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# A NOTE ON SIMULTANEOUS DIOPHANTINE APPROXIMATION

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Refining earlier investigations due to J.M.MACK [7] by a method of MORDELL it is proved that for any two irrational numbers $\alpha$, $\beta$ there exist infinitely many pairs of fractions p/r, q/r satisfying the inequalities

$$|\alpha - \frac{p}{r}| < \frac{8}{13} r^{-3/2}, \quad |\beta - \frac{q}{r}| < \frac{8}{13} r^{-3/2}.$$

## 1. Introduction and formulation of the results

By a well-known theorem of HURWITZ on diophantine approximation there exist for any irrational number $\alpha$ infinitely many (reduced) fractions p/r fulfilling the inequality

$$|\alpha - \frac{p}{r}| < 5^{-1/2} r^{-2}, \tag{1}$$

the constant $5^{-1/2}$ being best possible. It is much more difficult to find a constant c as small as possible such that for arbitrary irrational numbers $\alpha$ and $\beta$ the inequalities

$$|\alpha - \frac{p}{r}| < c r^{-3/2}, \quad |\beta - \frac{q}{r}| < c r^{-3/2} \tag{2}$$

hold simultaneously for infinitely many pairs of rationals (p/r, q/r). It follows readily from DIRICHLET's approximation theorem that $c = 1$ is an admissible value in the sense of this problem. DAVENPORT and MAHLER [5] proved all $c > 2^{1/2}\, 23^{-1/4}$, MULLENDER [8] all $c > 2^{7/2}\, 3^{-9/4}\, 23^{-1/8}$ to be admissible. Finally in 1951 DAVENPORT [3] established the

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assertion for $c > 46^{-1/8}$, the best result for a long time. The infimum $c_0$ of all admissible values for c has not yet been determined. CASSELS obtained the lower estimate $c_0 \geqq (2/7)^{1/2}$ [1].

A few years ago J.M.MACK succeeded in improving DAVENPORT's result by an ingenious new method [6], [7]. He showed that all $c > (2{,}6394)^{-1/2}$ are admissible values for the above assertion.

Adding a new idea to MACK's arguments we are able to establish the following slight improvement.

THEOREM 1. *For any pair of irrational numbers* $(\alpha,\beta)$ *there exist infinitely many pairs of fractions* $(p/r,\ q/r)$ *satisfying the inequalities*

$$|\alpha - \frac{p}{r}| < \frac{8}{13} r^{-3/2}, \quad |\beta - \frac{q}{r}| < \frac{8}{13} r^{-3/2}. \tag{3}$$

Simultaneously we obtain the corresponding dual result on linear forms.

THEOREM 2. *For any pair of irrational numbers* $(\alpha,\beta)$ *there exist infinitely many triples of integers* $(p,q,r)$ *fulfilling the inequalities*

$$|\alpha p + \beta q + r| < \left(\frac{8}{13}\right)^2 p^{-2}, \quad |\alpha p + \beta q + r| < \left(\frac{8}{13}\right)^2 q^{-2}, \quad p^2 + q^2 > 0. \tag{4}$$

By a well-known principle in the geometry of numbers (used first by DAVENPORT and MAHLER [5] and formulated in full generality by DAVENPORT [4]) both assertions are immediate consequences of the following estimate.

THEOREM 3. *For the critical determinant* $\Delta(K)$ *of the three-dimensional star body*

$$K := \{(x,y,z) \in \mathbb{R}^3 : x^2|z| < 1,\ y^2|z| < 1\} \tag{5}$$

*we have the inequality*

$$\Delta(K) > \left(\frac{13}{8}\right)^2 . \tag{6}$$

## 2. MACK's method and his results used in this paper

MACK starts by reducing the problem of finding a lower bound for $\Delta$ (K) to a two-dimensional problem. For $0 \leq t \leq 1$ he considers the set $S_o(t)$ of all points (x,y) of the plane with the property that for any real number $\kappa$ there exists a real number $\lambda$ congruent to $\kappa$ modulo 1 such that

