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SELFINJECTIVE AND SIMPLY CONNECTED ALGEBRAS

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In this paper, we present a new approach to the problem of classifying all basic finite-dimensional algebras over an algebraically closed field k which are connected, selfinjective and representation-finite. By [12], we can associate with such an algebra Λ a Dynkin-graph Δ, a subset C of vertices of $\mathbb{Z}\Delta$ (see fig.1) and a non-trivial automorphism group Π of $\mathbb{Z}\Delta$ stabilizing C, in such a way that these data uniquely determine the Auslander-Reiten quiver of Λ. Our main result is an alternate description of these sets C.

In general, there may be non-isomorphic basic algebras yielding the same data Δ, C and Π, but among them there is always exactly one standard algebra (1.3). In this article, we explicitly describe the standard selfinjective algebras by their quivers and relations. In addition, we give a sufficient (though not necessary) condition on Δ and Π, ensuring that all algebras with Δ and Π in their data are standard. We show that all algebras with $\Delta = E_6$, E_7 or E_8 are standard.

In order to state our main result, we need a few notations and definitions, which we introduce now. Throughout the paper we assume the field k to be algebraically closed. Unless stated otherwise, we consider right-modules, or, if we are dealing with representations of k-linear ca-

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tegories, contravariant k-linear functors to the category of vector-spaces. By Δ we always denote one of the Dynkin-graphs A_n, D_n, E_6, E_7 or E_8, and $\vec{\Delta}$ is a Dynkin-quiver, i.e. a quiver with underlying graph Δ. We identify $\vec{\Delta}$ with a full subquiver of $\mathbb{Z}\Delta$ which contains exactly one representative of each τ-orbit of vertices of $\mathbb{Z}\Delta$.

We associate with a subset C of vertices of $\mathbb{Z}\Delta$ the translation-quiver $\mathbb{Z}\Delta_C$ whose underlying quiver is obtained by adding a vertex c^* and arrows $c \to c^*$, $c^* \to \tau^{-1}c$ to $\mathbb{Z}\Delta$, for each $c \in C$. The translation of $\mathbb{Z}\Delta_C$ coincides with the translation of $\mathbb{Z}\Delta$ on the common vertices and is not defined on the remaining ones. In particular, each vertex c^* is projective and injective in $\mathbb{Z}\Delta_C$. We define C to be a <u>configuration of</u> $\mathbb{Z}\Delta$ if $\mathbb{Z}\Delta_C$ is a <u>representable translation-quiver</u>, i.e. a Riedtmann-quiver in the sense of [4],2.

We define a <u>$\vec{\Delta}$-section-algebra</u> to be a pair consisting of a simply connected algebra A ([4],6) and an isomorphism ω from $\vec{\Delta}$ onto a section of the Auslander-Reiten quiver Γ_A of A ([3],2.5).

Our main result is the following:

THEOREM. *For each Dynkin-quiver $\vec{\Delta}$, the configurations of $\mathbb{Z}\Delta$ correspond bijectively to the isomorphism classes of $\vec{\Delta}$-section-algebras.*

As a consequence of our main result, we find that <u>the configurations of $\mathbb{Z}\Delta$ correspond bijectively to the isomorphism classes of square-free tilting modules over $k\vec{\Delta}$</u>. In fact, using the criteria of [3], it is not hard to see that for a $\vec{\Delta}$-section-algebra (A,ω) with Auslander-Reiten quiver Γ_A the module $\oplus k(\Gamma_A)(p,\omega d)$ is a square-free tilting (left-) module over $k\vec{\Delta}$; here p ranges over the projective vertices of Γ_A and d over the vertices of

$\vec{\Delta}$. Conversely, each such tilting module gives rise to a $\vec{\Delta}$-section-algebra by [3],2.5.

After the completion of our results, we received a paper by D. Hughes and J. Waschbüsch [10] in which they state that each configuration of $\mathbb{Z}\Delta$ can be obtained from a tilting module over the quiver-algebra $k\vec{\Delta}$. A similar result has been announced by Tachikawa.

In [12] Chr. Riedtmann defined configurations as sets satisfying two combinatorial conditions; we will call these <u>combinatorial</u> configurations here. It is obvious that configurations as defined in this paper are combinatorial configurations. She obtained a classification of the combinatorial configurations by the end of 1977, under the further assumption of τ^{m_Δ}-periodicity (1.1) in case $\Delta = E_6$, E_7 or E_8 (the periodic combinatorial configurations of $\mathbb{Z}E_8$ were determined by her along with F. Jenni by computer). The resulting list of combinatorial configurations coincides with the list presented at the end of this paper.

The first question arising in her work on the classification of selfinjective representation-finite algebras was to what extent the data Δ , C and Π associated with such an algebra Λ actually determine Λ . The ordinary quiver is easily obtained from its Auslander-Reiten quiver, and in 1978 it seemed that the methods developped in [8] for A_n should yield the corresponding relations also for the other Dynkin-graphs, so that Δ , C and Π would determine Λ , up to isomorphism. This was still the perception at the Ottawa Conference in 1979, where Chr. Riedtmann presented her description of the algebras of tree class D_n by quivers and relations. At that time, problems arising in characteristic 2 went unnoticed, a mistake which she corrected at the end of 1979.

The second question was whether all combinatorial configurations C of $\mathbb{Z}\Delta$ and all admissible automorphism groups Π of $\mathbb{Z}\Delta$ stabilizing C actually arise. A first approach was to show that the residue quiver $\mathbb{Z}\Delta_C/\Pi$ actually is the Auslander-Reiten quiver of the bounden quiver given by the projective vertices of $\mathbb{Z}\Delta_C/\Pi$. As long as no algorithms for the computation of Auslander-Reiten quivers were known, it seemed simpler to try a second approach, which was to verify directly that the mesh-category associated with $\mathbb{Z}\Delta_C/\Pi$ satisfies Auslander's conditions for categories ind Λ. This was worked out in [12] for A_n, and since then also for D_n, by Chr. Riedtmann. It could be verified by computer for E_6, E_7 and E_8, since it suffices to consider $\Pi = \{1\}$.

In the meantime, the development of covering techniques has provided algorithms for the computation of Auslander-Reiten quivers ([15]). Using these techniques, it is not difficult to check by computer that the combinatorial configurations listed by Jenni and Riedtmann actually occur. It was O. Bretscher who discovered that the reduction to simply connected algebras given by covering techniques can also be used to obtain a new classification of the configurations. His work is recorded in a first and less complete version of this article, which appeared as a preprint ([5]). The lists given at the end have been established by Chr. Läser on the basis of the 1977 lists and computations of his own resting on Bretscher's results.

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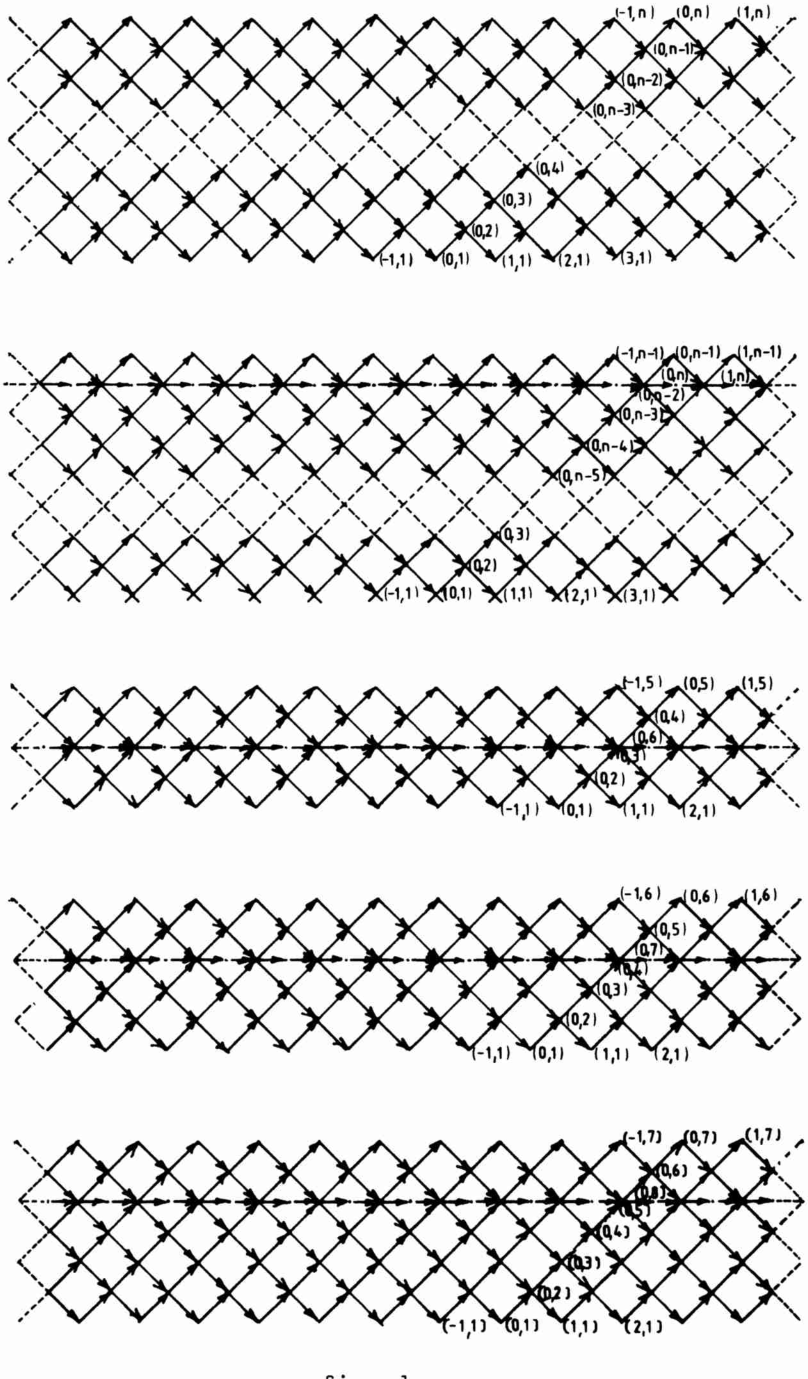
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(-1,1) (0,1) (1,1) (2,1)

fig. 1

1. Configurations and standard selfinjective algebras

1.1 We denote the Nakayama-permutation on $\mathbb{Z}\Delta$ by ν_Δ ([7],6.5; notice that in the case of E_6 the correct formula should be $\nu_{E_6} = (p+q+2, 6-q)$ if $q \leq 5$). For each vertex x of $\mathbb{Z}\Delta$, there is a path $w : \nu_\Delta^{-1}x \to x$ whose image $\bar{w}$ in the mesh-category $k(\mathbb{Z}\Delta)$ is not zero, and w is a longest path stopping at x with this property. We define m_Δ to be the smallest integer m such that $\bar{v} = 0$ in $k(\mathbb{Z}\Delta)$ for all paths v in $\mathbb{Z}\Delta$ whose length is greater than or equal to m. Thus $m_\Delta - 1$ is the common length of all paths from $\nu_\Delta^{-1}x$ to x, and a computation yields $m_{A_n} = n$, $m_{D_n} = 2n-3$, $m_{E_6} = 11$, $m_{E_7} = 17$ and $m_{E_8} = 29$.

1.2 Let C be a configuration of $\mathbb{Z}\Delta$, and denote by Λ the full subcategory of the mesh-category $k(\mathbb{Z}\Delta_C)$ whose objects are the projective vertices of $\mathbb{Z}\Delta_C$. Choosing as representatives for the indecomposable Λ-modules the restrictions $M(x) = k(\mathbb{Z}\Delta_C)(?,x)|\Lambda$ of the representable functors to Λ, we obtain an isomorphism M from $k(\mathbb{Z}\Delta_C)$ onto the category ind Λ ([4],2.4). The $M(c^*)$ for $c \in C$ are the only projective and the only injective modules in ind Λ.

PROPOSITION. Let c be a point of C. The injective envelope of the simple top of $M(c^*)$ is isomorphic to $M(d^*)$, where $d = \tau^{-m_\Delta}c \in C$; accordingly, C is stable under $\tau^{m_\Delta \mathbb{Z}}$.

Proof. Let s be the vertex of $\mathbb{Z}\Delta$ for which $M(s)$ is isomorphic to the simple module $M(c^*)/\mathrm{rad}\, M(c^*)$. As $\tau^{-m_\Delta} = \nu_\Delta^2 \tau^{-1}$ ([7],6.5), it suffices to show that $\nu_\Delta^{-1}s = \tau^{-1}c$ and $\nu_\Delta s = d$. We establish the first equality, the second one being similar.

Let x be a vertex of $\mathbb{Z}\Delta$. Since $M(c^*)$ is the projective cover of $M(s)$, $k(\mathbb{Z}\Delta)(x,s)$ is obtained from

$k(\mathbb{Z}\Delta_C)(x,s)$ by annihilating the morphisms which factor through c^*. On the other hand, all compositions $M(x) \to M(c^*) \xrightarrow{can.} M(c^*)/\mathrm{rad}\, M(c^*)$ are zero in $k(\mathbb{Z}\Delta_C)(x,s)$; accordingly, $k(\mathbb{Z}\Delta_C)(x,s)$ is identified with $k(\mathbb{Z}\Delta)(x,s)$.

Let $v : \nu_\Delta^{-1}s \to s$ be a path in $\mathbb{Z}\Delta$ with non-zero residue class $\bar{v}$ in $k(\mathbb{Z}\Delta)$. Identify $M(\tau^{-1}c)$ with $M(c^*)/\mathrm{soc}\, M(c^*)$ ([7],3.5). Since the canonical projection $M(\tau^{-1}c) \to M(s)$ factors through $M(\bar{v})$, there is a path $w : \tau^{-1}c \to \nu_\Delta^{-1}s$ with residue class $\bar{w}$ in $k(\mathbb{Z}\Delta)$ such that $M(\bar{v})\, M(\bar{w}) \neq 0$. By 1.1, w must be trivial.

Remark. Let $v : x \to y$ be a path in $\mathbb{Z}\Delta_C$ with non-zero residue class $\bar{v}$ in $k(\mathbb{Z}\Delta_C)$. Since $\bar{v}$ can be extended to a non-zero morphism $c^* \to x \xrightarrow{v} y \to d^*$ for some c in C and $d = \tau^{-m_\Delta}c$ ([4],2.8), the length of v is at most $2m_\Delta$.

1.3 Following [4],5.1, we call a representation-finite algebra Λ standard if Λ is basic and ind Λ is isomorphic to the mesh-category $k(\Gamma_\Lambda)$ associated with the Auslander-Reiten quiver Γ_Λ of Λ.

PROPOSITION. The standard representation-finite algebras which are connected and selfinjective, but not equal to k, are classified by the isomorphism classes of triples $(\mathbb{Z}\Delta, C, \Pi)$, where Δ is a Dynkin-graph, C is a configuration of $\mathbb{Z}\Delta$, and $\Pi \neq \{1\}$ is an admissible automorphism group of $\mathbb{Z}\Delta$ stabilizing C.

Admissible automorphism groups were defined in [11], 1.5. An isomorphism $f : (\mathbb{Z}\Delta, C, \Pi) \to (\mathbb{Z}\Delta', C', \Pi')$ is an isomorphism $f : \mathbb{Z}\Delta \to \mathbb{Z}\Delta'$ of translation-quivers such that $fC = C'$ and $f\Pi f^{-1} = \Pi'$.

Proof. It was shown in [12],2.5 that the Auslander-Reiten quiver Γ_Λ of a connected selfinjective representation-finite algebra Λ is isomorphic to $\mathbb{Z}\Delta_C/\Pi$, where

Δ is a Dynkin-graph, C is a set of vertices of $\mathbb{Z}\Delta$, and $\Pi \neq \{1\}$ is an admissible automorphism group of $\mathbb{Z}\Delta$ stabilizing C. By [4],2.9 $\mathbb{Z}\Delta_C/\Pi$ is a representable translation-quiver if and only if $\mathbb{Z}\Delta_C$ has this property. So the translation-quivers $\mathbb{Z}\Delta_C/\Pi$, or equivalently the isomorphism classes $(\mathbb{Z}\Delta, C, \Pi)$, where C is a configuration of $\mathbb{Z}\Delta$, classify the Auslander-Reiten quivers of connected representation-finite selfinjective algebras. But a standard representation-finite algebra is uniquely determined up to isomorphism by its Auslander-Reiten quiver.

