(I)$. A necessary and sufficient criterion is given for α to be an isomorphism,and sequential conditions on the symmetric algebra are studied.Some applications are given to Proj α:Proj $\mathcal{R}(I)\longrightarrow$ Proj Sym(I) and to the theory of approximation complexes.

0.INTRODUCTION

Throughout this paper we will be dealing with a ring R (commutative with identity) and a finitely generated ideal I of R.Mostly some generating sequence $\underline{x}=x_1,\dots,x_n$ of I is specified.

We think of the Koszul-complex $K.(\underline{x};R)$ associated to the sequence $\underline{x}$ in the following way:Fix some basis $\{e_1,\dots,e_n\}$ of R^n;then the exterior product $K_p(\underline{x};R)=\Lambda^p(R^n)$ is free on the basis $\{e_{i_1}\wedge\dots\wedge e_{i_p} \mid 1\leq i_1<\dots<i_p\leq n\}$,and the differential ∂ is defined by

$$\partial(e_{i_1}\wedge\dots\wedge e_{i_p}) = \textstyle\sum_{j=1}^{p} (-1)^{p-j}\cdot x_{i_j}\cdot e_{i_1}\wedge\dots\wedge\hat{e}_{i_j}\wedge\dots\wedge e_{i_p} .$$

Z_p resp. B_p will stand for $\mathrm{Ker}(\partial:K_p\to K_{p-1})$ resp. $\mathrm{Im}(\partial:K_{p+1}\to K_p)$.

In case R is (positively) graded and all x_i are homogeneous of degree 1, $K.(\underline{x};R)$ becomes a graded complex if we let $K_p(\underline{x};R)_d:=\oplus R_{d-p}\cdot e_{i_1}\wedge\dots\wedge e_{i_p}$.Thus p adds to the degree of the coefficient to get the degree of an element of K_p.

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Let $S:=R[e_1,\dots,e_n]$ denote a polynomial ring in the same set of symbols $\{e_1,\dots,e_n\}$. Then Z_1,B_1,K_1 may be identified with certain sets of linear forms in S. This being done, the symmetric algebra Sym(I) of the R-ideal I is isomorphic to S/Z_1S. We will abbreviate $a_i:=e_i+Z_1S$. Sym(I) is a graded R-algebra with $Sym_0(I)\simeq R, Sym_1(I)\simeq I$ where $a_i \leftrightarrow x_i$, and may be compared to the Rees-algebra $\mathcal{R}(I):=R\oplus I\oplus I^2\oplus I^3\oplus\dots$ via the canonical epimorphism $\alpha:Sym(I)\longrightarrow\mathcal{R}(I)$ induced by the isomorphism $\alpha_1:Sym_1(I)\simeq I$ mentioned above. This mapping has been studied in several papers (among which the references) and numerous criteria are known for α to be an isomorphism. In the present paper we give another necessary and sufficient condition in terms of the Koszul-complex ((1.2)). From the viewpoint of Algebraic Geometry it would be interesting to know when Proj α:Proj $\mathcal{R}(I)\longrightarrow$ Proj Sym(I) is an isomorphism. Sporadic results in this direction are given in (1.4),(2.4),(3.2).

In section 2, a relation is established between sequential conditions in R and in Sym(I) ((2.2)) which is applicable to Proj α as well ((2.4)) and to the theory of approximation complexes, which we briefly touch in section 3.

1. A CRITERION FOR $Sym(I)\simeq\mathcal{R}(I)$

As described in the introduction, $\underline{a}=a_1,\dots,a_n$ is the sequence of linear forms in the symmetric algebra Sym(I) corresponding to a generating sequence $\underline{x}=x_1,\dots,x_n$ of the ideal I.

We begin with a lemma, that, however trivial, will be used on several occasions:

(1.1) LEMMA. *For* $i\in\{1,\dots,n\}$ *and* $f_1,\dots,f_i\in S$ *arbitrary the following statements are equivalent:*

(i) $f_1e_1+\dots+f_ie_i\in(e_1,\dots,e_i)Z_1S$

(ii) $f_1(\underline{a})e_1+\dots+f_i(\underline{a})e_i\in B_1(a_1,\dots,a_i;Sym(I))$

In particular, $H_1(\underline{a};Sym(I))_d=0$ *for* $d>1$.