$$|\lambda| \max\{(x+\lambda t)^2, (y+\lambda)^2\} < 1 \tag{7}$$

and proves the estimate

$$\inf_{0 \leq t \leq 1} \Delta(t) \leq \Delta(K) \tag{8}$$

for the critical determinant $\Delta(t)$ of $S_o(t)$. To establish our inequality (6) it is therefore sufficient to show

$$\Delta(t) \geq 2{,}64065 \tag{9}$$

for $0 \leq t \leq 1$. For $0 \leq t \leq 0{,}69$ MACK already obtains estimates even better than (9), the greatest difficulties arise in the interval $0{,}9 \leq t \leq 0{,}91$. That is why we are going to describe in detail the set $S_o(t)$ (resp. a less complicated, only unessentially smaller subset $S(t)$) for t near 0,9 (see Fig.1, p.4). Since $S_o(t)$ and $S(t)$ are symmetric in the origin we consider only the upper half-plane, throughout the paper.

The boundary of $S(t)$ is given by the following smooth curves: (Convention: From now on numbers with four places of decimals are rounded-off values. For the exact analytical definitions we refer to MACK's paper; the actual calculations have been performed with ten-figure numbers. Numbers with more or less than four places of decimals are to be considered as exact values.)

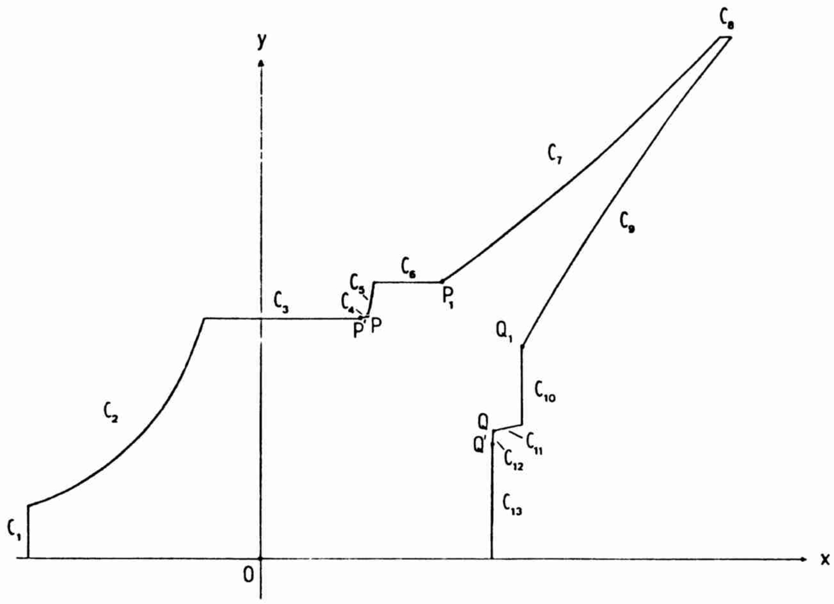


Fig. 1. The set S(t) for t = 0,9

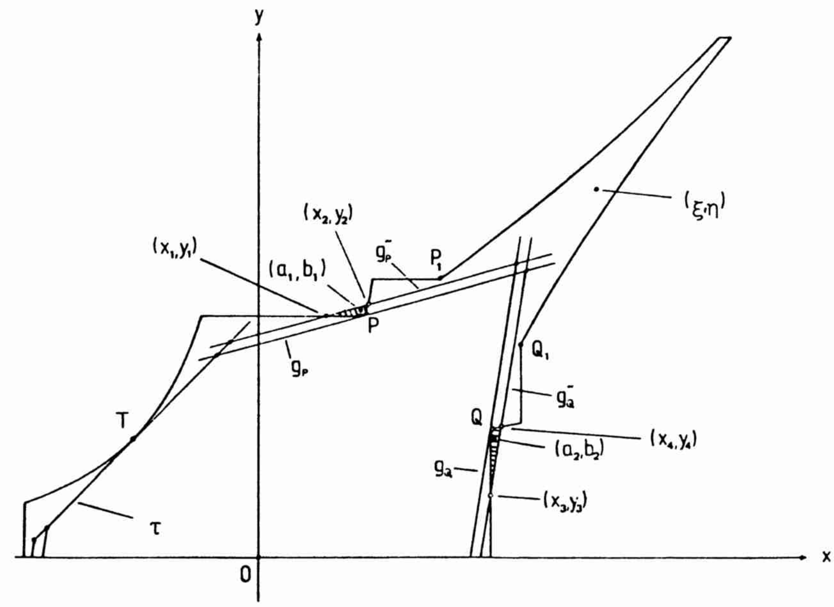


Fig. 2. The hexagons $H_1$ and $H_2$

$$C_1: \quad x = -3(t/4)^{1/3} \qquad \text{for } 0 \leqq y \leqq y^* \tag{10}$$
$$(y^* := -1 + 0{,}2238\, t^{-2/3} + (1 - 0{,}2238\, t^{-2/3})^{-1/2})$$

$$C_2: \quad x = (1-\lambda)t - (1-\lambda)^{-1/2}$$
$$y = -\lambda + \lambda^{-1/2} \qquad \text{for } y^* \leqq y \leqq 1{,}8899 \tag{11}$$

$$C_3: \quad y = 3/4^{1/3} = 1{,}8899$$

$C_4$ (the tiny curve between P' and P):

$$x = (1-\lambda)t - (1-\lambda)^{-1/2} \qquad (\lambda < -0{,}67)$$
