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CONTINUOUS CONVOLUTION SEMIGROUPS INTEGRATING A SUBMULTIPLICATIVE FUNCTION

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Let φ be a submultiplicative function on a locally compact group G and let S be a convolution semigroup on G with Lévy measure η. It is shown that the measures of S integrate φ if and only if η integrates φ outside some neighbourhood of the identity of G (Theorem 1). Moreover if $(X(t);t\geq 0)$ is the G-valued process with independent increments associated with the semigroup S it is shown that the measures of S integrate φ if and only if the random variable $\sup\{\varphi(X(t)): 0\leq t\leq 1\}$ is integrable (Theorem 2).

A Borel measurable function φ of a locally compact group G into the interval $]0,\infty[$ is said to be submultiplicative if $\varphi(xy) \leq \varphi(x)\varphi(y)$ for all $x,y \in G$ and if there exists a positive real number $c = c(\varphi)$ such that $\{x \in G: \varphi(x) \leq c\}$ is a neighbourhood of the identity e of G. Then one can consider the probability measures on the Borel subsets of G which integrate φ. This is an extension of the familiar moment conditions for measures on Euclidian spaces: If d is a left invariant continuous pseudometric on G then $(1 + d(.,e))^n$ $(n \in \mathbb{N})$ and $\exp d(.,e)$ are submultiplicative functions on G.

In the literature submultiplicative functions have been applied to estimate the decay at infinity of a measure (cf. [1,4,6,7,9]). In particular this has been done for the measures of a continuous convolution semigroup $(\mu_t)_{t\geq 0}$ on the group G. It turned out that the integra-

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bility of φ by the measures μ_t is closely related to the behaviour of the Lévy measure η of the semigroup defined by $\int f \, d\eta = \lim_{t \downarrow 0} (\int f \, d\mu_t)/t$ (for all real-valued bounded continuous functions f on G vanishing in some neighbourhood of e). It is the purpose of the present paper to establish the precise relation.

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Let us fix a locally compact group G and a continuous convolution semigroup $(\mu_t)_{t \geq 0}$ on G with generating functional A and Lévy measure η (cf. [3], Chapter IV). A is defined on the space $\mathcal{D}(G)$ of real-valued infinitely differentiable functions with compact support on G (cf. [3], 4.4.2). Furthermore let λ be a left Haar measure on G and $L^1(G)$ the Banach space of (equivalence classes of) real-valued λ-integrable functions on G. By $S_t f = \mu_t * f$ for all $f \in L^1(G)$ and $t \geq 0$ there is defined a strongly continuous semigroup $(S_t)_{t \geq 0}$ of contractions on $L^1(G)$ whose infinitesimal generator we denote by $(N_1, \mathcal{N}_1)$.

Now let φ be a submultiplicative function on G with corresponding constant $c = c(\varphi)$. We define functions $\tilde{\varphi}$ and φ^{-1} on G by $\tilde{\varphi}(x) = \varphi(x^{-1})$ and $\varphi^{-1}(x) = \varphi(x)^{-1}$ respectively (with the convention $\infty^{-1} = 0$). Then we obviously have $\varphi(xy) \geq \varphi(x) \, \tilde{\varphi}^{-1}(y)$ for all $x,y \in G$ (with the convention $0 \cdot \infty = 0$). Finally let 1_B denote the indicator function of a subset B of G.

THEOREM 1. The following assertions are equivalent:

(i) $\int \varphi \, d\mu_t < \infty$ for some $t > 0$.

(ii) $\sup\{\int \varphi \, d\mu_s : 0 \leq s \leq t\} < \infty$ for all $t > 0$.

(iii) $\int 1_{[\varphi > c]} \, \varphi \, d\eta < \infty$.

(iv) There exists some $f \in \mathcal{N}_1$ with $f \geq 0$ and $f \neq 0$ such that $\int f \varphi \, d\lambda < \infty$, $\int f \tilde{\varphi}^{-1} \, d\lambda > 0$, and $\int |N_1 f| \varphi \, d\lambda < \infty$.