1.4 The fundamental group Π of a connected representation-finite selfinjective algebra Λ is infinite cyclic ([11],4.2) with generator $\tau^r\phi$, where r is a positive integer and ϕ is an automorphism of $\mathbb{Z}\Delta$ which fixes at least one vertex. (It was shown in [12],3 that the only admissible subgroups of $\mathrm{Aut}\ \mathbb{Z}A_{2n}$ which occur as fundamental groups have this form.)

<u>PROPOSITION</u>. <u>Let</u> Λ <u>be a basic algebra with Auslander-Reiten quiver</u> $\mathbb{Z}\Delta_C/\Pi$. <u>If</u> $\Pi = (\tau^r\phi)^{\mathbb{Z}}$ <u>with</u> $r \geq m_\Delta$, <u>then</u> Λ <u>is standard</u>.

<u>Proof</u>. For each arrow $\alpha : x \to y$ in Γ_Λ, we choose an irreducible morphism $\underline{\alpha} \in \mathrm{Hom}_\Lambda(x,y)$. Modifying the $\underline{\alpha}$ by non-zero scalars as in [4],5.1, we can assume that $\Sigma\, \underline{\alpha}\, \underline{\sigma\alpha} \in \mathcal{R}^3(\tau x,x)$ for each x of $\mathbb{Z}\Delta/\Pi$, where $\mathcal{R}$ denotes the radical of the category $\mathrm{ind}\ \Lambda$, α ranges over all arrows stopping at x, and $\sigma\alpha$ is the arrow from τx to the source of α. We will show that $\mathcal{R}^3(\tau x,x) = 0$ for all x.

Let $F : k(\mathbb{Z}\Delta_C) \to \mathrm{ind}\ \Lambda$ be a covering functor. Then $\mathcal{R}^3(\tau x,x) \xleftarrow{\sim} \coprod_\psi k(\mathbb{Z}\Delta_C)(\tau y,\psi y)$, where $Fy = x$ and $\psi \in \Pi$ is such that the common length of all paths $\tau y \to \psi y$ is ≥ 3 ([4],3.2). Thus it suffices to prove that $k(\mathbb{Z}\Delta_C)(\tau y,\psi y) = 0$ for all y in $\mathbb{Z}\Delta$ and all $\psi \in \Pi\backslash\{1\}$. But if

$\psi \neq 1$, the length of a path $v : \tau y \to \psi y$ is at least $2r+2 > 2m_\Delta$, and hence $\bar{v} = 0$ in $k(\mathbb{Z}\Delta_C)$ (1.2).

1.5 PROPOSITION. *Any basic connected selfinjective and representation-finite algebra* Λ *of tree-class* $\Delta = E_6$, E_7 *or* E_8 *is standard.*

Proof. Let the Auslander-Reiten quiver of Λ be $\mathbb{Z}\Delta_C/\Pi$, where Π is generated by $\tau^r\phi$ for some positive integer r . Then $(\tau^r\phi)^2 = \tau^{2r}$ and τ^{m_Δ} both stabilize C . Since in our cases m_Δ is an odd prime (1.1), we conclude that m_Δ divides r , in which case we are done by 1.4, or that τ stabilizes C . But then the number of points of $C/\tau^{m_\Delta\mathbb{Z}}$ is divisible by m_Δ , whereas we will see in 2.3 that it equals the number of vertices of Δ : impossible.

1.6 In this section, we use the coordinates on $\mathbb{Z}D_n$ introduced in fig.1. We call a vertex (i,j) low if $j \leq n-2$ and high otherwise. By ψ we denote the automorphism of $\mathbb{Z}D_n$ which fixes the low vertices and exchanges (i,n) and $(i,n-1)$ for i in $\mathbb{Z}$.

PROPOSITION. *Let* Λ *be a basic algebra with Auslander-Reiten quiver* $(\mathbb{Z}D_n)_C/\Pi$. *If* C *is stable under* ψ , *then* Λ *is standard.*

Proof. The following lemma implies that Π is generated by $\tau^r\phi$ (notation as in 1.4), where r is a multiple of $m_{D_n} = 2n-3$.

LEMMA. *Let* C *be a configuration on* $\mathbb{Z}D_n$. *The representatives of the high vertices of* C *modulo* $\tau^{(2n-3)\mathbb{Z}}$ *can be chosen in one of the following two ways:* a) $(i,n-1)$ *and* (i,n) *for some* $i \in \mathbb{Z}$ *or* b) $(i_1,j_1),(i_2,j_2)$ *and* (i_3,j_3) , *where* $0 \leq i_1 < i_2 < i_3 < 2n-3$ *and* $j_1,j_2,j_3 \geq n-1$.

Proof. Remember that C is a combinatorial configuration as well (see introduction and [12],2.3). So the set

$$A_i = \{(p,q) : k(\mathbb{Z}D_n)((i,1),(p,q)) \neq 0\}$$

contains a point of C. It is not hard to see that A_i consists of the vertices (p,q) satisfying either $p = i$ or else $q \leq n-2$ and $p+q = i+n-1$. Since there are no non-zero morphisms in $k(\mathbb{Z}D_n)$ between distinct points of C, A_i either contains exactly one point of C or $(i,n-1)$ and (i,n) both lie in C. On the other hand, a low point (i,j) of C belongs to A_i as well as to $A_{i+j+1-n}$. Set $\overline{C} = C/\tau^{(2n-3)\mathbb{Z}}$ and $\overline{\mathbb{Z}} = \mathbb{Z}/(2n-3)\mathbb{Z}$. Denote by h the number of orbits $\overline{x} \in \overline{C}$ of high vertices $x \in C$, by d the number of residue classes $\overline{i}$ of integers such that $(i,n-1) \in C$ and $(i,n) \in C$. Finally, let S be the set of pairs $(\overline{i},\overline{x}) \in \overline{\mathbb{Z}} \times \overline{C}$, where $i \in \mathbb{Z}$ and $x \in A_i \cap C$. The cardinality $|S|$ of S is given by $|S| = 2n-3+d$ (consider the fibres of the first projection $\overline{\mathbb{Z}} \times \overline{C} \to \overline{\mathbb{Z}}$) and by $|S| = 2(n-h)+h$ (consider the second projection and use the equality $|\overline{C}| = n$, which we prove in 2.3 below). We infer that $d+h = 3$, or equivalently $3d+(h-2d) = 3$. Hence we have either $d = 1$ and $h = 2$, or $d = 0$ and $h = 3$.

Remarks. i) For $n \geq 5$, ψ-stable and ψ-unstable configurations of $\mathbb{Z}D_n$ cannot be isomorphic. The configurations of $\mathbb{Z}D_4$ are easy to determine. It turns out that there are only two isomorphism classes, and for each of them we can choose a ψ-stable representative (see §7, fig. 14).

ii) It was shown in [12] that all selfinjective algebras with $\Delta = A_n$ are standard. So non-standard algebras can occur only for $\Delta = D_n$, if $n = 3m$ (by part b of our lemma, $2n-3$ must be divisible by 3) and if Π is generated by τ^{2m-1} or τ^{4m-2}. In fact, non-standard algebras exist only for the group $\Pi = \tau^{(2n-1)\mathbb{Z}}$, and only if char $k = 2$ ([13]).

2. The section-algebra associated with a configuration

We fix a configuration $\mathcal{C}$ of $\mathbb{Z}\Delta$ and use the notations Λ and $M : k(\mathbb{Z}\Delta_{\mathcal{C}}) \xrightarrow{\sim} \operatorname{ind} \Lambda$ introduced in 1.2.

2.1 Write $x \leq \vec{\Delta}$ if $\mathbb{Z}\Delta_{\mathcal{C}}$ admits a path from the vertex x to some vertex d of $\vec{\Delta} \subset \mathbb{Z}\Delta$. The definitions of $\vec{\Delta} \leq x$, $\tau^{m_\Delta}\vec{\Delta} \leq x$ ... are analogous.

Let $A_{\mathcal{C}} = A$ be the full subcategory of Λ whose objects are the projective vertices p of $\mathbb{Z}\Delta_{\mathcal{C}}$ satisfying $\tau^{m_\Delta}\vec{\Delta} \leq p \leq \vec{\Delta}$. We denote by S the subset of vertices x of $\mathbb{Z}\Delta_{\mathcal{C}}$ for which $k(\mathbb{Z}\Delta_{\mathcal{C}})(p,x) = 0$ for all projective vertices p of $\mathbb{Z}\Delta_{\mathcal{C}}$ not belonging to A . Clearly, extension by 0 allows us to identify the indecomposable A-modules with the indecomposable Λ-modules whose support lies in A . As representatives of the indecomposable A-modules, we choose the $M(x)$ with $x \in S$. Notice that $\operatorname{Ext}^1_A(M(x),M(y)) \xrightarrow{\sim} \operatorname{Ext}^1_\Lambda(M(x),M(y))$ if x and y both belong to S .

For any arrow $\alpha : x \to y$ in $\mathbb{Z}\Delta_{\mathcal{C}}$ between points of S , the morphism $M(\bar{\alpha}) : M(x) \to M(y)$ is irreducible in the category $\operatorname{ind} A$ of indecomposable A-modules. Moreover, since the length of a path v of $\mathbb{Z}\Delta_{\mathcal{C}}$ yielding a non-zero morphism $\bar{v}$ in $k(\mathbb{Z}\Delta_{\mathcal{C}})$ is at most $2m_\Delta$ (1.2), $\vec{\Delta}_0$ lies in S . Therefore, M yields a morphism from $\vec{\Delta}$ into Γ_A , which we denote by ε . We denote by τ_A the Auslander-Reiten translation of Γ_A .

<u>PROPOSITION</u>. <u>The algebra</u> $\oplus k(\mathbb{Z}\Delta_{\mathcal{C}})(p,q)$, <u>where</u> p <u>and</u> q <u>range over the objects of</u> A , <u>together with</u> $\varepsilon : \vec{\Delta} \to \Gamma_A$, <u>is a</u> $\vec{\Delta}$-<u>section-algebra</u>.

<u>Proof</u>. We will show in 2.4 that $\varepsilon(\vec{\Delta})$ is a section through the Auslander-Reiten quiver Γ_A of our algebra. That Γ_A is simply connected will then follow from [3],2.5, provided that it contains no periodic τ-orbit. But since $\mathbb{Z}\Delta_C$ is simply connected, Γ_A contains no oriented cycle.

2.2 <u>LEMMA</u>. <u>Let</u> x <u>belong to</u> S . <u>Then we have</u>:

a) $\mathrm{Ext}^1_A(M(x),M(d)) = 0$ <u>for all vertices</u> d <u>of</u> $\vec{\Delta}$ <u>if and only if</u> $x \le \vec{\Delta}$.

b) $\mathrm{Ext}^1_A(M(d),M(x)) = 0$ <u>for all vertices</u> d <u>of</u> $\vec{\Delta}$ <u>if and only if</u> $x \ge \vec{\Delta}$.

<u>Proof</u>. We prove only a), the proof of b) being dual. It suffices to show that $x \le \vec{\Delta}$ if and only if $k(\mathbb{Z}\Delta)(\tau^{-1}d,x) = 0$ for all vertices d of $\vec{\Delta}$. Indeed, we have the following string of isomorphisms: $\mathrm{Ext}^1_A(M(x),M(d)) \xrightarrow{\sim} \mathrm{Ext}^1_\Lambda(M(x),M(d)) \xrightarrow{\sim} D\underline{\mathrm{Hom}}_\Lambda(\tau^{-1}M(d),M(x)) \xrightarrow{\sim} D\ k(\mathbb{Z}\Delta)(\tau^{-1}d,x)$. The second isomorphism is the Auslander-Reiten formula ([1], 2.2 and §3), and the third one expresses that M induces an isomorphism from $k(\mathbb{Z}\Delta)$ to the stable category $\underline{\mathrm{ind}}\ \Lambda$.

So let us assume that $x \nleq \vec{\Delta}$, or equivalently that $\tau^{-1}\vec{\Delta} \le x$. There is a path v in $\mathbb{Z}\Delta_C$ from a projective vertex p to x with non-zero residue class $\bar{v}$ in $k(\mathbb{Z}\Delta_C)$; by the definition of S , p satisfies $\tau^{m_\Delta}\vec{\Delta} \le p \le \vec{\Delta}$. So v crosses $\tau^{-1}\vec{\Delta}$, i.e. there is a factor $w : \tau^{-1}d \to x$ of v with $\bar{w} \neq 0$ for some d in $\vec{\Delta}_0$. The residue class of w in $k(\mathbb{Z}\Delta)$ cannot be zero, since otherwise w would pass through a projective vertex q not belonging to A , which would yield $k(\mathbb{Z}\Delta_C)(q,x) \neq 0$, a contradiction. Therefore, $k(\mathbb{Z}\Delta)(\tau^{-1}d,x) \neq 0$. The other implication is trivial.

2.3 <u>LEMMA</u>. <u>For all</u> d <u>in</u> $\vec{\Delta}_0$, <u>the projective dimension of</u> $M(d)$ <u>as an</u> A-<u>module is at most</u> 1 .

Proof. We will show that $\mathrm{Hom}_A(M(x),\tau_A M(d)) = 0$ for all vertices x in S yielding an injective A-module $M(x)$ and for all d in $\vec{\Delta}_0$, which is equivalent to our statement by [3],2.2. Clearly, we can assume that $M(d)$ is not projective, so that $\tau_A M(d)$ is isomorphic to $M(e)$ for some e in S. As $\mathrm{Ext}^1_A(M(d),M(e)) \neq 0$, we have $e \leq \tau\vec{\Delta}$ by 2.2, and thus it suffices to prove $\vec{\Delta} \leq x$.

Let s be the vertex in S for which $M(s)$ is isomorphic to the simple socle of $M(x)$, and let $M(c^*)$ be the projective cover of $M(s)$ as a Λ-module. For any y in S, we have $\dim_k k(\mathbb{Z}\Delta_C)(c^*,y) = \dim_k k(\mathbb{Z}\Delta_C)(y,x)$, both numbers being equal to the multiplicity of $M(s)$ as a composition factor of $M(y)$. Since $c^* \leq \vec{\Delta} \leq \tau^{-m_\Delta} c$, there is a $y \in \vec{\Delta}_0$ for which $k(\mathbb{Z}\Delta_C)(c^*,y) \neq 0$, and we conclude $\vec{\Delta} \leq x$.

COROLLARY. *The number of projective vertices p of $\mathbb{Z}\Delta_C$ such that $\tau^{m_\Delta}\vec{\Delta} \leq p \leq \vec{\Delta}$ equals the cardinality of $\vec{\Delta}_0$.*

Proof. We set $T = \oplus M(d)$, where d ranges over $\vec{\Delta}_0$. If $\mathrm{Ext}^1_A(M(x),T) = 0 = \mathrm{Ext}^1_A(T,M(x))$ for some x in S, we know by 2.2 that x lies in $\vec{\Delta}_0$. Since, in addition, $\mathrm{Ext}^1_A(T,T) = 0$ and $\mathrm{pdim}\, T \leq 1$, T is a tilting module over A ([3],2.1), and hence the number of its indecomposable summands equals the number of non-isomorphic simple A-modules, i.e. the number of projective vertices p of $\mathbb{Z}\Delta_C$ with $\tau^{m_\Delta}\vec{\Delta} \leq p \leq \vec{\Delta}$.

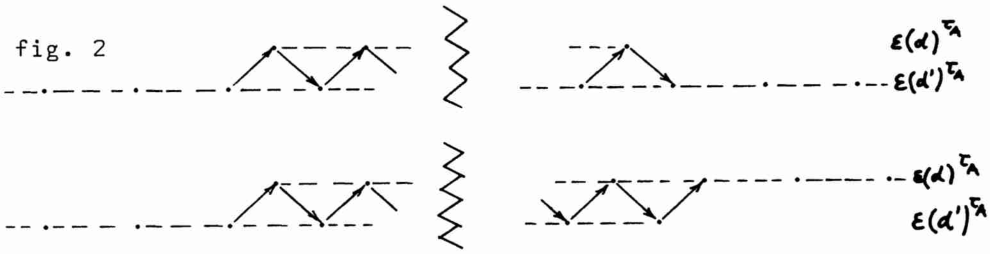
2.4 PROPOSITION. *The quiver $\varepsilon(\vec{\Delta})$ is a section through Γ_A.*

Proof. By definition ([3],2.5), we have to show that the vertices of $\varepsilon(\vec{\Delta})$ form a set of representatives for the τ_A-orbits of vertices of Γ_A and the arrows a set of representatives for the σ_A-orbits of arrows of Γ_A.