$$y = -\lambda + (-\lambda)^{-1/2} \qquad \text{for } 1{,}8899 \leqq y \leqq 1{,}9123 \tag{12}$$

$$C_5: \quad x = (2-\lambda)t - (2-\lambda)^{-1/2}$$
$$y = -\lambda + \lambda^{-1/2} \qquad \text{for } 1{,}9123 \leqq y \leqq 2{,}1652 \tag{13}$$

$$C_6: \quad y = 2{,}1652$$

$$C_7: \quad x = (1-\lambda)t - (1-\lambda)^{-1/2} \qquad (\lambda < -0{,}67)$$
$$y = -\lambda + (-\lambda)^{-1/2} \qquad \text{for } 2{,}1652 \leqq y \leqq 4{,}0800 \tag{14}$$

$$C_8: \quad y = 4{,}0800$$

$$C_9: \quad x = -t(1+\lambda) + (-1-\lambda)^{-1/2}$$
$$y = -\lambda - (-\lambda)^{-1/2} \qquad \text{for } 2-\lambda_1-(2-\lambda_1)^{-1/2} \leqq y \leqq 4{,}0800 \tag{15}$$

($\lambda_1 = \lambda_1(t)$ being defined by formula (12) of MACK's paper.)

$$C_{10}: \quad x = -t\lambda_1(t) + (-\lambda_1(t))^{-1/2}$$
$$\text{for } 2-\mu_1-(2-\mu_1)^{-1/2} \leqq y \leqq 2-\lambda_1-(2-\lambda_1)^{-1/2}$$

(For the definition of $\mu_1 = \mu_1(t)$ see MACK's formula (17).)

$$C_{11}: \quad x = -\lambda t + \lambda^{-1/2}$$

$$y = 2 - \lambda - (2-\lambda)^{-1/2} \qquad \text{for } y_Q(t) \leq y \leq 2 - \mu_1 - (2-\mu_1)^{-1/2} \tag{16}$$

$C_{12}$ (the small curve between Q and Q'):

$$x = (1-\lambda)t + (1-\lambda)^{-1/2}$$

$$y = 2 - \lambda - (2-\lambda)^{-1/2} \qquad \text{for } y^{\sim}(t) \leq y \leq y_Q(t) \tag{17}$$

$$y_Q(t) := 2 - \lambda_0(t) - (2-\lambda_0(t))^{-1/2}$$

$$\lambda_0(t) := (1-(1-4\gamma(t)^2)^{1/2})/2, \quad \gamma(t) := ((1+t^2)^{1/2}-1)t^{-2} \tag{18}$$

$$C_{13}: \quad x = 3(t/4)^{1/3} \qquad \text{for } 0 \leq y \leq y^{\sim}(t) \tag{19}$$

$$(y^{\sim}(t) := (2t)^{-2/3} + 1 - ((2t)^{-2/3}+1)^{-1/2})$$

The remaining part of the boundary of the O-symmetric set $S(t)$ is obtained from the part just described by reflecting it in the origin.