PROOF. '(i)$\Longrightarrow$(ii)' Since μ_r converges weakly to the unit mass in e if r tends to 0 and since $[\varphi \leq c]^{-1}$ is a neighbourhood of e there exists a $t_1 \in]0,t]$ such that $\mu_r([\varphi \leq c]^{-1}) \geq 1/2$ and hence $\int \tilde{\varphi}^{-1}\, d\mu_r \geq 1/2c$ if $0 \leq r \leq t_1$. But $\varphi(xy) \geq \varphi(x)\,\tilde{\varphi}^{-1}(y)$ for all $x,y \in G$ implies

$$(*)\quad \int \varphi\, d\mu_t \geq \left(\int \varphi\, d\mu_s\right)\left(\int \tilde{\varphi}^{-1}\, d\mu_{t-s}\right) \geq (1/2c)\int \varphi\, d\mu_s$$

and hence $\int \varphi\, d\mu_s < \infty$ if $0 \leq t - s \leq t_1$. By iteration this yields $\int \varphi\, d\mu_s < \infty$ if $0 \leq s \leq t$. Now (*) (with t replaced by t_1) yields

$$\sup\{\int \varphi\, d\mu_s : 0 \leq s \leq t_1\} \leq 2c \int \varphi\, d\mu_{t_1} < \infty .$$

Finally $\varphi(xy) \leq \varphi(x)\,\varphi(y)$ implies $\int \varphi\, d\mu_{r+s} \leq \left(\int \varphi\, d\mu_r\right)\left(\int \varphi\, d\mu_s\right)$ for all $r,s > 0$. This proves (ii).

'(ii)$\Longrightarrow$(iii)' Let $\gamma = 1_{[\varphi > c]} \cdot \eta$ and let $C(f) = \int (f - f(e))\, d\gamma$ for all $f \in \mathfrak{D}(G)$. Then C is the generating functional of a Poisson semigroup and $B = A - C$ is the generating functional of a continuous convolution semigroup $(\nu_t)_{t \geq 0}$ on G. Now the expansion of μ_t into a perturbation series according to the decomposition $A = B + C$ ([2], 61) yields the inequality

$$(**)\quad \mu_t \geq \exp(-bt) \int 1_{[0,t]} \cdot \nu_r * \gamma * \nu_{t-r}\, dr \qquad (t > 0)$$

where $b = \gamma(G) = \eta([\varphi > c])$.

There exists a $t > 0$ such that $\int \tilde{\varphi}^{-1}\, d\nu_s \geq 1/2c$ if $0 \leq s \leq t$ (see above). Moreover we have for all $x,y,z \in G$ $\varphi(xyz) \geq \tilde{\varphi}(x)^{-1}\, \varphi(y)\, \tilde{\varphi}(z)^{-1}$. Consequently (ii) implies

$$\exp(bt) \int \varphi\, d\mu_t \geq$$
$$\int 1_{[0,t]} \left(\int \varphi\, d(\nu_r * \gamma * \nu_{t-r})\right) dr \geq$$
$$\int 1_{[0,t]} \left(\int \tilde{\varphi}^{-1}\, d\nu_r\right)\left(\int \varphi\, d\gamma\right)\left(\int \tilde{\varphi}^{-1}\, d\nu_{t-r}\right) dr \geq$$
$$(t/4c^2) \int \varphi\, d\gamma .$$

Hence $\int 1_{[\varphi > c]}\, \varphi\, d\eta = \int \varphi\, d\gamma < \infty$.

'(iii)$\Longrightarrow$(iv)' Let $f \in \mathfrak{D}(G)$, $f \geq 0$, $f \neq 0$, with $f(x) = f(x^{-1})$ for all $x \in G$ and $f(x) = 0$ if $x \notin [\varphi \leq c] \cap [\varphi \leq c]^{-1}$. Then we obviously have $\int f \varphi\, d\lambda < \infty$ and $\int f \tilde{\varphi}^{-1}\, d\lambda > 0$.

Moreover $f \in \mathcal{N}_1$ and $N_1 f(x^{-1}) = A({}_x f)$ for all $x \in G$ where ${}_x f(y) = f(xy)$ (all $y \in G$).

[This can be easily derived from the fact that $\mathcal{D}(G)$ belongs to the domain of the infinitesimal generator of the semigroup of contractions on $\mathcal{C}^0(G)$ (= the space of real-valued continuous functions on G vanishing at infinity) induced by the semigroup $(\mu_t)_{t \geq 0}$ (cf. [3], 4.4.1 and 4.5.8).]