Proof.(i)=>(ii):Let $f_1e_1+....+f_ie_i=g_1e_1+.....+g_ie_i$, $g_j \varepsilon Z_1S$.Since $e_1,...,e_i$ is a regular sequence in S,there exists a skew-symmetric matrix $(h_{\nu\mu})_{\nu,\mu=1,...,i}$,$h_{\nu\mu}\varepsilon S$,such that $f_\nu=g_\nu + \sum_\mu h_{\nu\mu}e_\mu$ $(\nu=1,...,i)$

$$=> f_\nu(\underline{a})=g_\nu(\underline{a}) + \sum_\mu h_{\nu\mu}(\underline{a})a_\mu = \sum_\mu h_{\nu\mu}(\underline{a})a_\mu \qquad (\nu=1,...,i)$$

$$=> f_1(\underline{a})e_1+....+f_i(\underline{a})e_i = \sum_{\nu,\mu} h_{\nu\mu}(\underline{a})a_\mu e_\nu \varepsilon B_1(a_1,..,a_i;Sym(I))$$

(ii)=>(i):Let $(h_{\nu\mu})_{\nu,\mu=1,..,i}$ be a skew-symmetric matrix, $h_{\nu\mu}\varepsilon S$,for which $f_1(\underline{a})e_1+.....+f_i(\underline{a})e_i = \sum^i_{\nu,\mu=1} h_{\nu\mu}(\underline{a})a_\mu e_\nu$

$$=> f_\nu(\underline{a}) = \sum_\mu h_{\nu\mu}(\underline{a})a_\mu \qquad (\nu=1,...,i)$$

$$=> f_\nu - \sum_\mu h_{\nu\mu}e_\mu \varepsilon Z_1S \qquad (\nu=1,...,i) \text{ ,and}$$

$$f_1e_1+.....+f_ie_i = \sum_\nu (f_\nu - \sum_\mu h_{\nu\mu}e_\mu)e_\nu + \underbrace{\sum_{\nu,\mu} h_{\nu\mu}e_\mu e_\nu}_{=0}$$

$$\varepsilon\ (e_1,...,e_i)Z_1S .$$

As for the last statement,it is clear that (for $\sum f_\nu(\underline{a})a_\nu=0$) $f_1e_1+.....+f_ne_n \ \varepsilon\ (e_1,...,e_n)Z_1S$ if $f_1,...,f_n$ are homogeneous of degree $d-1\geq 1$,and therefore $f_1(\underline{a})e_1+....+f_n(\underline{a})e_n \varepsilon\ B_1(a_1,...,a_n;Sym(I))$ by what we have shown. ///

(1.2)THEOREM.The canonical epimorphism $\alpha:Sym(I) \longrightarrow \mathcal{R}(I)$ between the symmetric and the Rees-algebra of the ideal I is an isomorphism if and only if $Z_1 \cap I^kK_1 = I^{k-1}B_1$ for all $k\geq 1$ In fact,as graded R-modules:

$$Ker\ \alpha/Sym_+(I)\cdot Ker\ \alpha \simeq \bigoplus_{k\geq 2} Z_1 \cap I^{k-1}K_1/I^{k-2}B_1$$

Here it is understood that Z_1,B_1,K_1 refer to the Koszul-complex $K.(\underline{x};R)$ of some finite generating sequence $\underline{x}$ of the ideal I.The modules situated on the right-hand side are easily seen to be independent of the choice of $\underline{x}$.

Proof.Fix some generating sequence $\underline{x}=x_1,...,x_n$ for I.Since $\alpha_1:Sym_1(I) \xrightarrow{\sim} I$ is an isomorphism anyway,it will be sufficient to show

$$Ker\ \alpha_{k+1}/Sym_1(I)\cdot Ker\ \alpha_k \simeq Z_1 \cap I^kK_1\ /\ I^{k-1}B_1 \qquad (k\geq 1)$$