MACK proceeds by inscribing convex hexagons into $S(t)$ in two different ways for $0{,}69 \leq t \leq 0{,}9$ and for $0{,}91 \leq t \leq 1$. Applying MINKOWSKI's theorem he thereby obtains lower bounds for $\Delta(t)$. In the first case the hexagon is formed by the tangent to $C_2$ at the point corresponding to the value of the parameter $\lambda = 0{,}5$ and by the straight lines through P and $P_1$ resp. Q and $Q_1$, in the second case by the same tangent to $C_2$, the tangent to $C_4$ at P and the tangent to $C_{12}$ at Q (and by the images of the given lines in the origin).

For $0{,}69 \leq t \leq 0{,}9$ the area $A(t)$ of the hexagon decreases if t increases. The numerical calculation shows that for $t \leq 0{,}88$ it is large enough to ensure the validity of our inequality (9). For $0{,}91 \leq t \leq 1$ the area $A(t)$ increases with t and is sufficiently large for $t \geq 0{,}94$.

There remains to deal with the interval $0{,}88 \leq t \leq 0{,}94$. The principle of the improvement for this range is to be sketched in the following section.

## 3. The basic idea of our refinement

The salient point of our argumentation is the observation that the fourth vertex of the parallelogram generated by the points O, P and Q lies in the interior of S(t) for the values of t to be considered. We employ a method due to MORDELL a detailed exposition of which can be found in CASSELS' classical monograph [2] , page 84 - 98.

To this end we consider the tangent $\tau$ to $C_2$ at the point T corresponding to the value of the parameter $\lambda = 0{,}50470 =: \mu$ and the straight lines

$$g_P: \quad y = k_1 x + \alpha_0 \tag{20}$$

$$g_Q: \quad y = k_2 x + \beta_0 \tag{20'}$$

through P resp. Q with slope $k_1 = 0{,}269$ resp. $k_2 = 6{,}66$ (using the most "favorable" values for $k_1$, $k_2$ and $\mu$ due to MACK's computer calculations). Moreover we regard parallel lines

$$\tilde{g}_P: \quad y = k_1 x + \alpha_1 \tag{21}$$

$$\tilde{g}_Q: \quad y = k_2 x + \beta_1 \tag{21'}$$

lying outside S(t) near P resp. Q with small distance from P resp. Q (see Fig. 2).

Let $H_1$ be the hexagon formed by $\tau, \tilde{g}_P, g_Q$ (and their images in the origin) and analogously $H_2$ the hexagon formed by $\tau, g_P, \tilde{g}_Q$ (and their images). For their areas $A_1, A_2$ we obtain by some elementary, but rather lengthy analytic geometry the following formula:

$$A(t) = \frac{(\alpha-\beta)^2}{k_2-k_1} + \frac{(v(t)+\beta)^2}{u(t)-k_2} - \frac{(v(t)-\alpha)^2}{u(t)-k_1} \tag{22}$$

$$u(t) := (2+\mu^{-3/2})(2t+(1-\mu)^{-3/2})^{-1} \tag{23}$$

$$v(t) := ((1-\mu)^{-1/2} - (1-\mu)t)u(t) - \mu + \mu^{-1/2}. \tag{24}$$

(The indexes of $\alpha$, $\beta$ and A are to be added suitably according to the definitions of $H_1, H_2$ and to the equations of $g_P$, $g_P^{\sim}$, $g_Q$ and $g_Q^{\sim}$.)