Let $x \in G$ such that $\varphi(x) > c^2 = b$. If $y \in [\varphi \leq c]$ then $\varphi(y^{-1}x) \geq \varphi(x)\,\varphi(y)^{-1} > c$ and hence $f(x^{-1}y) = f(y^{-1}x) = 0$. Thus ${}_{x^{-1}} f$ vanishes in some neighbourhood of e. This implies

$$N_1 f(x) = A({}_{x^{-1}} f) = \int {}_{x^{-1}} f \, d\eta = \int f(x^{-1}y)\, \eta(dy).$$

Consequently:

$$\int |N_1 f| \varphi \, d\lambda =$$

$$\int 1_{[\varphi \leq b]} |N_1 f| \varphi \, d\lambda + \int 1_{[\varphi > b]} |N_1 f| \varphi \, d\lambda \leq$$

$$b \int |N_1 f| \, d\lambda + \int 1_{[\varphi > b]}(x) |N_1 f(x)| \, \varphi(x) \, \lambda(dx) =$$

$$b \int |N_1 f| \, d\lambda + \int 1_{[\varphi > b]}(x) \int f(x^{-1}y)\, \varphi(x)\, \eta(dy) \lambda(dx) =$$

$$b \int |N_1 f| \, d\lambda + \int \{\int 1_{[\varphi > b]}(x) f(x^{-1}y)\, \varphi(x)\, \lambda(dx)\}\, \eta(dy).$$

Let us consider the inner integral of the second summand:

$$\int 1_{[\varphi > b]}(x) f(x^{-1}y)\, \varphi(x)\, \lambda(dx) =$$

$$\int 1_{[\varphi > b]}(yx) f(x^{-1})\, \varphi(yx)\, \lambda(dx) =$$

$$\int 1_{[\varphi > b]}(yx)\, 1_{[\varphi \leq c]}(x) f(x)\, \varphi(yx)\, \lambda(dx) \leq$$

$$1_{[\varphi > c]}(y)\, \varphi(y) \{\int f \varphi \, d\lambda\}.$$

Since $\int 1_{[\varphi > c]} \varphi \, d\eta < \infty$ by assumption we finally have $\int |N_1 f| \varphi \, d\lambda < \infty$.

'(iv) $\Longrightarrow$ (i)' For a finite submultiplicative function φ with $\varphi = \tilde{\varphi}$ this has been proved by A.Hulanicki ([5], Theorem 4.1). But in the general case the proof can be carried out along the same lines. ⌟

REMARKS. 1. An analysis of the proof of Theorem 4.1 in [5] yields the following estimation valid for all $t > 0$ (where $\langle h,g\rangle$ means $\int hg\, d\lambda$):

$$\int \varphi \, d\mu_t \leq \{\langle f,\varphi\rangle/\langle f,\tilde{\varphi}^{-1}\rangle\} \exp\{t\langle |N_1 f|,\varphi\rangle/\langle f,\tilde{\varphi}^{-1}\rangle\}.$$

2. Let S be an open subsemigroup of G containing e. Then the Lévy measure η is supported by S if and only if all the measures μ_t are supported by S.

[Let $\varphi(x) = 1$ if $x \in S$ and $\varphi(x) = \infty$ if $x \in G \setminus S$. Then φ is a submultiplicative function on G with $c(\varphi) = 1$. If η is supported by S we have $\int 1_{[\varphi>1]} \varphi \, d\eta = 0$; hence (iii) of Theorem 1 is satisfied. Thus $\int \varphi \, d\mu_t < \infty$ for all $t \geq 0$ by this very theorem. But this can only hold if $\mu_t(G \setminus S) = 0$. The converse conclusion follows along the same lines.]