To establish this isomorphism, take any cycle $z\varepsilon Z_1 \cap I^k K_1$, which may be written $z=\lambda_1 e_1+\ldots+\lambda_n e_n, \lambda_i \varepsilon I^k, \Sigma\lambda_i x_i=0$. Choose $f_1,\ldots,f_n \varepsilon S$ homogeneous of degree k such that $\lambda_i=f_i(\underline{x})$. Then $\Sigma f_i(\underline{a})a_i$ is an element of Ker α_{k+1}, the residue class of which we denote by $\phi(z)$, thereby defining a mapping $\phi: Z_1 \cap I^k K_1 \longrightarrow \text{Ker } \alpha_{k+1}/\text{Sym}_1(I)\cdot\text{Ker } \alpha_k$. To see that ϕ is well defined, let $z=\Sigma\lambda_i e_i=\Sigma f_i(\underline{x})e_i=\Sigma g_i(\underline{x})e_i$, where the f_i, g_i are homogeneous polynomials of degree k.

$\Rightarrow$ $f_i(\underline{x})=g_i(\underline{x})$ for all i, i.e. $f_i(\underline{a})-g_i(\underline{a})$ εKer α_k

$\Rightarrow$ $\Sigma f_i(\underline{a})a_i - \Sigma g_i(\underline{a})a_i = \sum(f_i(\underline{a})-g_i(\underline{a}))\cdot a_i \varepsilon \text{Sym}_1(I)\cdot\text{Ker } \alpha_k$.

It is immediately seen that ϕ is an epimorphism and that ϕ vanishes on $I^{k-1}B_1$. To finish the proof, take any $z\varepsilon$ $Z_1 \cap I^k K_1$ with $\phi(z)=0$. Write $z=\Sigma\lambda_i e_i$, $\lambda_i=f_i(\underline{x})$, deg f_i=k, and choose $g_i(\underline{a})\varepsilon$Ker α_k such that $\Sigma f_i(\underline{a})a_i= \Sigma g_i(\underline{a})a_i$ $(g_i \varepsilon S_k, g_i(\underline{x})=0)$

$\Rightarrow$ $\sum(f_i(\underline{a})-g_i(\underline{a}))\cdot a_i = 0$

$\Rightarrow$ $\sum(f_i(\underline{a})-g_i(\underline{a}))\cdot e_i$ $\varepsilon B_1(\underline{a};\text{Sym}(I))$ by virtue of (1.1)

and hence, upon applying α : $z=\Sigma f_i(\underline{x})e_i$ $\varepsilon I^{k-1}B_1$. <u>qed</u>

(1.2) may be used to sharpen Lemma 3.3 of [2]:
As before, $\alpha:\text{Sym}(I) \longrightarrow \mathcal{R}(I)$ denotes the canonical epimorphism between the symmetric and the Rees-algebra of the finitely generated ideal I:

<u>(1.3) COROLLARY</u>. $I^{k-1}\cdot\text{Ker } \alpha_k = 0$ <u>for all</u> $k\geq 1$.

<u>Proof</u>. In view of the isomorphisms

$$\text{Ker } \alpha_{k+1}/\text{Sym}_1(I)\cdot\text{Ker } \alpha_k \simeq Z_1 \cap I^k\cdot K_1/I^{k-1}\cdot B_1 ,$$

it is enough to observe that these latter modules are annihilated by I. In fact,
$Z_1 \cap I^k K_1/I^{k-1}B_1 \simeq \text{Im}(H_1(\underline{x};I^k) \longrightarrow H_1(\underline{x};I^{k-1}))$, the morphism involved being induced from the embedding $I^k \hookrightarrow I^{k-1}$. It is well-known that any such homology-module is annihilated by $(\underline{x})=I$. ///

As was pointed out in the introduction, a question of interest for problems in Algebraic Geometry should rather be: When is Proj α an isomorphism? instead of When is α an isomorphism? .

It turns out that these questions are not equivalent:

(1.4) EXAMPLE. We are going to produce a ring R and an ideal I of R for which Proj α: Proj $\mathcal{R}(I) \longrightarrow$ Proj Sym(I)) is an isomorphism, whilst $\alpha: Sym(I) \longrightarrow \mathcal{R}(I)$ is not.

To this end, let k be field, $A := k[B_1 B_2, C]$ a polynomial ring (which is not considered to be graduated) an put

$$R := A[X_1,X_2]/(B_1X_1, B_2X_2, B_1X_2^2, B_2X_1^2, B_2X_1-CX_2, B_1X_2+CX_1) =: A[x_1,x_2]$$

$$I := (x_1,x_2)\cdot R \qquad (\Rightarrow R/I \simeq A).$$

Attach a graduation to R by deg x_1=deg x_2:=1, so that R becomes graduated with $R_+=I$ (in particular $gr_I(R)\simeq R$).