Assume that $M(d)$ is isomorphic to $\tau^r_A M(d')$ in mod A for some d and d' in $\vec{\Delta}_0$ and a natural number $r \geq 1$. Let x be the vertex in S for which $M(x)$ is isomorphic to $\tau_A M(d')$. Then on the one hand, there is a path in Γ_A from $M(d)$ to $M(x)$; hence there is a path from d to x in $\mathbb{Z}\Delta_C$ and $\vec{\Delta} \leq x$. On the other hand, $\mathrm{Ext}^1_A(M(d'),M(x)) \neq 0$ implies $\vec{\Delta} \nleq x$ (2.2). Hence the $\varepsilon(d)$ with $d \in \vec{\Delta}_0$ belong to different τ_A-orbits in Γ_A. As Γ_A contains no oriented cycle, the number of τ_A-orbits equals the number of projective A-modules, which coincides with the cardinality of $\vec{\Delta}_0$ (2.3). As a consequence, the vertices $\varepsilon(d)$, $d \in \vec{\Delta}_0$, form a complete set of representatives of the τ_A-orbits.

Since we know that $M(\bar{\alpha})$ is irreducible in mod A for any arrow α in $\vec{\Delta}$, it remains to show that each σ_A-orbit β^{σ_A} in Γ_A contains an arrow from $\varepsilon(\vec{\Delta})$. Let β^{σ_A} connect the τ_A-orbits of $\varepsilon(d)$ and $\varepsilon(d')$. Since Γ_A contains no oriented cycle, it is easily seen that the sub-translation-quiver of Γ_A formed by $\varepsilon(d)^{\tau_A}$, $\varepsilon(d')^{\tau_A}$ and β^{σ_A} has one of the two forms illustrated in fig.2. The figure shows that, given any two vertices $x \in \varepsilon(d)^{\tau_A}$ and $x' \in \varepsilon(d')^{\tau_A}$, β^{σ_A} contains either an arrow $\tau_A^{-r}x \to \tau_A^{s}x'$ or an arrow $\tau_A^{-r}x' \to \tau_A^{s}x$, where $r,s \in \mathbb{N}$. Set $x = \varepsilon(d)$, $x' = \varepsilon(d')$ and assume for instance that β^{σ_A} contains an arrow $\tau_A^{-r}\varepsilon(d) \to \tau_A^{s}\varepsilon(d')$, where $r,s \in \mathbb{N}$. If r was > 0, the vertex $c \in S$ such that $M(c) = \tau_A^{-1}\varepsilon(d)$ would satisfy the relation $c \nleq \vec{\Delta}$ by lemma 2.2a. So there could be no chain of irreducible morphisms from $M(c)$ to $\varepsilon(d')$ in mod A, a contradiction. We infer that $r = 0$, and similary that $s = 0$. As a consequence, we have an irreducible morphism $f : \varepsilon(d) \to \varepsilon(d')$ in mod A. If f admitted the decomposition $f = \sum_i M(h_i)\, M(g_i)$ in mod Λ, where $g_i \in k(\mathbb{Z}\Delta_C)(d,d_i)$, each vertex d_i would obviously belong to $\vec{\Delta}$. So f would be reducible in mod A too. We conclude that f is irreducible in $\operatorname{mod} \Lambda$ and that $\vec{\Delta}$ contains an arrow $d \to d'$.

fig. 2



3. The configuration associated with a section-algebra

By $\kappa : \mathbb{Z}\Delta \to \mathbb{Z}A_2$ we denote the unique morphism of translation-quivers such that the minimal value of κ on $\vec{\Delta}_0$ is zero ([4],1.6).

3.1 Let P be a set of representatives of the τ-orbits of $(\mathbb{Z}\Delta)_0$, and let $\mathbb{N}P$ be the full subquiver of $\mathbb{Z}\Delta$ whose vertices are the $\tau^{-r}p$ for p in P and r in $\mathbb{N}$. For x in $(\mathbb{N}P)_0$, we denote by x^- the set of tails of arrows of $\mathbb{N}P$ with head x . Notice that $\kappa(y) = \kappa(x)-1$ for $y \in x^-$. By induction on $\kappa(x)$, the following formulas define an integral-valued function $\delta_P = \delta$ on $(\mathbb{N}P)_0$:

$$\delta(x) = \begin{cases} 1 + \Sigma\delta(y) & \text{if } x \in P \text{ and } \delta(y) > 0 \text{ for all } y \in x^- \\ -\delta(\tau x) + \Sigma\delta(y) & \text{if } x \notin P \text{ and } 0 < \delta(\tau x) < \Sigma\delta(y) \\ 0 & \text{otherwise} \end{cases}$$

where y ranges over x^- in all summations. Denote by R_P the full sub-translation-quiver of $\mathbb{N}P$ whose vertices form the support of δ_P . Call P a $\vec{\Delta}$-pattern if R_P contains the subquiver $\vec{\Delta}$ of $\mathbb{Z}\Delta$.

We use notations and results of [4],§6 in order to establish a bijection between $\vec{\Delta}$-patterns and isomorphism classes of $\vec{\Delta}$-section-algebras. Clearly, a $\vec{\Delta}$-pattern $\mathcal{P}$ defines a unique grading $g_{\mathcal{P}}$ on Δ ($g_{\mathcal{P}}(d) = \kappa(p)-\kappa_o$, where $p \in \mathcal{P}$ lies in the τ-orbit of $d \in \Delta_o$ and κ_o is the minimal value of κ on $\mathcal{P}$), and our function δ is obtained by adding up the components of the dimension-map associated with $(\Delta, g_{\mathcal{P}})$. Since $\mathcal{P}$ is a $\vec{\Delta}$-pattern and Δ a Dynkin-graph, $g_{\mathcal{P}}$ is admissible and representation-finite. Thus $R_{\mathcal{P}}$ is the Auslander-Reiten quiver of the simply connected algebra $A_{\mathcal{P}} = \oplus k(R_{\mathcal{P}})(p,q)$ with $p,q \in \mathcal{P}$, and $A_{\mathcal{P}}$ together with the embedding of $\vec{\Delta}$ into $R_{\mathcal{P}}$ is a $\vec{\Delta}$-section-algebra. Conversely, for every $\vec{\Delta}$-section-algebra (A,ω) we can map the Auslander-Reiten quiver Γ_A into $\mathbb{Z}\Delta$ in such a way that $\omega(\vec{\Delta})$ is identified with $\vec{\Delta} \subset \mathbb{Z}\Delta$. The projective vertices of $\Gamma_A \subset \mathbb{Z}\Delta$ then form a $\vec{\Delta}$-pattern.

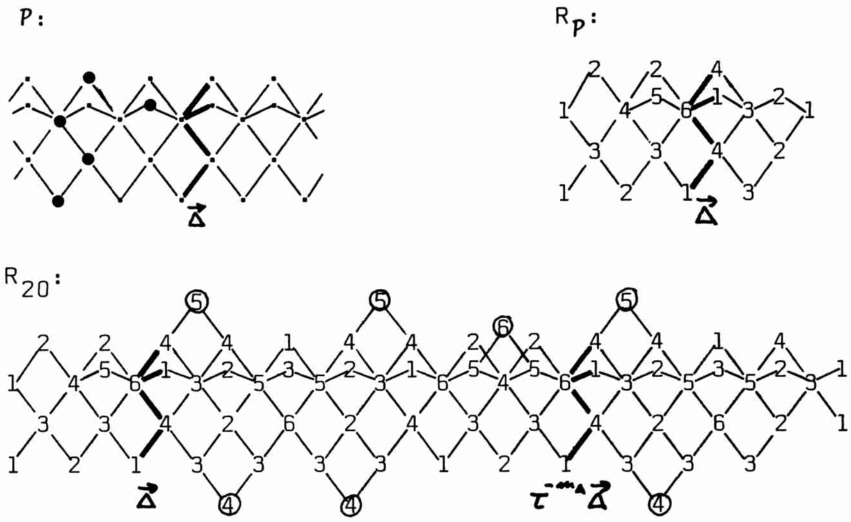
3.2 Let $\mathcal{P}$ be a $\vec{\Delta}$-pattern and $t \in \mathbb{N} \cup \{\infty\}$. By induction on $\kappa(x)$, the following formulas, in which y ranges over x^- , define an integral-valued function d_t on $(\mathbb{N}\mathcal{P})_o$:

$$d_t(x) = \begin{cases} 1 + \sum_y d_t(y) & \text{if } x \in \mathcal{P} \\ -d_t(\tau x) + \sum_y d_t(y) & \text{if } x \notin \mathcal{P} \text{ and } 0 < d_t(\tau x) < \sum_y d_t(y) \\ d_t(\tau x) \quad \text{if } x \notin \mathcal{P}, \ d_t(\tau x) > \sum_y d_t(y) \text{ and } \kappa(x) \leq t+1 \\ 0 \quad \text{otherwise} \end{cases}$$

We denote by $\mathcal{D}_t$ the set of vertices c of $\mathbb{N}\mathcal{P}$ for which $\kappa(c) \leq t-1$ and $d_t(c) > \sum d_t(y)$, where y ranges over the set c^+ of heads of arrows in $\mathbb{N}\mathcal{P}$ with tail c . The sets $\mathcal{D}_o \subset \mathcal{D}_1 \subset \mathcal{D}_2 \ldots$ form an increasing sequence with union $\mathcal{D}_\infty$. Our aim in this chapter is to prove the following

<u>PROPOSITION</u>. <u>$\mathcal{D}_\infty$ is stable under</u> τ^{-m_Δ} , <u>and</u> $C_{\mathcal{P}} = \tau^{m_\Delta \mathbb{Z}} \mathcal{D}_\infty$ <u>is a configuration of</u> $\mathbb{Z}\Delta$.

fig. 3



3.3 Let S_t be the full sub-translation-quiver of $\mathbb{N}\mathcal{P}$ whose vertices x satisfy $d_t(x) > 0$. With S_t we associate the translation-quiver R_t which is obtained by adding a vertex c^* and arrows $c \to c^*$, $c^* \to \tau^{-1}c$ to S_t for each $c \in \mathcal{D}_t$; the translation on R_t coincides with the translation of S_t on the common vertices and is not defined on c^* . We extend d_t to R_t by setting $d_t(c^*) = d_t(c)+1$. Similarly, we set $\kappa(c^*) = \kappa(c)+1$. Clearly, $\mathbb{N}\mathcal{P}$ and R_t are full subtranslation-quivers of R_∞ , and each vertex x of R_∞ such that $\kappa(x) \leq t$ belongs to R_t . As $\vec{\Delta}$ is a Dynkin-quiver, S_t is finite if $t \in \mathbb{N}$ (Consider the full subquiver of $\vec{\Delta}$ formed by the x such that $d_t(\tau^{-N}x) > 0$ for all $N \in \mathbb{N}$; let $\vec{D}$ be a connected component of this subquiver; by the definition of d_t , the restriction $d_t|\tau^{-N-1}\vec{D}$ is the Coxeter transform of $d_t|\tau^{-N}\vec{D}$ for large N ; on the other hand, a positive vector does not stay positive when acted upon by the powers of the Coxeter transformation).

In order to prove that the translation-quiver R_t is representable for all $t \in \mathbb{N}$, we first reformulate a result of [4],§6. Let Γ be a simply connected finite translation-quiver, $\kappa : \Gamma \to \mathbb{Z}A_2$ a quiver-morphism and $d_\Gamma : \Gamma_0 \to \mathbb{Z}$ the function which we define by induction on $\kappa(x)$ using the formulas

$$d_\Gamma(x) = \begin{cases} \sum_{y \in x^-} d_\Gamma(y) + 1 & \text{if } x \text{ is projective} \\ -d_\Gamma(\tau x) + \sum_{y \in x^-} d_\Gamma(y) & \text{otherwise.} \end{cases}$$

LEMMA. For a simply connected finite translation-quiver Γ the following statements are equivalent:

(i) Γ is the Auslander-Reiten quiver of an algebra.

(ii) $d_\Gamma(x) > 0$ for all $x \in \Gamma_0$ and $d_\Gamma(j) = 1 + \sum_{y \in j^+} d_\Gamma(y)$ for all injective $j \in \Gamma_0$.

(iii) $d_\Gamma(x) > 0$ for all $x \in \Gamma_0$ and $d_\Gamma(j) \geq \sum_{y \in j^+} d_\Gamma(y)$ for all injective $j \in \Gamma_0$.

Proof. (i) $\Rightarrow$ (ii) : If Γ is representable, $d_\Gamma(x)$ is the dimension of the module attached to $x \in \Gamma_0$.

(ii) $\Rightarrow$ (iii) : clear.

(iii) $\Rightarrow$ (i) : Let $T = G_\Gamma$ be the graph associated with Γ ([4],4.2). As Γ is simply connected, T is a tree (a simply connected locally finite translation-quiver has no periodic component; accordingly, the vertices of T are the τ-orbits of Γ, the edges are the σ-orbits). We assume that the minimum of κ on Γ_0 is 0, which is permissible, and we endow T with a grading g such that $g(p^\tau) = \kappa(p)$ for each projective $p \in \Gamma_0$. With the notations of [4],6.2, the map $\Gamma_0 \to (Q_T)_0$, $x \mapsto (\kappa(x), x^\tau)$ then extends to a full embedding of Γ into Q_T. We identify Γ with its image and extend d_Γ to a function $\bar{d} : (Q_T)_0 \to \mathbb{N}$ by setting $\bar{d}(y) = 0$ if $y \in (Q_T)_0 \setminus \Gamma_0$. Then it is clear by induction on $\kappa(x)$ that $\bar{d}(x)$ is the sum of the components of the dimension-vector $d(x)$ of [4],6.4 for each $x \in (Q_T)_0$. As a consequence, Γ is identified with R_T and we can apply [4], prop.6.4.

3.4 PROPOSITION. *For each $t \in \mathbb{N}$, R_t is the Auslander-Reiten quiver of a simply connected algebra. For each vertex x of R_t, we have*

$$d_t(x) = \sum_p \dim_k k(R_t)(p,x)$$

where the sum is taken over all projective vertices p of R_t.

Proof. By induction on t. For $t = 0$, R_t coincides with $R_{\mathcal{P}}$ and we can refer to 3.1. So we may assume that our statement is true for all integers which are strictly smaller than some $t > 0$. It suffices to prove

$$(*) \quad d_t(\tau^{-1}c) = d_{R_t}(\tau^{-1}c)$$

for all $c \in \mathcal{D}_t$ (3.3); indeed, (*) implies $d_t(x) = d_{R_t}(x)$ by induction on $\kappa(x)$; as d_t satisfies the conditions of lemma 3.3iii, R_t is representable and d_t is the dimension function (see the proof of the lemma).

If $\kappa(c) < t-1$, the equality (*) follows from the induction hypothesis: $d_t(\tau^{-1}c) = d_{t-1}(\tau^{-1}c) = d_{R_{t-1}}(\tau^{-1}c) = d_{R_t}(\tau^{-1}c)$. Assume that $\kappa(c) = t-1$ and notice that (*) is equivalent to $d_t(c) = 1 + \sum_y d_t(y)$, where y ranges over all successors of c in $\mathbb{N}\mathcal{P}$ (or in R_{t-1}). As c lies in $\mathcal{D}_t$, it must be injective in R_{t-1}. Using lemma 3.3ii, we infer that $d_t(c) = d_{R_{t-1}}(c) = 1 + \sum_y d_{R_{t-1}}(y) = 1 + \sum_y d_t(y)$.

3.5 For each $t \in \mathbb{N}$, let Λ_t be the full subcategory of the mesh-category $k(R_t)$ whose objects are the projective vertices of R_t, and let M_t be the isomorphism from $k(R_t)$ onto the category ind Λ_t of indecomposable Λ_t-modules, given by $M_t(x) = k(R_t)(?,x)|\Lambda_t$.

LEMMA. Let c be a point of $\mathcal{D}_t$, where $t \geq \kappa(c)+2m_\Delta+1$, and let $M_t(j)$ be isomorphic to the injective envelope of $M_t(c^*)/\mathrm{rad}\, M_t(c^*)$. Then $j = d^*$, where $d = \tau^{-m_\Delta}c$.

Proof. We adapt the proof of proposition 1.2: Now R_t plays the role of $\mathbb{Z}\Delta_C$, and the full subquiver R_t' obtained by deleting the projective injective vertices of R_t replaces $\mathbb{Z}\Delta$. Notice that for any two vertices x and y of R_t', $k(R_t')(x,y)$ is identified with $k(\mathbb{Z}\Delta)(x,y)$, if $\vec{\Delta} \leq x,y$ and $\kappa(x),\kappa(y) \leq t+1$.