We now suppose the critical determinant $\Delta(t)$ of $S(t)$ to be less than $\Delta_0 := 2{,}64065$ for some fixed value of $t$ between $0{,}88$ and $0{,}94$. Then there would exist a lattice $\Gamma$ admissible for $S(t)$ with lattice constant $d(\Gamma) < \Delta_0$. (For the definitions of the basic concepts in the geometry of numbers used here we refer to CASSELS [2].) If we choose $\alpha_1$ sufficiently large to ensure that $A_1(t) \geqq 10{,}5626 = 4\Delta_0$, MINKOWSKI's convex body theorem implies the existence of a lattice point $(a_1, b_1)$ of $\Gamma$ in the interior of $H_1$. Since $\Gamma$ is admissible for $S(t)$ and since $S(t)$ is symmetric with respect to the origin, this point may be supposed to lie in the shaded area near P (Fig. 2). The same reasoning applied to $H_2$ yields another lattice point $(a_2, b_2)$ of $\Gamma$ in the shaded area near Q.

If we have choosen the numbers $\alpha_1$ and $\beta_1$ cautiously enough (i.e. sufficiently near to $\alpha_0$ resp. $\beta_0$) we now can show that the lattice point $(\xi, \eta) = (a_1 + a_2, b_1 + b_2)$ of $\Gamma$ necessarily lies in the interior of $S(t)$. This contradicts our assumption that $\Gamma$ is admissible for $S(t)$ and thus our inequality (9) is established for the particular value of t.

## 4. The details of our proof

In carrying out the program outlined above an essential difficulty lies in the fact that it is very hard to proof that the various occurring functions are monotone on the interval $0{,}88 \leqq t \leqq 0{,}94$. We avoid this problem by dividing our interval into subintervals, where we can replace the functions involved by suitable upper or lower bounds.

Let $[t_1, t_2]$ be such a subinterval. Then we infer from (20) and (12)

$$\alpha_o(t) = y_P - k_1 x_P(t) = 1{,}9123 - k_1((1-\lambda_P)t - (1-\lambda_P)^{-1/2}), \tag{25}$$

$x_P(t)$, $y_P$ being the coordinates of the point P and $\lambda_P = -0{,}7800$ the corresponding value of the parameter in the equations (12). Hence $\alpha_o(t)$ decreases with increasing t; we choose $\alpha_o(t_2)$ as a lower bound for $\alpha_o(t)$ on $[t_1, t_2]$.

Similarly we obtain from (20'), (17) and (18)

$$\beta_o(t) = y_Q(t) - k_2 x_Q(t) = 2 - \lambda_o(t) - (2-\lambda_o(t))^{-1/2} - \\ -k_2 t(1-\lambda_o(t)) - k_2(1-\lambda_o(t))^{-1/2}. \tag{26}$$

By some elementary analysis we infer from (18) that $\lambda_o(t)$ is a monotone decreasing function of t and therefore

$$\beta_o(t_1, t_2) := 2 - \lambda_o(t_2) - (2-\lambda_o(t_2))^{-1/2} - \\ -k_2 t_1(1-\lambda_o(t_1)) - k_2(1-\lambda_o(t_2))^{-1/2} \tag{27}$$

is an upper bound for $\beta_o(t)$ on $[t_1, t_2]$. We choose $\alpha_o = \alpha_o(t_2)$, $\beta_o = \beta_o(t_1, t_2)$ in (20) resp. (20'); then near P resp. Q the straight lines $g_P$ and $g_Q$ lie even in the interior of S(t) for $t_1 \leqq t < t_2$.

Moreover by (23) and (24) u(t) and v(t) are monotone decreasing and the calculation shows

$$1{,}0085 \leqq u(t) \leqq 1{,}0346, \quad 1{,}8663 \leqq v(t) \leqq 1{,}9220 \tag{28}$$

for $0{,}88 \leqq t \leqq 0{,}94$. By (22) we therefore see easily that

$$A(t_1, t_2) := \frac{(\alpha-\beta)^2}{k_2 - k_1} + \frac{(v(t_2)+\beta)^2}{u(t_1)-k_2} - \frac{(v(t_1)-\alpha)^2}{u(t_2)-k_1} \tag{29}$$

is a lower bound for A(t) on $[t_1, t_2]$ for the values of $\alpha$ and $\beta$ involved. (See the third and fourth column of our table on page 12. Again the indexes of $\alpha$, $\beta$ and A are to be put in suitably in two different ways according to the definitions of $H_1$ and $H_2$ .)