3. Let the Lévy measure η of the convolution semigroup $(\mu_t)_{t\geq 0}$ be supported by an open neighbourhood U of e. Let $\varphi(x) = \exp[\inf\{n \in \mathbb{N} : x \in U^n\}]$ for all $x \in G$ (where the infimum over the void set is defined as ∞). Then φ is a submultiplicative function on G with $c(\varphi) = e$. Hence (iii) of Theorem 1 is fulfilled. Consequently by this very theorem there exists some constant $C > 0$ such that $\int \varphi \, d\mu_s \leq C$ for all $s \in [0,1]$. This yields

$$\mu_s(\complement U^n) \leq e^{-(n+1)} \int \varphi \, d\mu_s \leq C\, e^{-(n+1)}$$

for all $n \in \mathbb{N}$ and $s \in [0,1]$. Thus the decay of the mass of the measures μ_s following the filtration $(U^n)_{n\geq 1}$ is with exponential speed.

If U can be chosen relatively compact and such that $U^{-1} \subset U \cup U^2 \cup U^3 \cup \ldots$ then a better estimation of $\mu_s(\complement U^n)$ has been given by M.Duflo ([1], Prop.14).

Let now G be a locally compact group having a countable basis of its topology and let $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup of probability measures on G. Then there exists a stochastically continuous G-valued process $(\Omega, \mathfrak{A}, P, (X_t)_{t\geq 0})$ which has independent left incre-

ments $X_s^{-1} X_t$, $s < t$, such that μ_t is the distribution of the random variable X_t. Without loss we can assume that the process $(X_t)_{t\geq 0}$ has right continuous paths (cf. [3], 5.6). Finally let φ be a lower semicontinuous submultiplicative function on G with $c = c(\varphi)$. We put $\tau = \ln \varphi$ (with the convention $\ln \infty = \infty$) and $d = \ln c$.

THEOREM 2. The following assertions are equivalent:

(i) $\int \varphi(X_t)\, dP = \int \varphi \, d\mu_t < \infty$ for some $t > 0$.

(ii) $\int (\sup\{\varphi(X_s) : 0 \leq s \leq t\})\, dP < \infty$ for all $t > 0$.

Only '(i) $\Longrightarrow$ (ii)' has to be shown. We first prove an auxiliary result:

LEMMA. There exists some $t > 0$ such that for all $a > d$:

$$P(\sup\{\tau(X_s) : 0 \leq s \leq t\} > a) \leq 2\mu_t(\tau > a - d).$$

PROOF. Since $[\tau \leq d]^{-1}$ is a neighbourhood of e there exists a $t > 0$ such $\mu_s([\tau \leq d]^{-1}) \geq 1/2$ for all $s \in [0,t]$. Let us fix this t and let $a > d$ as well as $b = a - d$ be fixed too. We put for all $n \in \mathbb{N}$:

$$A_n = [\sup\{\tau(X_{kt/n}) : 1 \leq k \leq n\} > a] \quad \text{and}$$

$$A = [\sup\{\tau(X_s) : 0 \leq s \leq t\} > a].$$

Since $(X_t)_{t\geq 0}$ has right continuous paths and since φ is lower semicontinuous the sequence $(A_{n!})_{n\geq 1}$ ascends to A.

Moreover let $B = [\tau(X_t) > b]$. Finally for some fixed $n \in \mathbb{N}$ let $C_k = [\tau(X_t^{-1} X_{kt/n}) > d]$, $1 \leq k \leq n$. Then we have $A_n \subset B \cup C_1 \cup \ldots \cup C_n$: If ω is not in the set on the right hand side then

$$\tau(X_{kt/n}(\omega)) \leq \tau(X_t(\omega)) + \tau(X_t(\omega)^{-1} X_{kt/n}(\omega))$$
$$\leq b + d = a$$

for $k = 1,\ldots,n$; whence $\omega \notin A_n$. For $k = 1,\ldots,n$ let

$$D_k = [\tau(X_{kt/n}) > a] \cap \bigcap_{1\leq j<k} [\tau(X_{jt/n}) \leq a].$$

Then the sets D_k are pairwise disjoint and their union is

just A_n. As above one shows $D_k \subset B \cup C_k$ and hence $D_k \subset B \cup (C_k \cap D_k)$. This implies $A_n \subset B \cup (C_1 \cap D_1) \cup \ldots \cup (C_n \cap D_n)$. But C_k and D_k are stochastically independent (since the process (X_t) has independent left increments). This yields:

$$P(A_n) \leq P(B) + \sum_{k=1}^{n} P(C_k \cap D_k) = P(B) + \sum_{k=1}^{n} P(C_k)P(D_k)$$