We consider $\overline{\alpha}: Sym_{R/I}(I/I^2) \longrightarrow gr_I(R)\simeq R$, where obviously

$$\begin{aligned} Sym_{R/I}(I/I^2) &= Sym_A(I/I^2) \\ &= Sym_A(A\cdot X_1 \oplus A\cdot X_2/\langle B_1X_1, B_2X_2, B_2X_1-CX_2, B_1X_2+CX_1\rangle) \\ &\simeq A[X_1,X_2]/(B_1X_1, B_2X_2, B_2X_1-CX_2, B_1X_2+CX_1) \quad . \end{aligned}$$

We thus see that Ker $\overline{\alpha}=(B_1X_2^2, B_2X_1^2)\cdot Sym_{R/I}(I/I^2)\neq 0$, whence α cannot be an isomorphism ([5], Theorem 1.3). However

$$\begin{aligned} X_1\cdot B_1X_2^2 &= X_2^2\cdot(X_1B_1) \\ X_2\cdot B_1X_2^2 &= X_2^2\cdot(B_1X_2+CX_1)+X_1X_2\cdot(B_2X_1-CX_2)-X_1^2\cdot(B_2X_2) \\ X_1\cdot B_2X_1^2 &= X_1^2\cdot(B_2X_1-CX_2)+X_1X_2\cdot(B_1X_2+CX_1)-X_2^2\cdot(B_1X_1) \\ X_2\cdot B_2X_1^2 &= X_1^2\cdot(B_2X_2) \ , \end{aligned}$$

and hence $Sym_+(I/I^2)\cdot$Ker $\overline{\alpha}=0$, i.e. Ker $\overline{\alpha}_d=0$ for $d\geq 3$. By an argument similar to Theorem 3.1 of [2] we conclude Ker $\alpha_d=0$ for $d>>0$, that is, Proj α is an isomorphism. ///

2. SEQUENTIAL CONDITIONS ON THE SYMMETRIC ALGEBRA

Let $\underline{x}=x_1,\dots,x_n$ be a sequence in the ring R generating the ideal I.

(2.1) DEFINITION. ([2], [4] et al.)
$\underline{x}$ *is said to be a d-sequence if*

$$[(x_1,\dots,x_i):x_{i+1}]\cap I = (x_1,\dots,x_i) \quad \textit{for} \quad i=0,\dots,n-1$$

(where $(x_1,\dots,x_i)=0$ for $i=0$ by convention)

$\underline{x}$ *is said to be a proper sequence if*

$$x_{i+1}\cdot H_1(x_1,\dots,x_i;R)=0 \quad \textit{for} \quad i=1,\dots,n-1 .$$

Remarks.

1) Some authors require that $\underline{x}$ should, in addition, generate the ideal I minimally.
2) In [2] $\underline{x}$ is called proper if $x_{i+1}\cdot H_p(x_1,\dots,x_i;R)=0$ for $i=1,\dots,n-1$ and all $p\geq 1$. It is by no means trivial to see that these notions of "proper" coincide; see (2.3).
3) The following general implications hold true: $\underline{x}$ regular sequence => $\underline{x}$ d-sequence => $\underline{x}$ proper sequence (proof straightforward)

The setup for the following theorem is that of the introduction, i.e. $\underline{a}=a_1,\dots,a_n$ is the sequence of linear forms in the symmetric algebra Sym(I) corresponding to a generating sequence $\underline{x}=x_1,\dots,x_n$ of the ideal I via $\mathrm{Sym}_1(I)\simeq I$.

(2.2) THEOREM. *The following conditions are equivalent:*

(i) $\underline{x}$ *is a proper sequence (in* R)
(ii) $x_{i+1}\cdot H_p(x_1,\dots,x_i;R)=0$ *for all* $i=1,\dots,n-1, p\geq 1$.
(iii) $\underline{a}$ *is a proper sequence (in* Sym(I))
(iv) $\underline{a}$ *is a d-sequence (in* Sym(I))

Proof. As the implication (ii)=>(i) is trivial and (iv)=>(iii) is generally true (see remark above), we will successively show (i)=>(iv), (iii)=>(iv), (iv)=>(ii).