For our given subinterval $[t_1, t_2]$ we now determine numbers $\alpha_1$ and $\beta_1$ (see our table) such that

$$\min\{A_1(t_1,t_2),\ A_2(t_1,t_2)\} \geqq 10{,}5626 = 4\Delta_0 . \tag{30}$$

Let t be any value between $t_1$ and $t_2$ and let $\Gamma$ be a lattice admissible for $S(t)$ with lattice constant $d(\Gamma)<\Delta_0$. Then by the arguments given in section 3 there exist lattice points $(a_1,b_1)$ and $(a_2,b_2)$ of $\Gamma$ in the areas near P resp. Q which are shaded in Fig. 2.

In order to establish estimates for $a_1$ and $b_1$ we simply determine the points of intersection $(x_1,y_1)$ of $C_3$ with $\tilde{g_P}$ and $(x_2,y_2)$ of $C_5$ with $\tilde{g_P}$.

Obviously we have

$$y_1 = 3/4^{1/3}, \quad x_1 = (y_1 - \alpha_1)/k_1 \tag{31}$$

independently of t. By (13) and (21) we obtain the equation

$$k_1(2-\lambda)t - k_1(2-\lambda)^{-1/2} + \alpha_1 + \lambda - \lambda^{-1/2} = 0 \tag{32}$$

to determine the value of the parameter $\lambda$ corresponding to $(x_2,y_2)$ by (13). Since the left-hand side of (32) increases both with $\lambda$ and with t, the value of $\lambda$ defined by (32) decreases with increasing t. By (13) y is a decreasing function of $\lambda$ and so both $x_2(t)$ and $y_2(t)$ are monotone increasing functions of t. This yields the estimates

$$x_1 \leqq a_1 \leqq x_2(t_2), \qquad y_1 \leqq b_1 \leqq y_2(t_2) \tag{33}$$

for any value of t in the interval $[t_1, t_2]$ .

In an analogous way we determine the points of intersection $(x_3,y_3)$ of $C_{13}$ with $\tilde{g_Q}$ and $(x_4,y_4)$ of $C_{11}$ with $\tilde{g_Q}$.

By (19) we get

$$x_3(t) = 3(t/4)^{1/3}, \quad y_3(t) = k_2 x_3(t) + \beta_1 \tag{34}$$

and from (16) and (21') we deduce the equation

$$-k_2\lambda t + k_2\lambda^{-1/2} + \beta_1 - 2 + \lambda + (2-\lambda)^{-1/2} = 0 \tag{35}$$

to determine the value of the parameter $\lambda$ corresponding to $(x_4,y_4)$ by (16). The left-hand side of (35) is a monotone decreasing function in both variables $\lambda$ and t. Hence the value of $\lambda$ defined by (35) decreases with increasing t. Since by (16) x is a decreasing function of $\lambda$, both $x_4(t)$ and $y_4(t)$ increase with t. Thus we get the estimates

$$x_3(t_1) \leqq a_2 \leqq x_4(t_2)\ , \quad y_3(t_1) \leqq b_2 \leqq y_4(t_2) \tag{36}$$

for any value of t between $t_1$ and $t_2$.

Adding the inequalities (33) and (36) we obtain the following bounds for the coordinates of the point $(\xi,\eta) := (a_1 + a_2,\ b_1 + b_2)$ :

$$\xi_1 := x_1 + x_3(t_1) \leqq \xi \leqq x_2(t_2) + x_4(t_2) =: \xi_2 \tag{37}$$

$$\eta_1 := y_1 + y_3(t_1) \leqq \eta \leqq y_2(t_2) + y_4(t_2) =: \eta_2 \tag{38}$$