$$\leq P(B) + \sup\{P(C_k): 1 \leq k \leq n\} \sum_{k=1}^{n} P(D_k)$$

$$= P(B) + \sup\{P(C_k): 1 \leq k \leq n\}\, P(A_n).$$

Consequently:

$$P(B) \geq P(A_n)\,(1 - \sup\{P(C_k): 1 \leq k \leq n\})$$

$$= P(A_n)\, \inf\{P(\complement C_k): 1 \leq k \leq n\}.$$

But $P(B) = \mu_t(\tau > b)$ and

$$P(\complement C_k) = \mu_{(n-k)t/n}([\tau \leq d]^{-1}) \geq 1/2 \qquad (1 \leq k \leq n)$$

imply $2\mu_t(\tau > b) \geq P(A_n)$. Since $n \in \mathbb{N}$ was arbitrary and since $\sup\{P(A_n): n \in \mathbb{N}\} = P(A)$ our assertion is proved. ⌟

REMARK. The inequality of the lemma is some kind of a continuous analogue to the classical Lévy inequality for partial sums of independent random variables. Similar inequalities appear in many places (see for example [8], proof of Theorem 2).

PROOF OF THEOREM 2. Let $t > 0$ as in the lemma and put $Y = \sup\{\tau(X_s): 0 \leq s \leq t\}$. Then with the aid of the lemma we obtain:

$$\int e^Y\, dP = 1 + \int_0^\infty P(Y > a)\, e^a\, da$$

$$\leq 1 + de^d + \int_d^\infty P(Y > a)\, e^a\, da$$

$$\leq 1 + cd + 2\int_d^\infty \mu_t(\tau > a - d)\, e^a\, da$$

$$= 1 + cd + 2e^d \int_0^\infty \mu_t(\tau > b)\, e^b\, db$$

$$\leq 1 + cd + 2c(1 + \int_0^\infty \mu_t(\tau > b)\, e^b\, db)$$

$$= 1 + cd + 2c \int \varphi \, d\mu_t .$$

Let now $c_r = \int (\sup\{\varphi(X_s): 0 \leq s \leq r\})\, dP$ for all $r > 0$. As we have just shown $c_r < \infty$ if $r \leq t$. To finish the proof let $Y_s = X_t^{-1} X_{t+s}$ for all $s \geq 0$. Obviously $(\Omega, \mathfrak{A}, P, (Y_s)_{s\geq 0})$ is a stochastic process which is stochastically equivalent to the original process $(\Omega, \mathfrak{A}, P, (X_s)_{s\geq 0})$. Thus also $c_r = \int (\sup\{\varphi(Y_s): 0 \leq s \leq r\})\, dP$. If $t \leq s \leq 2t$ we have

$$\varphi(X_s) = \varphi(X_t X_t^{-1} X_s) \leq \varphi(X_t)\, \varphi(Y_{s-t})$$

and hence

$$\sup\{\varphi(X_s): t < s \leq 2t\} \leq \varphi(X_t)\, \sup\{\varphi(Y_r): 0 < r \leq t\}.$$

Since the process $(X_s)_{s\geq 0}$ has independent left increments X_t and $\{Y_s: s \geq 0\}$ are stochastically independent. Thus

$$\int \sup\{\varphi(X_s): t < s \leq 2t\})\, dP \leq c_t^2 .$$

Consequently $c_{2t} \leq c_t + c_t^2 < \infty$. By induction we obtain $c_{nt} \leq c_t + \ldots + c_t^n < \infty$ for all $n \in \mathbb{N}$. Thus the proof of Theorem 2 is complete. ⌟

REMARKS. 1. A.Tortrat has proved a result related to Theorem 2 for infinitely divisible probability measures on a separable Banach space [10]. In fact the proof of Theorem 2 is independent of the local compactness of G. Thus the result holds for every polish group.

2. Let φ be a continuous submultiplicative function on G such that $\int \varphi \, d\mu_t < \infty$ for some $t > 0$. Since $\lim_{s\downarrow 0} X_s = e$ with probability one the theorem of the majorized convergence (which can be applied in view of Theorem 2) yields $\lim_{s\downarrow 0} \int \varphi \, d\mu_s = \varphi(e)$.

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