Let $M_t(s)$ be isomorphic to $M_t(c^*)/\mathrm{rad}\, M_t(c^*)$. We have $\tau^{-1}\vec{\Delta} \leq \nu_\Delta^{-1}s$: Otherwise we could find a non-zero morphism in $k(\mathbb{Z}\Delta)$, and hence in $k(R_t)$, from some x on $\vec{\Delta}$ to s, and thus $c^* \leq \vec{\Delta}$, a contradiction. We refer to 1.2 for the proof of $\tau^{-1}c = \nu_\Delta^{-1}s$.

Since $\kappa(\nu_\Delta s) = \kappa(s)+m_\Delta-1 = \kappa(c)+2m_\Delta \leq t-1$, there is a path $v : s \to \nu_\Delta s$ in R_t' with non-zero residue class $\bar{\bar{v}}$ in $k(R_t')$. We can find a morphism $\phi : M_t(\nu_\Delta s) \to M_t(j)$ such that $\phi M_t(\bar{v}) \neq 0$, where $\bar{v}$ is the image of v in $k(R_t)$, and we can even assume that $\phi = M_t(\bar{u})$ for some path $u : \nu_\Delta s \to j$ in R_t. Suppose $\nu_\Delta s \notin \mathcal{D}_t$ or else $j \neq (\nu_\Delta s)^*$, The injection $M_t(\overline{uv})$ does not factor through an injective other than $M_t(j)$, so that the head x of the first arrow $\alpha : \nu_\Delta s \to x$ in u lies in $(\mathbb{N}\mathcal{P})_0$. Since $\kappa(x) \leq t$, x belongs to R_t' and is not injective. Since $\overline{\alpha v} \in k(R_t)(s,x)$ does not factor through an injective, the image $\bar{\bar{\alpha v}}$ of $\overline{\alpha v}$ in $k(R_t')(s,x) = k(\mathbb{Z}\Delta)(s,x)$ is not zero, which is impossible (1.1).

3.6 LEMMA. If $k(R_t)(x,y) \neq 0$, then $\kappa(y) \leq \kappa(x)+2m_\Delta$.

Proof. Let $\mu : x \to y$ be a non-zero morphism in $k(R_t)$. Then there is a projective vertex p and a morphism $\rho : p \to x$ such that $\mu\rho \neq 0$. Let $M_t(j)$ be the

injective envelope of the simple top $M_t(s)$ of $M_t(p)$. Since $\mu\rho$ is not zero, there is a morphism $\tau : y \to j$ such that $\tau\mu\rho \neq 0$. Therefore, it suffices to prove that $\kappa(j) \leq \kappa(p)+2m_\Delta$. In fact we shall prove that
$\kappa(s) \leq \kappa(p)+m_\Delta$ and $\kappa(j) \leq \kappa(s)+m_\Delta$.

Let us prove the first inequality, the second being proved similarly. First we notice that, for any two vertices u,v of R'_t , the space $k(R'_t)(u,v)$ is identified with the quotient of $k(\mathbb{Z}\Delta)(u,v)$ by the morphisms which factor through a vertex of $\mathbb{Z}\Delta$ lying outside R_t ([4],2.5). Therefore, if p belongs to $\mathcal{P} \subset (R'_t)_o$, s is a vertex of $\mathbb{Z}\Delta$ such that $k(\mathbb{Z}\Delta)(p,s) \neq 0$; as a consequence we obtain $\kappa(s) \leq \kappa(p)+m_\Delta-1$. On the contrary, if $p = c^*$, the relation $k(\mathbb{Z}\Delta)(\tau^{-1}c,s) \neq 0$ yields $\kappa(s) \leq \kappa(\tau^{-1}c)+m_\Delta-1$ $= \kappa(c^*)+m_\Delta$.

3.7 Now we are ready to prove proposition 3.2. Lemma 3.5 implies that $\mathcal{D}_\infty$ is stable under τ^{-m_Δ} . It remains to be shown that $\mathbb{Z}\Delta_C$ is a representable translation-quiver for $C = \tau^{m_\Delta\mathbb{Z}}\mathcal{D}_\infty$. For any t , R_t is a full subtranslation-quiver of $\mathbb{Z}\Delta_C$, which contains all vertices x of $\mathbb{Z}\Delta_C$ satisfying $\vec{\Delta} \leq x$ and $\kappa(x) \leq t$. If a vertex x of R_t is projective with $x \nleq \vec{\Delta}$ or injective with $\kappa(x) \leq t-1$, then x is projective and injective in $\mathbb{Z}\Delta_C$.

Our proof consists in checking that $\mathbb{Z}\Delta_C$ satisfies the conditions stated in [4],2.8. Using the automorphism τ^{-Nm_Δ} of $\mathbb{Z}\Delta_C$, $N \in \mathbb{N}$, we are reduced to a verification within R_t for a convenient choice of $t \in \mathbb{N}$. As an example, we show that each y in $\mathbb{Z}\Delta_C$ satisfies the inequality $\sum_x \dim_k k(\mathbb{Z}\Delta_C)(x,y) < \infty$, where x ranges over all vertices of $\mathbb{Z}\Delta_C$. For a given x , we have $k(\mathbb{Z}\Delta_C)(x,y)$ $=k(\mathbb{Z}\Delta_C)(\tau^{-Nm_\Delta}x,\tau^{-Nm_\Delta}y) = k(R_t)(\tau^{-Nm_\Delta}x,\tau^{-Nm_\Delta}y)$ if $N,t \in \mathbb{N}$ are chosen in such a way that $t \geq \kappa(x)+2Nm_\Delta \geq m_\Delta$

and $t \geq \kappa(y)+2Nm_\Delta \geq m_\Delta$. We infer that $\kappa(y)-\kappa(x) = \kappa(\tau^{-Nm_\Delta}y)-\kappa(\tau^{-Nm_\Delta}x) \leq 2m_\Delta$ whenever $k(\mathbb{Z}\Delta_C)(x,y) \neq 0$ (3.6).

4. Proof of the main result

4.1 LEMMA. a) *A configuration* C *of* $\mathbb{Z}\Delta$ *is uniquely determined by the map* D_C *from* $\vec{\Delta}_0 \subset (\mathbb{Z}\Delta)_0$ *to* $\mathbb{N}$ *given by*

$$D_C(d) = \Sigma\dim_k k(\mathbb{Z}\Delta_C)(p,d) \quad ,$$

where p *runs over the projective vertices of* $\mathbb{Z}\Delta_C$.

b) *A* $\vec{\Delta}$*-pattern* P *is uniquely determined by the map* D_P *from* $\vec{\Delta}_0 \subset (\mathbb{N}P)_0$ *to* $\mathbb{N}$ *given by*

$$D_P(d) = \Sigma\dim_k k(R_P)(p,d) \quad ,$$

where p *runs over the points of* P .

Proof. a) Let d_C be the function assigning to an $x \in (\mathbb{Z}\Delta)_0$ the value

$$d_C(x) = \Sigma\dim_k k(\mathbb{Z}\Delta_C)(p,x) = \dim_k M(x) \quad ,$$

where p ranges over the projective vertices of $\mathbb{Z}\Delta_C$, and where M is defined as in 1.2. Clearly, C is uniquely determined by d_C , since a vertex c of $\mathbb{Z}\Delta$ lies in C if and only if $d_C(c) > \Sigma d_C(y)$, where y belongs to c^+ , computed in $\mathbb{Z}\Delta$. But d_C in turn is given by D_C , its restriction to $\vec{\Delta}$: In order to compute $d_C(x)$ for some vertex $x \geq \tau^{-1}\vec{\Delta}$ we use induction on $\kappa(x)$ and the formulas

$$d_C(x) = \begin{cases} -d_C(\tau x) + \Sigma d_C(y) & \text{if } d_C(\tau x) < \Sigma d_C(y) \\ d_C(\tau x) & \text{otherwise} \end{cases}$$

where y lies in $x^- \subset (\mathbb{Z}\Delta)_0$. The construction in case $x \leq \tau\vec{\Delta}$ is dual.

b) Let δ_P be the function assigning to an $x \in (\mathbb{Z}\Delta)_0$ the value

$$\delta_P(x) = \begin{cases} \Sigma \dim_k k(R_P)(p,x) & \text{if } x \in (R_P)_0 \\ 0 & \text{otherwise} \end{cases} ,$$

where p ranges over P. Then P is uniquely determined by δ_P. On the other hand, we can compute $\delta_P(x)$ for any $x \in (\mathbb{Z}\Delta)_0$ if we know D_P, the restriction of δ_P to $\vec{\Delta}_0$: If $x \le \tau\vec{\Delta}$ we use induction on $-\kappa(x)$ and the formulas

$$\delta_P(x) = \begin{cases} -\delta_P(\tau^{-1}x) + \Sigma\delta_P(y) & \text{if } 0 < \delta_P(\tau^{-1}x) < \Sigma\delta_P(y) \\ 0 & \text{otherwise} \end{cases} ,$$

where y lies in $x^+ \subset (\mathbb{Z}\Delta)_0$. The construction in case $x \ge \tau^{-1}\vec{\Delta}$ is dual. The following proposition provides an alternate proof.

4.2 Let D be a map from $\vec{\Delta}_0$ to $\mathbb{N}$ and set

$$M^D = \prod_{\phi} \mathrm{Hom}_k(k^{D(t\phi)}, k^{D(h\phi)}) ,$$

where ϕ runs through the arrows of $\vec{\Delta}$, has tail $t\phi$ and head $h\phi$. The group $G^D = \prod_{i \in \vec{\Delta}_0} GL(D(i),k)$ operates on M^D in such a way that the G^D-orbits correspond to the isomorphism classes of left $k\vec{\Delta}$-modules of dimension-type D. Since the quiver-algebra $k\vec{\Delta}$ is representation-finite, there is a module $X^D \in M^D$ whose G^D-orbit is open in M^D.

<u>PROPOSITION</u>. <u>Given a map $D : \vec{\Delta}_0 \to \mathbb{N}$, there is a $\vec{\Delta}$-pattern P such that $D = D_P$ if and only if X^D is a direct sum of n non-isomorphic indecomposables, where n is the cardinality of $\vec{\Delta}_0$. Then P is uniquely determined.</u>

<u>Proof</u>. Using the bijection between $\vec{\Delta}$-patterns and isomorphism classes of square-free tilting modules over $k\vec{\Delta}$, we see that $D = D_P$ for some $\vec{\Delta}$-pattern P if and only if there is such a tilting module T of dimension-type D. Since the module $X^D \in M^D$ is characterized by the property

$\mathrm{Ext}^1_{k\vec{\Delta}}(X^D,X^D) = 0$ (D. Voigt; see [6],[14]), $T = X^D$ is the only candidate, and X^D actually is a square-free tilting module if and only if it is the direct sum of n non-isomorphic indecomposables ([9],4.5;[3],2.1). Hence there is at most one square-free tilting module of dimension-type D, up to isomorphism, which shows that $\mathcal{P}$ is uniquely determined.

4.3 In view of 4.2, it suffices to prove the two following claims, in order to establish our main result:

a) *For each configuration* $\mathcal{C}$ *of* $\mathbb{Z}\Delta$, *we have* $D_{\mathcal{C}} = D_{\mathcal{P}}$, *where* $\mathcal{P} = \mathcal{P}_{\mathcal{C}}$ *is the* $\vec{\Delta}$*-pattern associated with the* $\vec{\Delta}$*-section-algebra* $A = A_{\mathcal{C}}$ (2.1 *and* 3.1).

b) *For each* $\vec{\Delta}$*-pattern* $\mathcal{P}$, *we have* $D_{\mathcal{P}} = D_{\mathcal{C}}$, *where* $\mathcal{C} = \mathcal{C}_{\mathcal{P}}$ *is the configuration of* $\mathbb{Z}\Delta$ *assigned to* $\mathcal{P}$ (3.2).

Proof. We use the notations Λ, M and R_t, M_t introduced in 1.2,3.3 and 3.5.

a) For each $d \in \vec{\Delta}_0$, $D_{\mathcal{P}}(d)$ is the dimension of the A-module $\varepsilon(d) = M(d)$, which is by construction the direct sum $\bigoplus_p k(\mathbb{Z}\Delta_{\mathcal{C}})(p,d)$, where p ranges over the projective vertices of $\mathbb{Z}\Delta_{\mathcal{C}}$ satisfying $\tau^{m_\Delta}\vec{\Delta} \le p \le \vec{\Delta}$. Since the other projective vertices admit only trivial morphisms into d, the dimension of $M(d)$ coincides with $D_{\mathcal{C}}(d)$.

b) Choose $t \ge 3m_\Delta$. Let $p \in \mathcal{P}$, and let $M_t(q)$ be the injective envelope of the top of $M_t(p)$. By 3.6, we have $\vec{\Delta} \le q \le \tau^{-m_\Delta}\vec{\Delta}$. Since $\kappa(q) \le t-1$, q is projective and injective in R_t as well as in $\mathbb{Z}\Delta_{\mathcal{C}}$. On the other hand, all projective injective vertices q in R_t with $\vec{\Delta} \le q \le \tau^{-m_\Delta}\vec{\Delta}$ arise in this way from points of $\mathcal{P}$. Using [4],2.8, first for R_t and then for $\mathbb{Z}\Delta_{\mathcal{C}}$, we obtain $D_{\mathcal{P}}(d) = \Sigma\dim_k k(R_{\mathcal{P}})(p,d) = \Sigma\dim_k k(R_t)(p,d)$ $= \Sigma\dim_k k(R_t)(d,q) = \Sigma\dim_k k(\mathbb{Z}\Delta_{\mathcal{C}})(d,q) = \Sigma\dim_k k(\mathbb{Z}\Delta_{\mathcal{C}})(p',d)$ $= D_{\mathcal{C}}(d)$, where p ranges over $\mathcal{P}$, q over the injective vertices of R_t with $\vec{\Delta} \le q \le \tau^{-m_\Delta}\vec{\Delta}$, and p' over the projective vertices of $\mathbb{Z}\Delta_C$ with $\tau^{m_\Delta}\vec{\Delta} \le p' \le \vec{\Delta}$, or equivalently over all projective vertices.

5. Description of the standard representation-finite selfinjective algebras by quivers and relations

5.1 PROPOSITION. *Let Λ be a locally representation-finite category with quiver Q , and assume the Auslander-Reiten quiver Γ of Λ is simply connected. Then Λ is isomorphic to kQ/I for some ideal I in the quiver-category kQ such that, for any two objects x and y of Λ , either $I(x,y)$ is spanned by the differences of paths from x to y or else $I(x,y) = kQ(x,y)$.*

Proof. First we show that the results on "Zykellose Algebren" by Bongartz ([2]) hold in our situation, i.e. that $\dim_k \Lambda(x,y) \le 1$ and that $\Lambda \xrightarrow{\sim} kQ/I$, where $I(x,y)$ is spanned by paths and differences of paths for any x and y . The first assertion is clear, since Q contains no oriented cycle and since Λ is locally representation-finite.

We find a suitable ideal I by the following procedure. Choose some $x_o \in \Gamma_o$, and let $\kappa : \Gamma \to \mathbb{Z}A_2$ be the morphism of translation-quivers with $\kappa(x_o) = 0$ ([4],1.6). Viewing Λ as a full subcategory of $k(\Gamma)$, we assign the integer $\kappa(y)$ to each object y of Λ . For $m \in \mathbb{N}$, let Λ_m be the full subcategory of Λ whose objects are the y with $-m \le \kappa(y) \le m$. Denote the quiver of Λ_m by Q_m . It suffices to construct for each m an ideal $I_m \subset kQ_m$ which is generated by paths and differences of paths in such a way that $\Lambda_m \xrightarrow{\sim} kQ_m/I_m$ and $I_{m-1} = I_m \cap kQ_{m-1}$. We extend I_m to an ideal in kQ by setting $I_m(x,y) = kQ(x,y)$ if x or y does not belong to Q_m . Then $I = \cap I_m$ has the desired property.

The case $m = 0$ being trivial, we assume we already found I_p for all $p < m$. Notice that any object y of Λ_m with $\kappa(y) = m$ or $\kappa(y) = -m$ is a sink or a source in Q_m, respectively. Since Q is locally finite, there are at most countably many such y. Hence we can use an inductive procedure reducing the problem to the following (or its dual):

<u>CLAIM</u>. Let Λ be a locally representation-finite category whose quiver Q contains no oriented cycle, and suppose there is a sink y in Q. Assume the full subcategory Λ' containing all objects of Λ but y is isomorphic to kQ'/I', where Q' is the quiver of Λ' and I' is generated by paths and differences of paths. Then $\Lambda \xrightarrow{\sim} kQ/I$, where I has the same property and $I \cap kQ' = I'$.

By [2], the claim is true if we replace Λ by the full subcategory Λ_y whose objects are the x with $\Lambda(x,y) \neq 0$, since Λ_y is finite. Then it is easy to see for Λ, too.