Preliminary Remark.

Note that the isomorphisms $Sym_0(I) \simeq R$ and $Sym_1(I) \simeq I, a_i \leftrightarrow x_i$, give rise to natural identifications:

$$(*) \quad \begin{cases} K_p(x_1,\ldots,x_i;R) \simeq K_p(a_1,\ldots,a_i;Sym(I))_p \\ I\cdot K_p(x_1,..,x_i;R) \simeq K_p(a_1,\ldots,a_i;Sym(I))_{p+1} \\ Z_p(x_1,\ldots,x_i;R) \simeq Z_p(a_1,\ldots,a_i;Sym(I))_p \\ B_p(x_1,\ldots,x_i;R) \simeq B_p(a_1,\ldots,a_i;Sym(I))_{p+1} \end{cases}$$

for all $i=1,\ldots,n$ and $p\geq 1$.

(i)=>(iv): By virtue of (1.1) and (*)

$$e_{i+1}\cdot Z_1(x_1,\ldots,x_i;R) \subseteq (e_1,\ldots,e_i)Z_1S$$
$$\iff a_{i+1}\cdot Z_1(a_1,\ldots,a_i;Sym(I))_1 \subseteq B_1(a_1,\ldots,a_i;Sym(I))_2$$
$$\iff x_{i+1}\cdot Z_1(x_1,\ldots,x_i;R) \subseteq B_1(x_1,\ldots,x_i;R)\ .$$

This last statement holds true by assumption.

We claim next

$(Z_1S \cap (e_1,\ldots,e_i)S)_d = ((e_1,\ldots,e_i)Z_1S)_d$ for all $d\geq 2$ and $i=1,\ldots,n$.

Proof by induction on d (and all i). Given an arbitrary element $z \in (Z_1S \cap (e_1,\ldots,e_i)S)_d$, let $t := \min\{j\in\{i,\ldots,n\} \mid z\in(e_1,\ldots,e_j)Z_1S\}$. We need t=i, so suppose we had t>i. Write $z=e_1z_1+\ldots+e_tz_t$ with $z_1,\ldots,z_t \in (Z_1S)_{d-1}$

$\Rightarrow\ e_tz_t \in (e_1,\ldots,e_{t-1})S$

$\Rightarrow\ z_t \in (e_1,\ldots,e_{t-1})S$

$\Rightarrow\ z_t \in (Z_1S \cap (e_1,\ldots,e_{t-1})S)_{d-1}$.

In case d=2, this last formula reads $z_t\in Z_1(x_1,\ldots,x_{t-1};R)$, so that $e_tz_t \in (e_1,\ldots,e_{t-1})Z_1S$ by what we have just seen.

In case d>2, the statement being true for d-1, we conclude $z_t \in (e_1,\ldots,e_{t-1})Z_1S$ and certainly $e_tz_t \in (e_1,\ldots,e_{t-1})Z_1S$.

Anyway, $z \in (e_1,\ldots,e_{t-1})Z_1S$, contradicting the minimal choice of t. Therefore t=i, and we have shown our claim to which we now apply (1.1) to conclude: $H_1(a_1,\ldots,a_i;Sym(I))_d=0$ for $d\geq 2$ and $i=1,\ldots,n$. But this is just another way to express $[(a_1,\ldots,a_i):a_{i+1}]\cap Sym_+(I) = (a_1,\ldots,a_i)$ $(i=0,\ldots,n-1)$.

(iii)=>(iv): It is well-known that there are exact sequences of Koszul-homology

$$\ldots \to H_1(a_1,\ldots,a_i;\mathrm{Sym}(I)) \xrightarrow{a_{i+1}} H_1(a_1,\ldots,a_i;\mathrm{Sym}(I)) \to H_1(a_1,\ldots,a_{i+1};\mathrm{Sym}(I)) \to \ldots$$

where the first mapping is multiplication with a_{i+1} (and hence the zero-mapping by assumption). Thus

$$H_1(a_1,\ldots,a_i;\mathrm{Sym}(I)) \hookrightarrow H_1(a_1,\ldots,a_{i+1};\mathrm{Sym}(I)) \quad (i=1,\ldots,n-1)$$

These inclusions, for all i, taken together with $H_1(a_1,\ldots,a_n;\mathrm{Sym}(I))_d=0$ for all $d\geq 2$ (cf. (1.1)), yield $H_1(a_1,\ldots,a_i;\mathrm{Sym}(I))_d=0$ for all $d\geq 2, i=1,\ldots,n$ as well.