In order to show that the lattice point $(\xi,\eta)$ really lies inside S(t) we first determine the interval, say $[\lambda_7(\eta_2), \lambda_7(\eta_1)]$, of all values of $\lambda$ corresponding to points (x,y) of the curve $C_7$ with $\eta_1 \leqq y \leqq \eta_2$. (For the values involved y is a decreasing function of $\lambda$.) Then (14) yields the following bounds for the corresponding values of x:

$$\underline{x}_7 \leqq x \leqq \overline{x}_7\ , \tag{39}$$

$$\underline{x}_7 = \underline{x}_7(t_1,t_2) := (1-\lambda_7(\eta_1))\,t_1 - (1-\lambda_7(\eta_1))^{-1/2} \tag{39'}$$

$$\overline{x}_7 = \overline{x}_7(t_1,t_2) := (1-\lambda_7(\eta_2))\,t_2 - (1-\lambda_7(\eta_2))^{-1/2} \tag{39''}$$

In the same way we determine the interval $[\lambda_9(\eta_2), \lambda_9(\eta_1)]$ (say) which consists of the values of $\lambda$ corresponding to points (x,y) of the curve $C_9$ with $\eta_1 \leqq y \leqq \eta_2$. Thereby we get as bounds for the corresponding values of x

$$\underline{x}_9 \leqq x \leqq \overline{x}_9 \tag{40}$$

$$\underline{x}_9 = \underline{x}_9(t_1, t_2) := -t_1(1+\lambda_9(\eta_1)) + (-1-\lambda_9(\eta_1))^{-1/2} \tag{40'}$$

$$\overline{x}_9 = \overline{x}_9(t_1, t_2) := -t_2(1+\lambda_9(\eta_2)) + (-1-\lambda_9(\eta_2))^{-1/2} \ . \tag{40''}$$

Carrying out the numerical calculations for each subinterval to be considered leads to the results given in the table below.

| $t_1$ | $t_2$ | $\alpha_1$ | $\beta_1$ | $\overline{x}_7$ | $\xi_1$ | $\xi_2$ | $\underline{x}_9$ |
|---|---|---|---|---|---|---|---|
| 0,88 | 0,89 | 1,704 | -11,18 | 2,3601 | 2,5021 | 2,6678 | 2,6983 |
| 0,89 | 0,90 | 1,699 | -11,221 | 2,3949 | 2,5275 | 2,6920 | 2,7246 |
| 0,90 | 0,91 | 1,693 | -11,271 | 2,4290 | 2,5566 | 2,7175 | 2,7445 |
| 0,91 | 0,92 | 1,688 | -11,32 | 2,4642 | 2,5819 | 2,7430 | 2,7647 |
| 0,92 | 0,93 | 1,683 | -11,37 | 2,4994 | 2,6072 | 2,7686 | 2,7840 |
| 0,93 | 0,94 | 1,678 | -11,42 | 2,5347 | 2,6324 | 2,7942 | 2,8029 |

The chain of inequalities

$$\overline{x}_7 < \xi_1 < \xi_2 < \underline{x}_9 \tag{41}$$

being satisfied in each case, the lattice point $(\xi,\eta)$ necessarily lies in the interior of $S(t)$ for any value of $t$ in the interval $0,88 \leqq t \leqq 0,94$. (The corresponding values of $\eta_2$ are between 2,93 and 2,95 hence much smaller than 4,0800.) Thus we have reached the desired contradiction and therefore proved our inequality (9).

There is no difficulty in showing that our shaded areas are in fact the only parts of the hexagons $H_1$ resp. $H_2$ which are not contained in $S(t)$. To this end one has to verify (for the values of $\alpha_1, \beta_1, t_1$ and $t_2$ occurring in our table) that the points $P_1$, $Q_1$ lie above $g_P^{\sim}$ resp. below $g_Q^{\sim}$, that the slope of $C_4$ at P is less than $k_1$ and that the slope of $C_{12}$ at Q is greater than $k_2$. All these facts can be deduced easily from the given equations of the curves involved.

Obviously it would be possible to prove all our theorems for a constant slightly smaller than 8/13, by a refinement of the mere numerical details of the method described. However such an improvement would be only of a rather small order of magnitude.

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