The end of the proof of our proposition is based on an idea of Bongartz. It remains to be seen that, if some path v from x to y lies in $I(x,y)$, then they all do. If not, we choose a v whose residue class $\bar{v}$ in Λ is zero and such that there is a path w starting and stopping at the same vertices as v with $\bar{w} \neq 0$. Let Λ' be the full subcategory of Λ whose objects are the x_o, $x_1, \ldots, x_n$ through which v passes. We can assume n minimal. Then the quiver of Λ' has the form

$$\begin{tikzcd}
& x_1 \arrow[r] & \cdots & \cdots \arrow[r] & x_{n-1} \arrow[dr] & \\
x_o \arrow[ur] \arrow[rrrrr] & & & & & x_n
\end{tikzcd}$$

We choose the following Λ'-modules:

$$V_0 = \begin{tikzcd}[column sep=small, row sep=small] & k \arrow[dl, "\mathbb{1}"'] & 0 \arrow[l] & \cdots & 0 & k \arrow[l] & \\ k & & & & & & k \arrow[ul, "\mathbb{1}"'] \arrow[llllll, "\mathbb{1}"] \end{tikzcd}, \quad V_1 = \begin{tikzcd}[column sep=small, row sep=small] & k \arrow[dl] & k \arrow[l, "\mathbb{1}"'] & 0 \arrow[l] & \cdots & 0 & \\ 0 & & & & & & 0 \arrow[ul] \arrow[llllll] \end{tikzcd}$$

$$V_i = \begin{tikzcd}[column sep=small, row sep=small] & 0 \arrow[dl] & \cdots & k & k \arrow[l, "\mathbb{1}"'] & \cdots & 0 & \\ 0 & & & & & & & 0 \arrow[ul] \arrow[lllllll] \end{tikzcd}, \quad V_{n-2} = \begin{tikzcd}[column sep=small, row sep=small] & 0 \arrow[dl] & \cdots & 0 & k \arrow[l] & k \arrow[l, "\mathbb{1}"'] & \\ 0 & & & & & & 0 \arrow[ul] \arrow[llllll] \end{tikzcd}$$

We obtain an oriented cycle $V_o \to V_1 \to \dots \to V_{n-2} \to V_o$ of non-zero non-invertible morphisms. Since the left-adjoint functor L to the restriction $R : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda'$ is fully faithful, applying L yields an oriented cycle in Γ , which is impossible.

5.2 Let C be a configuration on $\mathbb{Z}\Delta$, and let Λ be the full sub-category of projective objects of $k(\mathbb{Z}\Delta_C)$. The Auslander-Reiten quiver of Λ is isomorphic to $\mathbb{Z}\Delta_C$, which is simply connected. Therefore, we can identify Λ with kQ/I , where Q is the quiver of Λ and I an ideal as in 5.1. Let A be the full sub-category of Λ defined in 2.1. We identify A with kK/J , where K is a full subquiver of Q and $J = I \cap kK$. A path $v : y \to x$ of K will be called <u>complete</u> if it does not belong to J , whereas γv and $v\beta$ do for each arrow γ of K with tail x and each β with head y .

We want to recover Q and I from K and J . Define a quiver $\mathbb{Z}K$ by completing the disjoint union $\mathbb{Z} \times K$ of copies $m \times K$ of K with arrows $(m,x) \to (m+1,y)$ for all m , whenever there is a complete path $y \to x$ in K . For any two vertices (m,x) and p of $\mathbb{Z}K$, we define $\mathbb{Z}J((m,x),p)$ to be the subspace of $k\mathbb{Z}K((m,x),p)$ spanned by the differences of the paths $(m,x) \to p$ if there is a path $p \to (m+1,x)$ in $\mathbb{Z}K$, or by all paths $(m,x) \to p$ otherwise.

<u>PROPOSITION</u>. Λ <u>is isomorphic to</u> $k\mathbb{Z}K/\mathbb{Z}J$.

Proof. Let Λ_m be the full subcategory of Λ whose objects are the projective vertices p of $\mathbb{Z}\Delta_C$ for which $\tau^{-(m-1)m_\Delta}\vec{\Delta} \leq p \leq \tau^{-mm_\Delta}\vec{\Delta}$. The automorphism τ^{-m_Δ} of $\mathbb{Z}\Delta_C$ induces an isomorphism $\tau^{-m_\Delta} : \Lambda_m \xrightarrow{\sim} \Lambda_{m+1}$, and hence we can identify the quiver of Λ_m with $m \times K$. Then $\tau^{-m_\Delta}(m,x) = (m+1,x)$ and $\tau^{-m_\Delta}(m,\alpha) = (m+1,\alpha)$, where x is a vertex and α an arrow of K.

The $m \times K$ are full subquivers of Q and their union $\mathbb{Z} \times K$ contains all vertices of Q. Let $\alpha : (m,x) \to p$ be an arrow of Q not belonging to $\mathbb{Z} \times K$. By 1.2, there is a path $w : p \to (m+1,x)$ in Q such that $w\alpha$ does not lie in I. Since the first coordinates of vertices cannot decrease along paths in Q, we see that $p = (m+1,y)$ and $w = (m+1,v)$ for some path $v : y \to x$ in K. We claim that v is complete in K: Assume that $w\beta \notin I$ for some arrow β of $(m+1) \times K$. By 1.2, the associated morphism $\overline{w\beta}$ of Λ can be extended to a non-zero morphism $\overline{w\beta u} : (m,x) \to (m+1,x)$, where u is a path in Q. This implies $\overline{\beta u} \neq 0$, hence $\overline{\alpha} = \overline{\beta u}$, a contradiction. Similarly, suppose that $\gamma w \notin I$ for some arrow γ of $(m+1) \times K$. By 1.2, $\tau^{m_\Delta}(\overline{\gamma w})$ can be extended to a non-zero morphism $\overline{u}\tau^{m_\Delta}(\overline{\gamma w}) : \tau^{m_\Delta}p \to p$. This implies $\overline{u}\tau^{m_\Delta}(\overline{\gamma}) \neq 0$, hence $\overline{\alpha} = \overline{u}\tau^{m_\Delta}(\overline{\gamma})$, a contradiction. Conversely, let $v : y \to x$ be a complete path in K. We can extend $(m+1,v)$ to a path $w = (m+1,v)\alpha(m,u) : (m,x) \to (m+1,x)$, where $u : x \to z$ is a path in K and $\alpha : (m,z) \to (m+1,y)$ an arrow in Q; by 5.1 w does not lie in I. Since, by 1.2 and 5.1, $(m+1,uv)\alpha : (m,z) \to (m+1,z)$ does not lie in I either, uv cannot belong to J. Therefore, u is trivial, and $\alpha : (m,x) \to (m+1,y)$ is an arrow of Q. We conclude that Q is isomorphic to $\mathbb{Z}K$.

By 1.2 and 5.1, I is uniquely determined by Q, since a path $(m,x) \to p$ lies in I if and only if it cannot be extended to $(m,x) \to p \to (m+1,x)$.

Remark. With K and J we associate the "short-circuited" quiver K_{sc}, which is obtained by adding to K an arrow $x \to y$ whenever there is a complete path $v : y \to x$ in K. If we distinguish between the arrows of K_{sc} already present in K and the new arrows, say by calling the former solid and the latter broken, we can recover K and J - and therefore Q and I - from K_{sc}. Clearly, K_{sc} is the residue quiver of Q modulo $\tau^{m_\Delta \mathbb{Z}}$, the solid arrows being the images of the arrows in $\mathbb{Z} \times K$.

5.3 Let $\bar{\Lambda} \neq k$ be a connected algebra which is standard, representation-finite and selfinjective. We want to describe $\bar{\Lambda}$ by quiver and relations. The Auslander-Reiten quiver $\Gamma_{\bar{\Lambda}}$ of $\bar{\Lambda}$ is isomorphic to $\mathbb{Z}\Delta_C/\Pi$ for some configuration C of $\mathbb{Z}\Delta$ and a non-trivial admissible automorphism group Π of $\mathbb{Z}\Delta_C$. Let $\Lambda \subset k(\mathbb{Z}\Delta_C)$, Q and I be defined as in 5.2. The fundamental group Π of $\Gamma_{\bar{\Lambda}}$ acts on Q and stabilizes I. By Q_Π we denote the residue quiver of Q modulo Π and by I_Π the ideal generated by the image of I in kQ_Π.

PROPOSITION. $\bar{\Lambda}$ is isomorphic to kQ_Π/I_Π.

Proof. Let x,y be two vertices of Q. The canonical projection $\pi : Q \to Q_\Pi$ induces isomorphisms $\coprod_{g\in\Pi} kQ(x,gy) \xrightarrow{\sim} kQ_\Pi(\pi x,\pi y)$, $\coprod_{g\in\Pi} I(x,gy) \xrightarrow{\sim} I_\Pi(\pi x,\pi y)$ and $\coprod_{g\in\Pi} \Lambda(x,gy) \xrightarrow{\sim} (kQ_\Pi/I_\Pi)(\pi x,\pi y)$. In other words, the induced functor $E : \Lambda \to kQ_\Pi/I_\Pi$ is a covering functor.

On the other hand, the canonical projection $\rho : \mathbb{Z}\Delta_C \to \mathbb{Z}\Delta_C/\Pi$ induces a covering functor $k(\rho) : k(\mathbb{Z}\Delta_C) \to k(\mathbb{Z}\Delta_C/\Pi)$ between the associated mesh-categories. By restriction to the projective vertices of $\mathbb{Z}\Delta_C$ and $\mathbb{Z}\Delta_C/\Pi$, we obtain a covering functor $F : \Lambda \to \bar{\Lambda}$. The composition $kQ \to kQ/I \xrightarrow{\sim} \Lambda \xrightarrow{F} \bar{\Lambda}$ maps I onto 0. So it induces a functor $D : kQ_\Pi/I_\Pi \to \bar{\Lambda}$ such that $DE = F$. The functor D is an isomorphism, because it is bijective on the objects, and E,F are both covering functors.

6. Selfinjective algebras of tree-class A_n

Our purpose is to present a new access to the configurations C of $\mathbb{Z}A_n$ and to the basic algebras with Auslander-Reiten quiver $(\mathbb{Z}A_n)_C/\tau^{n\mathbb{Z}}$. Algebras of this form will be called Brauer-algebras here.

6.1 Start with a Brauer-quiver Q ([8],1.4), i.e. with a finite connected quiver satisfying the following conditions: a) each arrow of Q belongs to a (simple, oriented) cycle; b) each vertex belongs to two cycles exactly; c) two cycles meet in one vertex at most.

Orient Q by dividing its arrows into an α- and a β-camp in such a way that two arrows belong to the same camp if they are parts of the same cycle, to different camps if they are parts of two neighbouring cycles (fig.6). Denote by A_Q the algebra defined by the quiver Q and the following relations:

a) $\alpha\beta = \beta\alpha = 0$

b) $\alpha^{a_x} = \beta^{b_x}$ for each vertex x of Q, where a_x (resp. b_x) denotes the length of the α-cycle (resp. β-cycle) running through x.

Clearly, A_Q does not depend on the chosen orientation, and its quiver is derived from Q by erasing the loops.

PROPOSITION. The map $Q \mapsto A_Q$ yields a bijection between the isoclasses of Brauer-quivers and the isoclasses of Brauer-algebras.

The proposition is already proved in [8] and [12]. A new proof is given in 6.3 below.

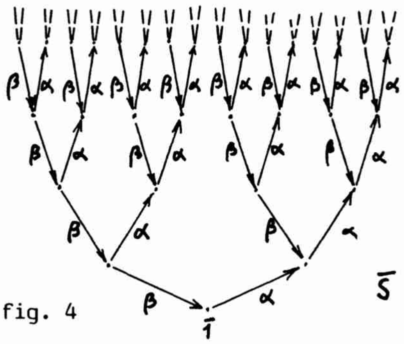


fig. 4

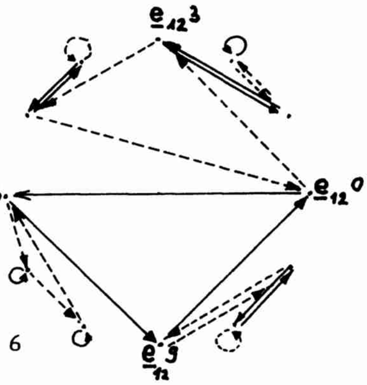


fig. 6

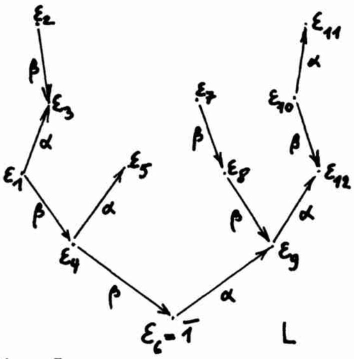
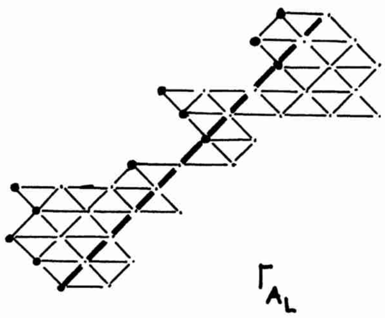