(iv)=>(ii): The d-sequence $\underline{a}=a_1,\ldots,a_n$ satisfies $a_{i+1}\cdot H_p(a_1,\ldots,a_i;\mathrm{Sym}(I))=0$ $(i=1,\ldots,n-1$, all $p\geq 1)$ (this is due to Fiorentini [1]).
The identification (*) above then yields
$x_{i+1}\cdot H_p(x_1,\ldots,x_i;R)=0$ $(i=1,\ldots,n-1, p\geq 1)$ qed

(2.3) COROLLARY 1. *A sequence $\underline{x}=x_1,\ldots,x_n$ in any ring R is proper* (i.e. $x_{i+1}\cdot H_1(x_1,\ldots,x_i;R)=0$ *for* $i=1,\ldots,n-1$) *iff* $x_{i+1}\cdot H_p(x_1,\ldots,x_i;R)=0$ *for all* $i=1,\ldots,n-1$, $p\geq 1$. *Hence definition* (2.1) *coincides with the original definition of proper sequence as given in* [2].

(2.4) COROLLARY 2. *Suppose the ideal I of the noetherian ring R is generated by a proper sequence. Then the canonical epimorphism* $\alpha:\mathrm{Sym}(I)\to \mathcal{R}(I)$ *is an isomorphism if and only if* $\mathrm{Proj}\ \alpha:\mathrm{Proj}\ \mathcal{R}(I)\to \mathrm{Proj}\ \mathrm{Sym}(I)$ *is.*

Proof. If I is generated by a proper sequence, $\mathrm{Sym}_+(I)$ is generated by a d-sequence $a_1,\ldots,a_n$ according to (2.2). In particular $(0:a_1)\cap\mathrm{Sym}_+(I)=0$, so (because $\mathrm{Ker}\ \alpha\subseteq\mathrm{Sym}_+(I)$):

$$\begin{aligned}\mathrm{Proj}\ \alpha \text{ is an isomorphism} &\iff \mathrm{Sym}_+(I)^d\cdot\mathrm{Ker}\ \alpha=0 \quad (d\gg 0)\\ &\iff \mathrm{Ker}\ \alpha=0\\ &\iff \alpha \text{ is an isomorphism}\end{aligned}$$

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3. APPROXIMATION COMPLEXES

We begin with a brief outline on the construction of the approximation complexes $\mathcal{Z}.$ and $\mathcal{M}.$ .For details and background information the reader is referred to [2] and [3].

Again we are dealing with the Koszul-complex $K.(\underline{x};R)$ (differentiation denoted by ∂) of a sequence $\underline{x}=x_1,\dots,x_n$ generating the ideal I of R.

$S = R[e_1,\dots,e_n]$ denotes a polynomial ring.We agree to write down the Koszul-complex $\mathcal{L}.$ of the (regular) sequence $\underline{e}=e_1,\dots,e_n$ in S in the following form:

$$\mathcal{L}.:=0 \longrightarrow K_n\otimes_R S(-n) \xrightarrow{\partial'} \dots\dots \xrightarrow{\partial'} K_1\otimes_R S(-1) \xrightarrow{\partial'} K_o\otimes_R S \longrightarrow 0$$

(note the shift in the degree to make $\mathcal{L}.$ a graded complex) where ∂' is the usual (homogeneous) differentiation

$$\partial'(e_{i_1}\wedge\dots\wedge e_{i_p}\otimes f) = \textstyle\sum_{j=1}^{p} (-1)^{j-1} e_{i_1}\wedge\dots\wedge\hat{e}_{i_j}\wedge\dots\wedge e_{i_p}\otimes e_{i_j}\cdot f \ .$$