fig. 5

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              2  3  2  1  2  2 • 2  3  1  1  3  2 • 2  3
           1  5  5  3 • 3  4  1  5  4  2  4  5  1  5
              4  7  6  1  5  3  4  6  5 • 5  6  4 • 4
      1  1  2  6  8  4  3  4  6  5  7  2  7  5  2
         2  3  4  7  6  6  2  7  7  6  4  4  6  3
            4  5  5  5  8  5 • 5  8  8  3  6  3  4
      2  1  6  6  3  7  7  2  6  9  5  5  5  1
1  2  1  3  3  7  4  5  6  4  3  7  6  7  4  3
  3  3  2  5  4  5  6  4  3  5  4  4  8  6  2
1  4  4 • 4  6  2  7  5  1  4  6  1  6  7  4
  2  5  1  5  4 • 4  6  2  2  5  3 • 3  5  5
     3  2 • 2  3  1  3  3 • 3 • 3  2  1  2  3
```

fig. 7

6.2 Let $\overline{S}$ be the oriented tree of fig.4. A pedigree L is by definition a subtree of $\overline{S}$ which contains the lowest vertex $\overline{1}$. With each such L we associate the algebra A_L defined by the quiver L and all possible relations $\alpha\beta = 0$. With each vertex x of L we associate the indecomposable (contravariant) representation M_x of L such that $M_x(y) = k$ or 0 according as y lies between $\overline{1}$ and x or not; the transition maps of M_x are zeros and identities. Then we get a section $M_{\varepsilon_1} \to M_{\varepsilon_2} \to \dots \to M_{\varepsilon_n}$ of the Auslander-Reiten quiver Γ_{A_L} by enumerating the vertices of L from the left to the right as indicated in fig.5 (for the construction of Γ_{A_L} we refer to [4],7.2 and 7.5). In fact, it follows easily from [16] or [4],§7 that the preceding construction yields a bijection between the pedigrees with n vertices and the isoclasses of $\vec{A}_n$-section-algebras, where $\vec{A}_n$ denotes the quiver

$$1 \to 2 \to \dots \to n \quad .$$

Remark. The $\vec{A}_n$-pattern $\mathcal{P}_L$ associated with the section-algebra A_L (3.1) can be described as follows: Let us identify each vertex x of L with the shortest walk from $\overline{1}$ to x in L, and write such a walk as a word in β^{-1} and α. For instance, we identify the vertices ε_1, ε_6 and ε_{11} of fig.5 with the words $\beta^{-1}\beta^{-1} = \beta^{-2}$, $\overline{1}$ (=the empty word) and $\alpha^2\beta^{-1}\alpha$ respectively. We order these words lexicographically in such a way that $\beta^{-1} < \alpha$; so we have in particular $\overline{1} < \beta^{-1} < \beta^{-2} < \beta^{-1}\alpha < \alpha < \alpha\beta^{-1}$. The lexicographic order assigns to each vertex x an ordinal $\sigma(x) \in \{1, \dots, n\}$ and a vertex $(-p(x),\sigma(x))$ of $\mathbb{Z}A_n$, where $p(x)$ is the number of letters β^{-1} in the word identified with x ; in fig.5 for instance, the vertices ε_1, ε_6 and ε_{11} are mapped onto the vertices $(-2,3)$, $(0,1)$ and $(-1,12)$ respectively. The pattern $\mathcal{P}_L$ equals the set $\{(-p(x),\sigma(x)): x \in L_o\}$ ([6],7.2).

6.3 Proof of proposition 6.1. Let L be a pedigree with n vertices, $P = P_L$ the associated $\vec{A}_n$-pattern and $C = C_P$ the configuration of $\mathbb{Z}A_n$ assigned to P (3.2). Denote by A the Brauer-algebra with Auslander-Reiten quiver $(\mathbb{Z}A_n)_C/\tau^{n\mathbb{Z}}$. According to 5.2, the quiver L_{sc} of A can be described as follows: Set $y = \alpha x$ if L contains an α-arrow from x to y, and define an α-orbit with origin x and terminus y to be a set of vertices $\{ x, \alpha x, \ldots, \alpha^r x = y \}$ such that no α-arrow stops at x or starts at y. Define β-orbits in a similar way. Then L_{sc} is obtained from L by adding a "broken" (α- or β-) arrow $y \dashrightarrow x$ for each (α- or β-) orbit of cardinality > 1 with origin x and terminus y.

The Brauer-algebra A is defined by the quiver L_{sc} and the relations produced in 5.2 and 5.3. In fact, we can get a more convenient description of these relations by adding to L_{sc} a loop at each vertex x such that $\{x\}$ is an (α- or β-) orbit of cardinality 1. In this way we clearly obtain a Brauer-quiver $\bar{L}$. It is easy to see that the relations of 5.2 and 5.3 are equivalent to the relations a) and b) of 6.1. Accordingly, A is identified with $A_{\bar{L}}$.

This shows that each Brauer-algebra Λ is isomorphic to some A_Q, where the Brauer-quiver Q is uniquely determined up to isomorphism by Λ (obtain Q by adding loops to the quiver of Λ). It remains to show that, conversely, each A_Q is a Brauer-algebra. For this sake, we orient Q and choose a vertex, which we denote by $\bar{1}$. Consider the walks of Q of the form

$$\bar{1} \xrightarrow{\alpha} \cdot \ \ldots \ \cdot \xrightarrow{\alpha} \cdot \xleftarrow{\beta} \cdot \ \ldots \ \cdot \xleftarrow{\beta} \cdot \xrightarrow{\alpha} \cdot \ \ldots$$

or

$$\ldots \ \cdot \xrightarrow{\beta} \cdot \xleftarrow{\alpha} \cdot \ \ldots \ \cdot \xleftarrow{\alpha} \cdot \xrightarrow{\beta} \cdot \ \ldots \ \cdot \xrightarrow{\beta} \bar{1}$$

which end with an α- or β-arrow and which do not run twice through the same vertex. The vertices of Q and the arrows occuring in the above walks form a pedigree L such that $Q = \overline{L}$. As a consequence, A_Q is the Brauer-algebra associated with the $\vec{A}_n$-section-algebra A_L.

Remark. Define an arrowed Brauer-quiver to be an oriented Brauer-quiver together with a distinguished α-arrow. Attach the arrowed Brauer-quiver $(\overline{L}, \varepsilon_n \to \overline{1})$ to each pedigree L with n vertices. The last part of the above proof then shows that L can be recovered from $(\overline{L}, \varepsilon_n \to \overline{1})$. So we get a bijection between the pedigrees and the isoclasses of arrowed Brauer-quivers.

On the other hand, we have a bijection between the isoclasses of arrowed Brauer-quivers and of arrowed planar trees. The arrowed planar tree T attached to an arrowed Brauer-quiver P has the cycles of P as vertices and the vertices of P as edges; the cyclic order that each cycle c of P defines on its vertices v, coincides with the clockwise cyclic order which the immersion of T into the plane defines on the edges v of T with extremity c ([8],1.4). The head of the distinguished arrow of P is an edge of T, which we distinguish within T by "orienting" it towards its "α-extremity".

Composing the bijections $L \mapsto (\overline{L}, \varepsilon_n \to \overline{1})$ and $P \mapsto T$, we obtain a bijection $L \mapsto T$ between the pedigrees and the isoclasses of arrowed planar trees: The vertices of T are the α- and β-orbits of L; the edges of T are the vertices of L ... Being "self-symmetric", the bijection $L \mapsto T$ is different from the bijection first obtained by Harary, Prins and Tutte [17].

6.4 For the sake of completeness, we end this section with a description of the configuration C assigned to a pedigree L (6.3). As C is periodic, it is enough to describe the vertices $(p,q) \in \mathcal{C}$ such that $0 \leq p < n$. These are the points $(i-1,j-i)$ and $(i-1,n+j-i)$ of the following proposition.

We use the following notations: $\mathcal{P} = \mathcal{P}_L$ is the $\vec{A}_n$-pattern associated with L (6.2), R_∞ the translation-quiver constructed from $\mathcal{P}$ as in 3.3, Λ_∞ the full subcategory of the mesh-category $k(R_\infty)$ formed by the projective vertices, $M_\infty(v)$ the indecomposable Λ_∞-module $k(R_\infty)(?,v)|\Lambda_\infty$ attached to a vertex v of R_∞; (obviously, R_∞ is representable by 3.4). The points of L are labelled as in 6.2 and fig.5.

PROPOSITION. *Let* ε_i *be a vertex of the pedigree* L, $(-p(\varepsilon_i),\sigma(\varepsilon_i)) \in \Lambda_\infty$ *the associated point of* $\mathcal{P}$ (6.2) *and* $M_\infty(c^*)$ *the injective envelope of the top* T *of* $M_\infty(-p(\varepsilon_i),\sigma(\varepsilon_i))$. *Then we have either* $c = (i-1,j-i)$ *if* L *contains a* β-*arrow from* ε_i *to* ε_j, *or else* $c = (i-1,n+j-i)$ *if* ε_j *is the origin of the* β-*orbit of* L *with terminus* ε_i.

Proof. First assume that L contains an arrow $\varepsilon_i \overset{\beta}{\to} \varepsilon_j$. Then the simple A_L-module k_{ε_i} with support $\{\varepsilon_i\}$ occurs in the socle of M_{ε_h} (6.2) iff $i \leq h < j$. By lemma 6.5 below, we infer that $k(\mathbb{Z}A_n)((0,h),c) \xrightarrow{\sim} k(\mathbb{N}\mathcal{P})((0,h),c) \xrightarrow{\sim} \overline{\mathrm{Hom}}_{\Lambda_\infty}(M_\infty(0,h),M_\infty(c)) \neq 0$ iff $i \leq h < j$, or equivalently that $c = (i-1,j-i)$ (fig.7a).

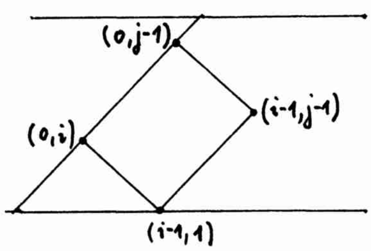


fig. 7a

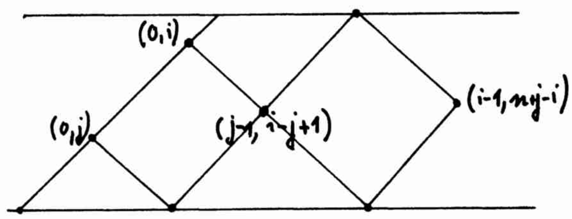


fig. 7b

Now assume that ε_i is the terminus and ε_j the origin of a β-orbit of L . In this case, k_{ε_i} occurs in the top $M_{\varepsilon_h}/\mathrm{rad}\, M_{\varepsilon_h}$ of M_{ε_h} iff $j \le h \le i$. Accordingly, we have $\overline{\mathrm{Hom}}_{\Lambda_\infty}(M_\infty(0,h),T) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda_\infty}(M_\infty(0,h),T)$ $\xrightarrow{\sim}$ $\mathrm{Hom}_{A_L}(M_{\varepsilon_h},k_{\varepsilon_i}) \neq 0$ iff $j \le h \le i$. In particular, we have $\mathrm{Hom}_{\Lambda_\infty}(M_\infty(0,i),T) \neq 0$ and therefore $T \xrightarrow{\sim} M_\infty(r,s)$ with $r \ge 0$. We infer that $k(\mathbb{Z}A_n)((0,h),(r,s))$ $\xrightarrow{\sim}$ $k(\mathbb{N}P)((0,h),(r,s)) \xrightarrow{\sim} \overline{\mathrm{Hom}}_{\Lambda_\infty}(M_\infty(0,h),T) \neq 0$ iff $j \le h$ $\le i$, or equivalently that $(r,s) = (j-1,i-j+1)$ (fig.7b). Using the fact that $M_\infty(c)$ is the radical of the projective-injective Λ_∞-module $M_\infty(c^*)$, and that every non-zero $\mu \in k(\mathbb{Z}A_n)((r,s),(p,q)) \xrightarrow{\sim} \overline{\mathrm{Hom}}_{\Lambda_\infty}(T,M_\infty(p,q))$ $\xrightarrow{\sim}$ $\mathrm{Hom}_{\Lambda_\infty}(T,M_\infty(p,q))$ can be extended to a non-zero $\nu \in k(\mathbb{Z}A_n)((r,s),c) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda_\infty}(T,M_\infty(c))$, we prove as in 1.2 and 3.5 that $c = \nu_{A_n}(r,s) = (i-1,n+j-i)$ (fig.7b).

6.5 In the following lemma, Λ is a locally finite-dimensional category and $\overline{\mathrm{Hom}}_\Lambda(M,N)$ the quotient of $\mathrm{Hom}_\Lambda(M,N)$ by the subspace formed by the morphisms $\mu : M \to N$ which factor through an injective Λ-module.

LEMMA. *Let* M *be a non-injective indecomposable* Λ-*module with socle* S *and* I *a projective-injective indecomposable* Λ-*module with socle* $T \neq I$ *and radical* R . *Then,* $\mathrm{Hom}_\Lambda(S,T)$ *is canonically isomorphic to* $\overline{\mathrm{Hom}}_\Lambda(M,R)$.

Proof. Consider the map $f : \mathrm{Hom}_\Lambda(M,R) \to \mathrm{Hom}_\Lambda(S,T)$ which assigns to each $\mu : M \to R$ the induced morphism $f(\mu) : S \to T$. Since each morphism $S \to T$ extends to a (non-surjective) morphism $M \to I$, the map f is surjective. It remains to show that a morphism $\mu \in \mathrm{Hom}_\Lambda(M,R)$ belongs to $f^{-1}(0)$, i.e. vanishes on S , iff it factors through an injective. The condition is sufficient because a morphism from an injective J to R must vanish on the socle of J . Conversely, if $\mu(S) = 0$, let $\nu : E \to I$ be an extension of μ to the injective envelope E of M .

Since ν vanishes on the socle S of E, its restriction to any non-trivial direct summand of E is non-injective. As a consequence, ν cannot be surjective; hence ν factors through R.

6.6 Remark. In [12],3.4 Riedtmann assigns to each configuration C of $\mathbb{Z}A_n$ a Brauer-quiver Q_C having $\sqrt[n]{1}$ as set of vertices. We can interpret the description of the configurations C of 6.4 by saying that the arrowed Brauer-quiver $(Q_C, \underline{e}_n(\alpha_C^{-1}0) \overset{\alpha}{\to} \underline{e}_n(0))$ is isomorphic to $(\bar{L}, \varepsilon_n \overset{\alpha}{\to} \bar{1})$.

7. Selfinjective algebras of tree-class D_n, $n \geq 4$

Our purpose is to classify up to isomorphism the configurations C of $\mathbb{Z}D_n$ or, equivalently, the basic algebras with Auslander-Reiten quiver $(\mathbb{Z}D_n)_C/\tau^{(2n-3)\mathbb{Z}}$. If $n \geq 5$, we call such an algebra two- or three-cornered according as C contains two or three high vertices modulo τ^{2n-3} (1.3). If $n = 4$, we agree to call it two-cornered.

7.1 Description of the two-cornered algebras. Let P be an arrowed Brauer-quiver with $n-2$ vertices and distinguished arrow $x \overset{\alpha}{\to} y$ (fig.8). Let a_z (resp. b_z) be the length of the α-cycle A_z (resp. the β-cycle B_z) of P running through the vertex z. We denote by P^{++} the quiver obtained from P by replacing $x \overset{\alpha}{\to} y$ by

$$\begin{tikzcd} & \cdot \arrow[dr, "\gamma_2"] & \\ x \arrow[ur, "\gamma_1"] \arrow[dr, "\delta_1"'] & & y \\ & \cdot \arrow[ur, "\delta_2"'] & \end{tikzcd}$$

(see fig.9). We denote by D_P the algebra defined by P^{++} and the following relations:

a) $0 = \alpha\beta = \beta\alpha = \gamma_1\beta = \beta\gamma_2 = \delta_1\beta = \beta\delta_2 = \delta_1\alpha^{a_x-1}\gamma_2$
$= \gamma_1\alpha^{a_x-1}\delta_2$

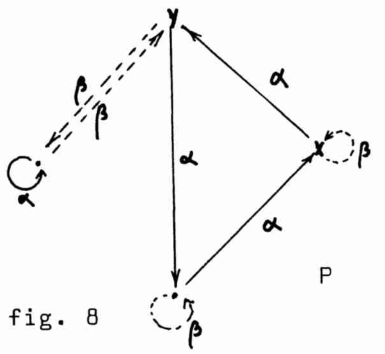


fig. 8

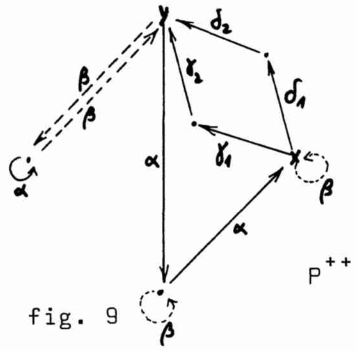


fig. 9

b) $\alpha^{a_z} = \beta^{b_z}$ for each vertex z of P lying outside A_x

c) $\gamma_2\gamma_1 = \delta_2\delta_1$

d) $\alpha^{a_z^+}\gamma_2\gamma_1\alpha^{a_z^-} = \beta^{b_z}$ for each vertex z of A_x ; here we denote by a_z^- (resp. by a_z^+) the length of the α-path of P^{++} from z to x (resp. from y to z).

Of course, the ordinary quiver of D_P is derived from P^{++} by deleting the loops.

PROPOSITION. The map $P \mapsto D_P$ *yields a bijection between the isoclasses of arrowed Brauer-quivers having at least 2 vertices and the isoclasses of two-cornered algebras. Accordingly, there are* $\bar{c}_{n-1} = \frac{1}{n-1}\binom{2n-4}{n-2}$ *isoclasses of two-cornered algebras of tree-class* D_n .

The proof of the proposition is given in 7.4 below.

7.2 *Description of the three-cornered algebras.* Consider a *looped* Brauer-quiver P , i.e. an oriented Brauer-quiver with a distinguished β-loop. By P^+ we denote the quiver derived from P by splitting the distinguished vertex x of P into an α-sink x^- and an α-source x^+ : the arrow $\alpha^{-1}x \overset{\alpha}{\to} x$ (resp. $x \overset{\alpha}{\to} \alpha x$) of P is replaced by an arrow $\alpha^{-1}x \overset{\alpha}{\to} x^-$ (resp. $x^+ \overset{\alpha}{\to} \alpha x$) if $\alpha^{-1}x \neq x \neq \alpha x$; if P has only one vertex, the α-loop $\alpha \circlearrowright x$ is replaced by an arrow $x^+ \overset{\alpha}{\to} x^-$; in all cases, the distinguished loop $\beta \circlearrowright x$ is replaced by an arrow $x^- \to x^+$, which we denote by γ (see fig.10).

Now let P_1 , P_2 , P_3 be three looped Brauer-quivers with distinguished vertices x_1 , x_2 , x_3 . By $P_1P_2P_3$ we denote the quiver derived from $\overset{+}{\underset{-}{P_1}}$, $\overset{+}{\underset{-}{P_2}}$ and $\overset{+}{\underset{-}{P_3}}$ by identifying x_2^+ with x_3^- , x_3^+ with x_1^- , x_1^+ with x_2^- (fig.10). By $D(P_1P_2P_3)$ we denote the algebra defined by the quiver $P_1P_2P_3$ and the following relations:

a) $0 = \alpha_i\beta_i = \beta_i\alpha_i = \alpha_2\alpha_3 = \alpha_1\alpha_2 = \alpha_3\alpha_1$; $i = 1,2,3$

b) $\alpha_i^{a_z} = \beta_i^{b_z}$ for each i and each vertex z of P_i lying outside the α-cycle Z_i of P_i which contains x_i .

c) $\gamma_3\gamma_2 = \alpha_1^{a_{x_1}}$, $\gamma_1\gamma_3 = \alpha_2^{a_{x_2}}$, $\gamma_2\gamma_1 = \alpha_3^{a_{x_3}}$.

d) $\alpha_i^{a^-}\gamma_i\alpha_i^{a^+} = \beta^{b_z}$ for each i if $z \in Z_i \setminus \{x_i\}$; here we denote by a^- (resp. by a^+) the length of the α-path of P_i^+ from z to x_i^- (resp. from x_i^+ to z).

Of course, the ordinary quiver of $D(P_1P_2P_3)$ is derived from $P_1P_2P_3$ by deleting the loops, and the arrow α_i if P_i has only one vertex.

In the following proposition, $|Q|$ denotes the number of vertices of a quiver Q .

PROPOSITION. Each algebra $D(P_1P_2P_3)$ such that $|P_1| + |P_2| + |P_3| \geq 5$ is three-cornered. Conversely, each three-cornered algebra is isomorphic to some $D(P_1P_2P_3)$, where the looped Brauer-quivers P_1 , P_2 , P_3 are uniquely determined up to isomorphism and to a cyclic permutation.

The proof is given in 7.5 below.

7.3 Let $\vec{D}_n$ be the quiver $1 \to 2 \to \dots \to n-2 \begin{smallmatrix} \nearrow n-1 \\ \searrow n \end{smallmatrix}$ and $\vec{A}_{n-1}$ be the full subquiver of $\vec{D}_n$ whose vertices are $1,2,\dots,n-1$. The inclusion $\vec{A}_{n-1} \to \vec{D}_n$ extends to a trans-

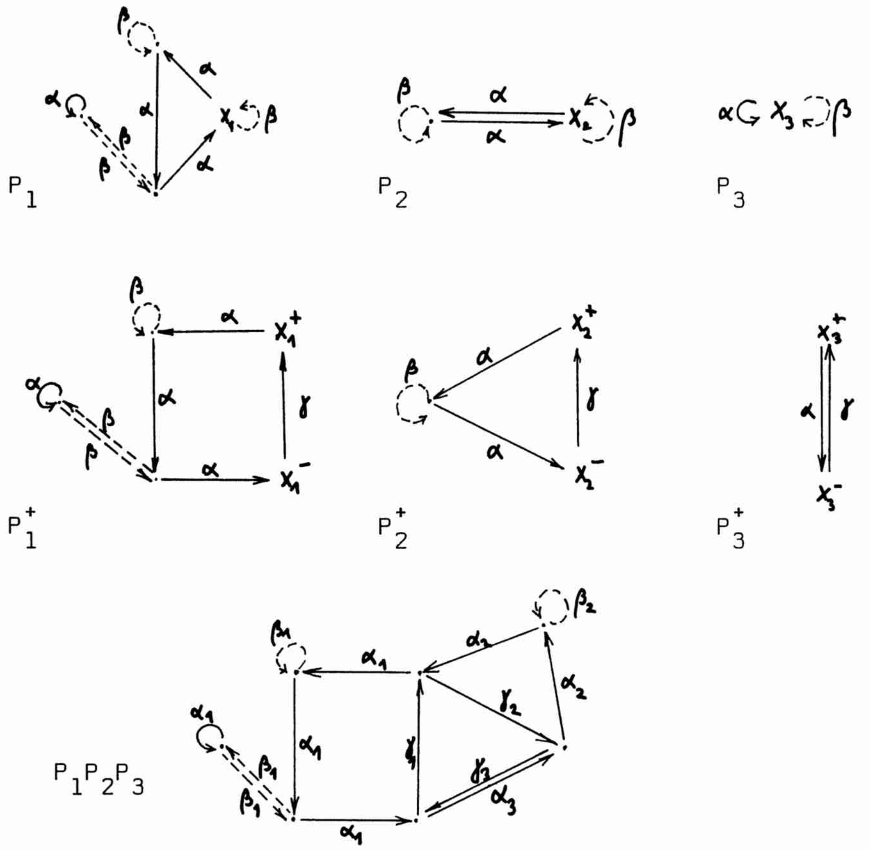


fig. 10

lation-quiver morphism $\mathbb{Z}A_{n-1} \to \mathbb{Z}D_n$, which allows us to identify $\mathbb{Z}A_{n-1}$ with a full sub-translation-quiver of $\mathbb{Z}D_n$. A vertex of $\mathbb{Z}D_n$ is represented by two coordinates as indicated in fig.1.

With each $\vec{A}_{n-1}$-pattern P we associate the subset $P^+ = P \cup \{(0,n)\}$ of $(\mathbb{Z}D_n)_o$. As $(0,n)$ lies in $\vec{D}_n$, P^+ is a $\vec{D}_n$-pattern.

LEMMA. Each configuration $\mathcal{C}$ of $\mathbb{Z}D_n$ is isomorphic to some $\mathcal{C}_{P^+}$ (3.2).

Proof. By 1.6, $\mathcal{C}$ contains high vertices. Using an automorphism of $\mathbb{Z}D_n$ if necessary, we are therefore reduced to the case where $(-1,n) \in \mathcal{C}$. Then the dimension-function $D_{\mathcal{C}}$ (4.1) satisfies $D_{\mathcal{C}}(n) = D_{\mathcal{C}}(n-2)+1$, and the pattern $P_{\mathcal{C}}$ (4.3) must contain $(0,n)$. The vertices of $P_{\mathcal{C}}$ other than $(0,n)$ form an $\vec{A}_{n-1}$-pattern P such that $P^+ = P_{\mathcal{C}}$.

7.4 Proof of proposition 7.1. Let $\mathcal{C}$ be a configuration of $\mathbb{Z}D_n$ containing $(-1,n-1)$ and $(-1,n)$. According to 7.3, we have $\mathcal{C} = \mathcal{C}_{P^+}$, where P is an $\vec{A}_{n-1}$-pattern containing $n-1$. The set $P^- = P\setminus\{n-1\}$ is an $\vec{A}_{n-2}$-pattern, whose section-algebra is identified with A_L for some pedigree L (6.1; see fig.11). If a is the terminus of the α-orbit of 1 in L, the quiver K of the section-algebra A_{P^+} is obtained by adding two arrows $a \to n-1$ and $a \to n$ to L (in fig.11 K is the quiver formed by the full arrows of K_{sc}). The two-cornered algebra with Auslander-Reiten quiver $(\mathbb{Z}D_n)_{\mathcal{C}}/\tau^{(2n-3)\mathbb{Z}}$ is defined by the quiver K_{sc} (fig.11) and the relations described in 5.2 and 5.3. Clearly, this implies that it is identified with D_P (7.1), where $P = (\bar{L}, a \to 1)$ is the arrowed Brauer-quiver associated with L (6.2).

Since L can be any pedigree having at least two vertices, P is an arbitrary arrowed Brauer-quiver with at least two vertices. Therefore, the map $P \mapsto D_P$ induces a surjection on the isoclasses, and it remains for us to show that D_P determines P up to isomorphism. We leave it to the reader to ascertain that the quiver K_{sc} of D_P determines L (and hence P) up to an isomorphism (in fact, even the subquiver L of K_{sc} can be recovered from K_{sc} if $n \geq 5$).

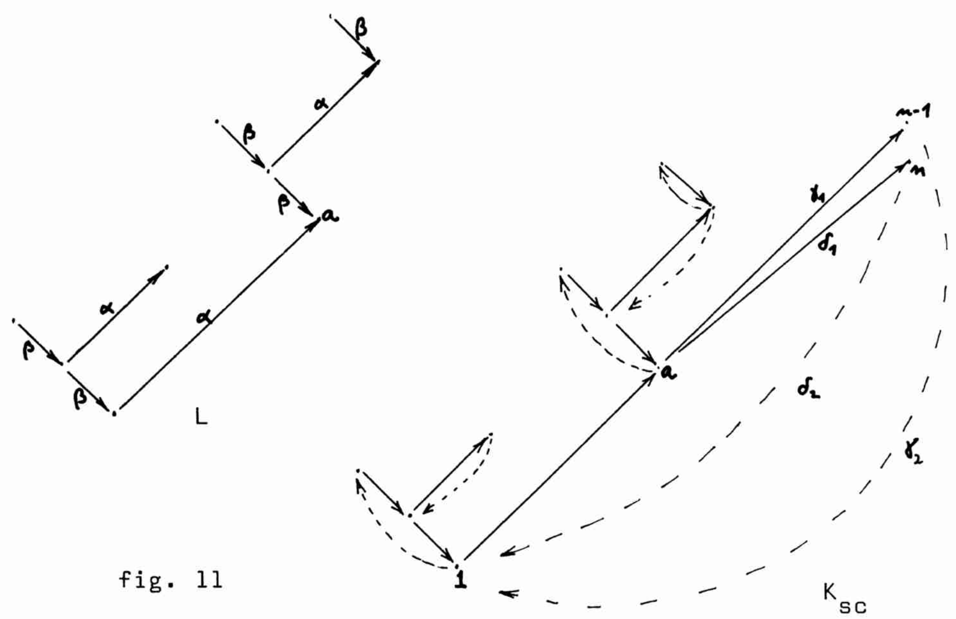


fig. 11

7.5 <u>Proof of proposition</u> 7.2. Let C be a configuration of $\mathbb{Z}D_n$ such that $(-1,n-1) \notin C$ and $(-1,n) \in C$. According to 7.3, we have $C = C_{P^+}$, where P is an $\vec{A}_{n-1}$-pattern such that $n-1 \notin P$. The section-algebra A_P of P (3.1) is identified with A_L for some pedigree L (6.1), whose vertices we identify with the points of P (fig.12). Then the terminus of the α-orbit of $\bar{1}$ in L is identified with a vertex $(0,a) \tilde{\to} a \neq n-1$ of $\vec{A}_{n-1}$. We denote by L_3 the pedigree formed by the vertices $(p,q) \in P \tilde{\to} L$ such that $q \leq a$.

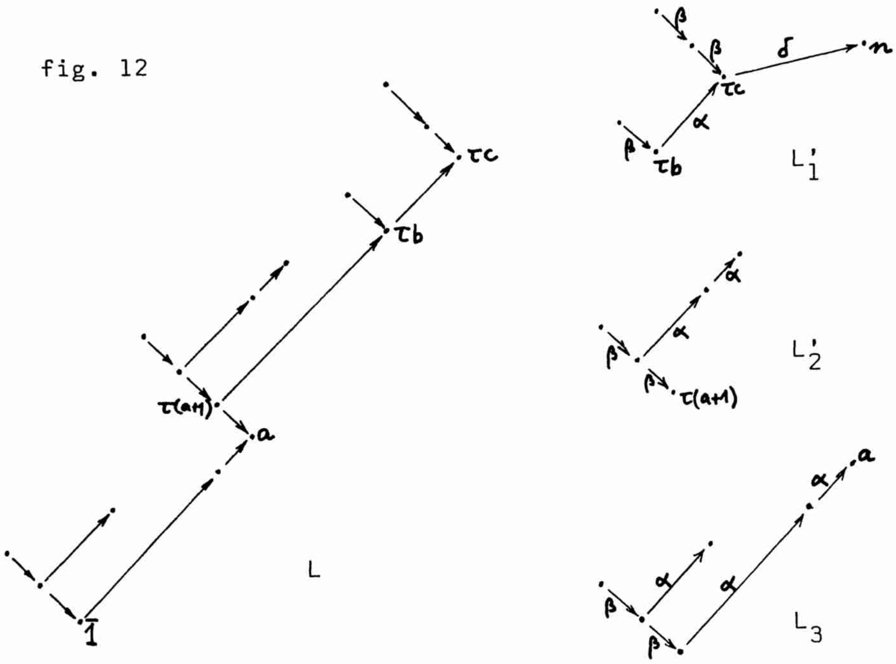
Since $a < n-1$, the vertex $(-1,a+1) \tilde{\to} \tau(a+1)$ also belongs to P. We denote by L_2' the full subquiver of L formed by $\tau(a+1)$ and the "paternal ancestors" of $\tau(a+1)$ in L (see fig.12). Then there is a unique embedding of L_2' into $\bar{S}$ (fig.4) which maps $\tau(a+1)$ onto $\bar{1}$, α-arrows onto β-arrows and β-arrows onto α-arrows. We denote by L_2 the image of this embedding, which is a pedigree.

Let us finally examine the quiver L'_1 of the full subcategory of $k(R_{P^+})$(3.1) formed by the points of P^+ lying outside L_3 and L'_2. If L'_1 is reduced to the vertex n, we denote by L_1 the pedigree reduced to the lowest vertex $\bar{1}$ of $\bar{S}$. Otherwise, the α-orbit of $\tau(a+1)$ in L has the form $\{\tau(a+1),\tau(b),\dots,\tau(c)\}$, where $a+1 < b \leq \dots \leq c$, and the vertices of L'_1 are the points $x = (p,q)$ of P^+ such that $b \leq q$ (fig.12). Such a point x satisfies $k(R_{P^+})(x,n) \neq 0$ if $p = -1$ or $x = n$. Accordingly, we obtain L'_1 by adding one arrow $\tau c \xrightarrow{\delta} n$ to the subtree of L formed by the vertices $(p,q) \in P$ such that $b \leq q \leq n-1$. Moreover, there is a unique embedding of L'_1 into $\bar{S}$ which maps τb onto $\bar{1}$, each β-arrow onto a β-arrow, δ and each α-arrow onto an α-arrow. We denote by L_1 the image of this embedding.

The pedigrees L_3, L_2 and L_1 give rise to Brauer-quivers $\bar{L}_3$, $\bar{L}_2$ and $\bar{L}_1$ (6.1), in which we distinguish the β-loops with extremities $a, \tau(a+1) \xrightarrow{\sim} \bar{1}$ and $n \in L'_1 \xrightarrow{\sim} L_1$ respectively. In this way we produce three looped Brauer-quivers P_1, P_2, P_3 and an algebra $D(P_1P_2P_3)$. It follows easily from 5.2 and 5.3 that $D(P_1P_2P_3)$ is isomorphic to the three-cornered algebra with Auslander-Reiten quiver $(\mathbb{Z}D_n)_C/\tau^{(2n-3)\mathbb{Z}}$ (construct the quiver K_{sc} by "short-circuiting" the quiver K of A_{P^+}; see fig.13).