The differentiation ∂ of $K.(\underline{x};R)$ induces a homogeneous mapping $\partial\otimes 1: K_p\otimes S(-p) \longrightarrow K_{p-1}\otimes S(-p+1)$ of degree -1 which is denoted by ∂ as well.It turns out that ∂ and ∂' commute,i.e. there are commutative diagramms

$$(\dagger) \qquad \begin{array}{ccc} K_p\otimes S(-p)_d & \xrightarrow{\quad\partial'\quad} & K_{p-1}\otimes S(-p+1)_d \\ \partial\downarrow & /// & \downarrow\partial \\ K_{p-1}\otimes S(-p+1)_{d-1} & \xrightarrow{\quad\partial'\quad} & K_{p-2}\otimes S(-p+2)_{d-1} \end{array}$$

and hence $\mathcal{L}.$ induces complexes

$$\mathcal{Z}.:=0 \longrightarrow Z_n\otimes S(-n) \xrightarrow{\partial'} Z_{n-1}\otimes S(-n+1) \xrightarrow{\partial'} \dots\dots \xrightarrow{\partial'} Z_o\otimes S \longrightarrow 0$$

$$\mathcal{M}.:=0 \longrightarrow H_n\otimes S(-n) \xrightarrow{\partial'} H_{n-1}\otimes S(-n+1) \xrightarrow{\partial'} \dots\dots \xrightarrow{\partial'} H_o\otimes S \longrightarrow 0$$

The homologies of these complexes are independent of the generating sequence $\underline{x}$ for I.Note that $H_o(\mathcal{Z}.)\simeq \mathrm{Sym}(I)$. α is an isomorphism if $H_1(\mathcal{M}.)=0$.Herzog-Simis-Vasconcelos showed ([3]) that $\mathcal{M}.$ is acyclic (i.e. $H_p(\mathcal{M}.)=0$ for $p\geq 1$) if and only if I can be generated by a d-sequence (provided R is noetherian, local, with infinite residue class field). Moreover, $\mathcal{Z}.$ is acyclic if I is generated by a proper sequence. We are now going to show that the converse holds:

(3.1) COROLLARY 3. (to Theorem (2.2))
Suppose R is a noetherian local ring with infinite residue class field and I is an ideal in R such that the $\mathcal{Z}.$-complex (for some generating set of I) is acyclic.
Then I is generated by a proper sequence.

Proof. Let $\underline{x}=x_1,\dots,x_n$ be a sequence generating I for which the $\mathcal{Z}.$-complex is acyclic, i.e. the following sequence of $S=R[e_1,\dots,e_n]$ -modules is exact:

$$0 \longrightarrow Z_n \otimes S(-n) \rightarrow \dots\dots\dots \rightarrow Z_1 \otimes S(-1) \longrightarrow Z_o \otimes S \longrightarrow Sym(I) \longrightarrow 0$$

$\underline{e}=e_1,\dots,e_n$ being a regular sequence of the polynomial-ring S, all the $H_q(\underline{e};Z_p \otimes S(-p))$ vanish $(p=0,\dots,n, q\geq 1)$. Standard homological arguments then yield isomorphisms (where C_p stands for $Im(\partial':Z_{p+1}\otimes S(-p-1) \longrightarrow Z_p \otimes S(-p))$:

$$\begin{aligned} H_{p+1}(\underline{a};Sym(I)) &\equiv H_{p+1}(\underline{e};Sym(I)) \\ &\simeq Ker(H_o(\underline{e};C_p) \longrightarrow H_o(\underline{e};Z_p \otimes S(-p)) \\ &\simeq H_o(\underline{e};C_p) \\ &\simeq Z_{p+1} \qquad \text{(annihilated by } Sym_+(I)) \end{aligned}$$

$\Rightarrow H_p(\underline{a};Sym(I)) \otimes_{Sym(I)} Sym(I)[e_1,\dots,e_n] \simeq Z_p \otimes_R R[e_1,\dots,e_n]$, that is to say, the $\mathcal{M}.$-complex for the sequence $\underline{a}=a_1,\dots,a_n$ of Sym(I) is isomorphic to the $\mathcal{Z}.$-complex for the sequence $\underline{x}=x_1,\dots,x_n$ of R, which is acyclic by assumption. Using the result of Herzog-Simis-Vasconcelos mentioned above, we find that $Sym_+(I)$ is generated by a d-sequence of linear forms, which gives rise to a proper sequence in R generating I according to (2.2). (Note: Although Sym(I) is not local, the proof given in [loc.cit.] extends to rings of this type)///

In order to study the projective schemes associated with the graded rings Sym(I) and $\mathcal{R}(I)$, one could introduce "sheafified" versions of approximation complexes. To be more precise, for R noetherian, $I=(\underline{x})=(x_1,\dots,x_n)$, $S=R[e_1,\dots,e_n]$ denote $P:=\mathrm{Proj}\ S \simeq P_R^{n-1}$. Then

$$\tilde{\mathcal{Z}}.=\ 0 \longrightarrow \widetilde{Z_n \otimes S(-n)} \longrightarrow \ \dots\dots \longrightarrow \widetilde{Z_1 \otimes S(-1)} \longrightarrow \widetilde{Z_o \otimes S} \longrightarrow 0$$

and

$$\tilde{\mathcal{M}}.=\ 0 \longrightarrow \widetilde{H_n \otimes S(-n)} \longrightarrow \ \dots\dots \longrightarrow \widetilde{H_1 \otimes S(-1)} \longrightarrow \widetilde{H_o \otimes S} \longrightarrow 0$$

are complexes of $\mathcal{O}_P$-modules. It is clear that, for instance, Proj α is an isomorphism if $H_1(\tilde{\mathcal{M}}.)=0$ etc..
However:

(3.2)PROPOSITION. a) $\tilde{\mathcal{Z}}.$ acyclic $<=>$ $\mathcal{Z}.$ acyclic

b) $\tilde{\mathcal{M}}.$ acyclic $<=>$ $\mathcal{M}.$ acyclic

For (3.2) we will need a standard homological lemma that we give without proof:

(3.3)LEMMA. Let R be a noetherian ring, I an ideal of R, $0 \longrightarrow F_k \longrightarrow \dots\dots \longrightarrow F_1 \longrightarrow M \longrightarrow 0$ be a complex of finitely generated R-modules $(k\geq 1)$ such that there exists $n\geq k$ with $\mathrm{depth}_I(F_i)\geq n$ for $i=1,\dots,k$.

Then $\mathrm{depth}_I(M)\geq n+1-k$.

Proof of (3.2). We will only deal with b), as the proof of a) is very similar. To be definite, write $I=(\underline{x})=(x_1,\dots,x_n)$.
We know $H_p(\tilde{\mathcal{M}}.)=0$ for all $p\geq 1$, hence $S_+^d \cdot H_p(\mathcal{M}.)=0$ for $d>>0$.
To show $0=H_p(\mathcal{M}.)$ for $p=1,\dots,n$, we argue by backward induction on p.
p=n: $H_n(\mathcal{M}.)\subseteq H_n\otimes S(-n)$, so $\mathrm{depth}_{S_+}(H_n(\mathcal{M}.)\geq 1$. From $S_+^d \cdot H_n(\mathcal{M}.)=0$ $(d>>0)$ we derive $H_n(\mathcal{M}.)=0$.
$1\leq p<n$: Suppose $0=H_n(\mathcal{M}.)=\dots\dots\dots=H_{p+1}(\mathcal{M}.)$.
The exact sequence

$$0 \longrightarrow H_n\otimes S(-n) \rightarrow \dots\dots\dots \rightarrow H_{p+1}\otimes S(-p-1) \longrightarrow \mathrm{Im}\ \partial'_{p+1} \longrightarrow 0$$

yields, together with (3.3), an estimate

$$\operatorname{depth}_{S_+} \operatorname{Im} \partial'_{p+1} \geq (n+1)-(n-p)=p+1\geq 2$$

As $\operatorname{Ker} \partial'_p \subseteq H_p \otimes S(-p)$, we find $\operatorname{depth}_{S_+} \operatorname{Ker} \partial'_p \geq 1$

These modules can be inserted into the exact sequence

$$0 \longrightarrow \operatorname{Im} \partial'_{p+1} \longrightarrow \operatorname{Ker} \partial'_p \longrightarrow H_p(\mathcal{M}.) \longrightarrow 0 ,$$

which shows $\operatorname{depth}_{S_+} H_p(\mathcal{M}.)\geq 1$.

By consequence, $S_+^d \cdot H_p(\mathcal{M}.)=0$ $(d>>0)$ forces $H_p(\mathcal{M}.)=0$. ///

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