So each three-cornered algebra is isomorphic to some $D(P_1P_2P_3)$. Since the pedigrees L_1, L_2, L_3 occuring in the above argumentation are subjected to the only restriction $|L_1| + |L_2| + |L_3| \geq 5$, we infer that, conversely, each $D(P_1P_2P_3)$ is three-cornered if $|P_1| + |P_2| + |P_3| \geq 5$. The algebra $D(P_1P_2P_3)$ determines P_1, P_2, P_3 up to isomorphism and to a cyclic permutation, because we can recover L_1, L_2, L_3 up to a cyclic permutation from the

fig. 12



quiver Q of $D(P_1P_2P_3)$ (show that we can recover the triangle T formed by γ_1, γ_2, γ_3 : Let $Q\setminus D$ be the quiver obtained from Q by deleting the arrows - but not the vertices - of some triangle D ; if $Q\setminus D$ is connected for some D, then $D = T$; otherwise, T is the only triangle D such that $Q\setminus D$ has two connected components, out of which one is reduced to a point).

Remark. The number of looped Brauer-quivers with n vertices is $\bar{c}_n = \frac{1}{n}\binom{2n-2}{n-1}$ (if $n>1$, delete the distinguished β-loop and unite the contiguous α-arrows so as to obtain an arrowed Brauer-quiver with $n-1$ vertices). It follows by classical arguments that the number of isoclasses of three-cornered algebras with $n \geq 5$ equals

K_{sc}

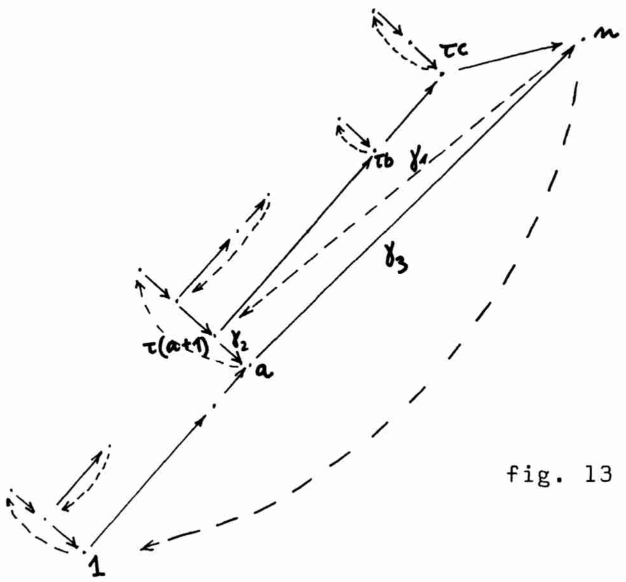


fig. 13

$\frac{1}{3}(\bar{c}_n-\bar{c}_{n-1}) + \frac{2}{3}\bar{c}_r$ if $n = 3r$ and $\frac{1}{3}(\bar{c}_n-\bar{c}_{n-1})$ if n is not divisible by 3 . As a consequence, we get that the number of configurations of $\mathbb{Z}D_n$ is $\frac{3n-4}{n}\binom{2n-3}{n-2} = \frac{3n-4}{2}\bar{c}_n$.

7.6 <u>Configurations on</u> $\mathbb{Z}D_4$. Fig.14 shows representatives of the two isoclasses of configurations on $\mathbb{Z}D_4$.

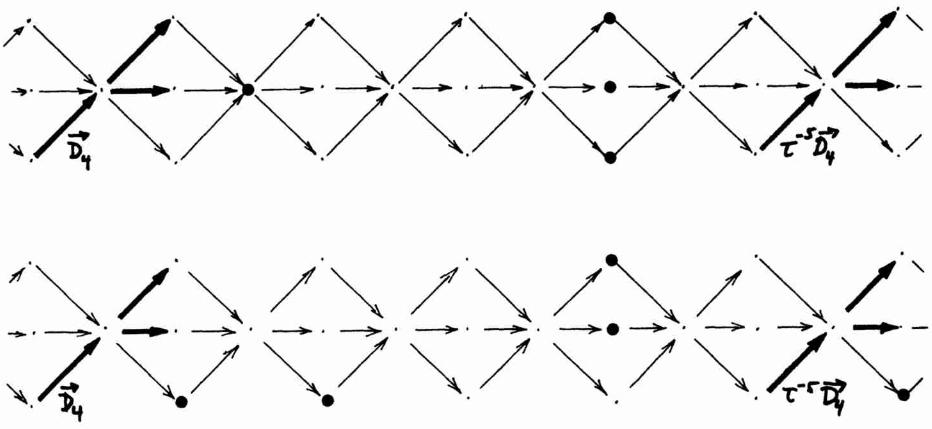


fig. 14

8. Selfinjective algebras of tree-class E_n , $n = 6,7,8$

The following lists provide a complete classification of the configurations of $\mathbb{Z}E_n$. The numerical data contained in these lists require a short explanation. To save room, we set $\underline{t} = 10+t$ and $\underline{\underline{t}} = 20+t$ for all $t \in \mathbb{N}$.

We order the vertices (y,x) of $\mathbb{Z}E_n$ lexicographically reading them from the right to the left. So we have $(y,x) < (y',x')$ either if $x < x'$ or else if $x = x'$ and $y < y'$. Since each configuration C of $\mathbb{Z}E_n$ is periodic, it is sufficient to list the vertices $(y,x) \in C$ such that $0 \le y < m_{E_n}$. Accordingly, C is characterized by an increasing sequence of configuration-points $(y_1,x_1), \dots ,$ (y_n,x_n) , which we present as a matrix

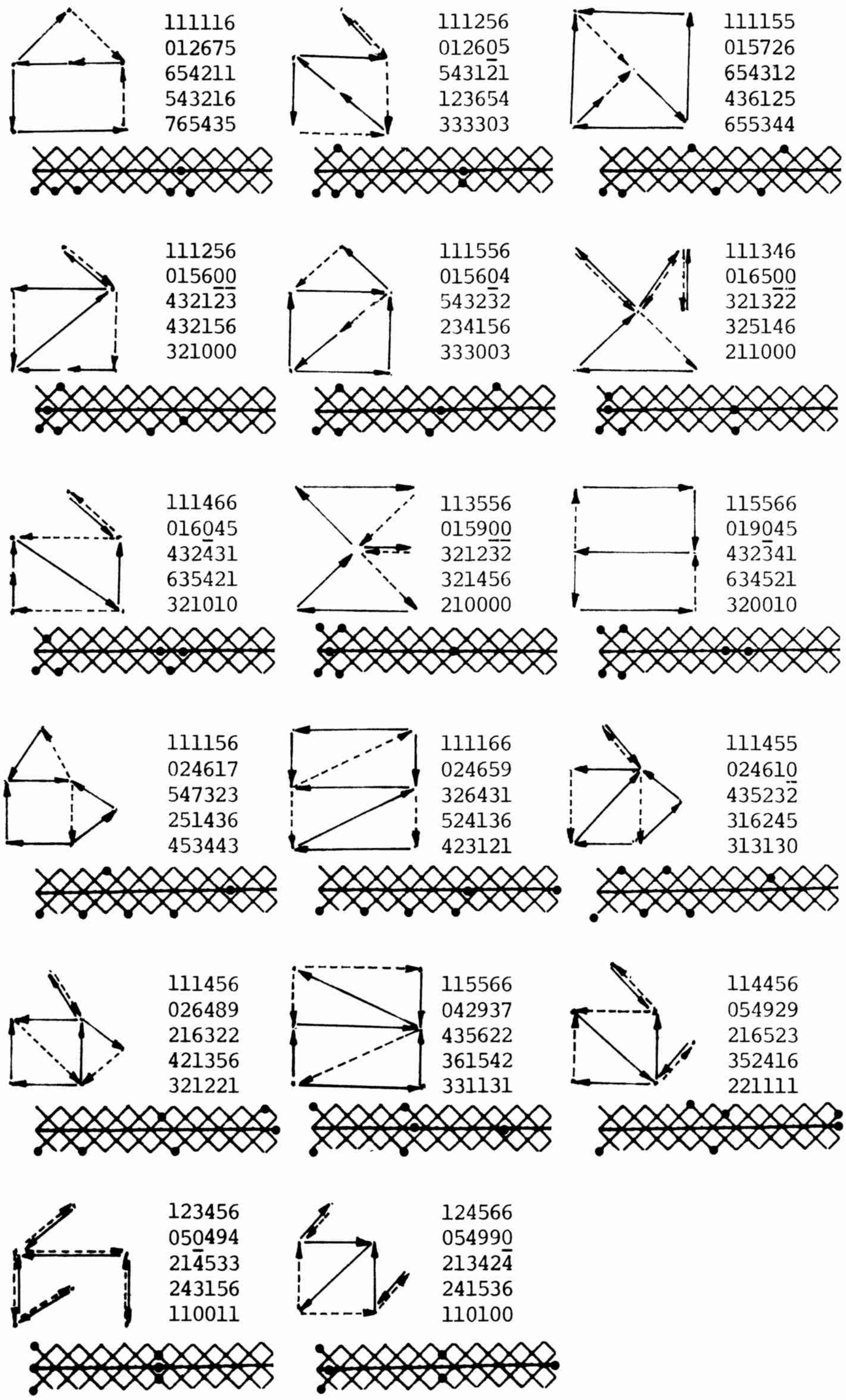
$$\begin{matrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{matrix}$$

For instance, the first listed configuration of $\mathbb{Z}E_8$ contains the vertices $(0,1) < (1,1) < (2,1) < (3,1) < (4,1) < (14,6) < (28,7) < (14,8)$. If we interpret a representing matrix as a word whose letters are columns consisting of two numbers, we obtain a lexicographic order on the set of all configurations. The configurations are listed according to that order.

In the cases $n = 6$ and $n = 7$, the $2 \times n$ - matrix representing a configuration C is formed by the first two rows of the listed $5 \times n$ - matrix. The three last rows provide us with the following additional information: Let $[x_i\ y_i\ z_i\ t_i\ u_i]^T$ be the i-th column of the listed matrix. Then z_i equals $D_C(i)$ (4.1), and $M_t(-u_i,t_i)$ is the projective cover of the socle of $M_t(y_i,x_i)^*$ if t is large enough. Here, M_t denotes the functor associated with the pattern $P = P_C$ (3.5 and 4.3).

By reflection along the vertical line through $(0,1)$ and $(-1,3)$, each configuration $\mathcal{C}$ gives rise to an <u>opposite configuration</u> $\mathcal{C}^{op}$. We call two configurations $\mathcal{C}$ and $\mathcal{D}$ <u>equivalent</u> if $\mathcal{D}$ is isomorphic to $\mathcal{C}$ or to $\mathcal{C}^{op}$, and we only list the smallest configuration of each equivalence class. Of course, lists of isomorphism classes of configurations would be larger, since there are 22, 143 and 598 isoclasses for $n = 6,7,8$ respectively.

Let us explain briefly how we established the list in case $n = 8$. Assume that we know the configurations of $\mathbb{Z}E_7$. Given such a configuration $\mathcal{C}$, we first compute the associated dimension-function $d = d_{\mathcal{C}} : (E_7)_0 \to \mathbb{N}$ (4.1). Setting $d'(1) = d(1)+1$ and $d'(i) = d(i-1)$ for $i = 2, \ldots, 8$, we clearly define the dimension-function of a configuration $\mathcal{C}'$ of $\mathbb{Z}E_8$. The map $\mathcal{C} \mapsto \mathcal{C}'$ yields a bijection between the configurations of $\mathbb{Z}E_7$ and those configurations of $\mathbb{Z}E_8$ which contain $(0,1)$. Now it is easy to prove (or to verify in the list of combinatorial configurations established by Jenni-Riedtmann) that each configuration of $\mathbb{Z}E_8$ contains some vertex $(x,1)$, $x \in \mathbb{Z}$. Our construction therefore provides us with representatives of all isoclasses. In particular, it furnishes all the listed configurations.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321000
111556
015604
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234156
333003
111346
016500
321322
325146
211000
111466
016045
432431
635421
321010
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015900
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210000
115566
019045
432341
634521
320010
111156
024617
547323
251436
453443
111166
024659
326431
524136
423121
111455
024610
435232
316245
313130
111456
026489
216322
421356
321221
115566
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331131
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054929
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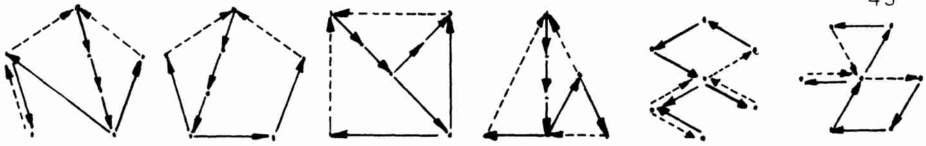
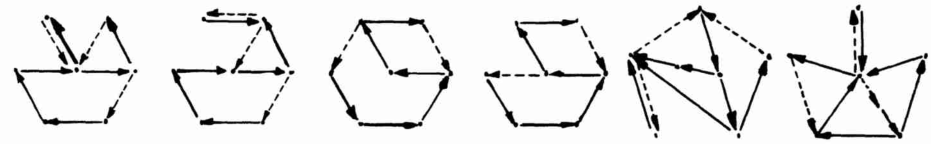
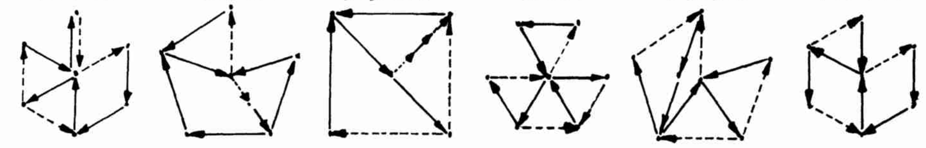


1111567	1111667	1111166	1111667	1114567	1114667
0123868	0123788	0127268	0127858	0128676	0128566
6543121	7654211	7654312	6543411	4321322	4321232
1234567	1234567	2345176	1234567	4321567	4321567
4444404	8888888	8888488	4444414	3210010	3210000



1115667	1115677	1116667	1116677	1111266	1111266
0127856	0126778	0126786	0125678	0136128	0136156
5432313	5432431	6543214	5432341	6546312	5435232
5432167	7435621	6543217	7435621	3417256	3417256
4321010	4320110	5432100	4320010	8858588	4414140



1111566	1111666	1111666	1111366	1111667	1111667
0136846	0136571	0136784	0135846	0135288	0135568
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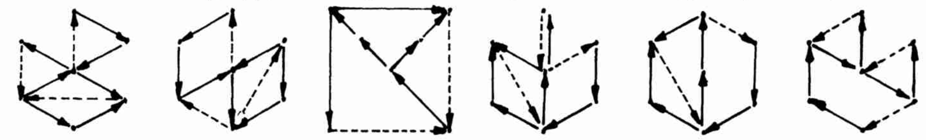

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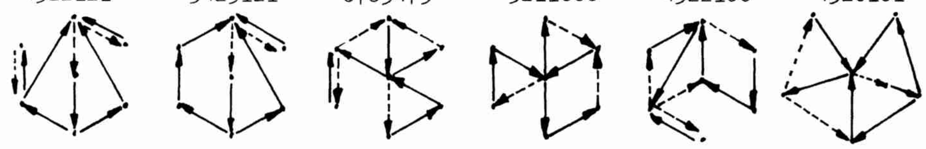

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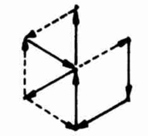

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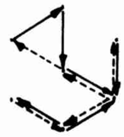

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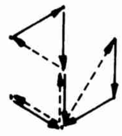

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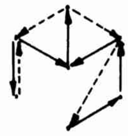

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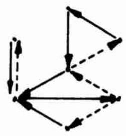

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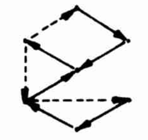

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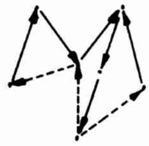

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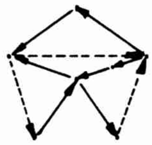

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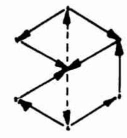

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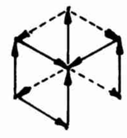

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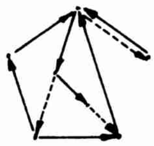

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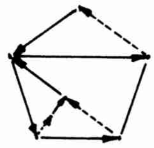

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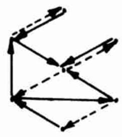

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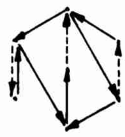

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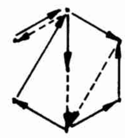

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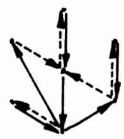

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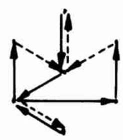

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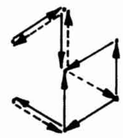

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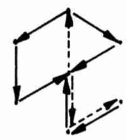

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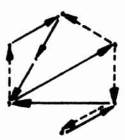

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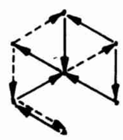

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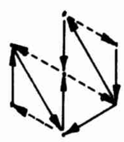

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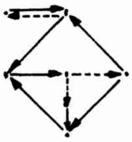			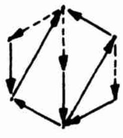		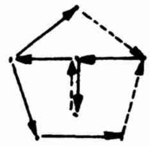
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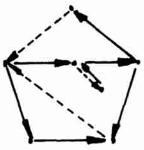		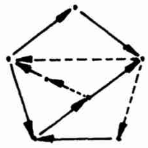	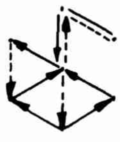	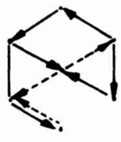	
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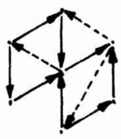	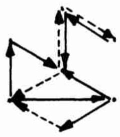		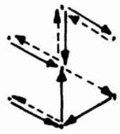	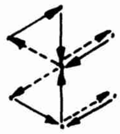	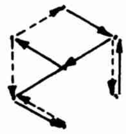
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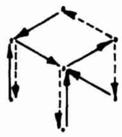		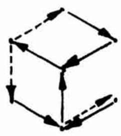			
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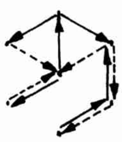		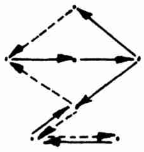	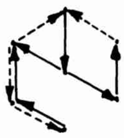		
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02683136	02683684	02686834	02686784	02808169	02829441	02891468	02031791
11112277	11112677	11112777	11112677	11112777	11112778	11112377	11112777
02036796	02036724	02036467	02037246	02037124	02086478	02284813	02284148
11112778	11117777	11117777	11123778	11127778	11122678	11122778	11124678
02284688	02294466	02299668	02248363	02244683	02346727	02346677	02347262
11124778	11126778	11127778	11127788	11123778	11124778	11127788	11225678
02347122	02342467	02341247	02341725	02361474	02372145	02379125	02468373
11225778	11226788	11226778	11227788	11227778	11235678	11235778	11236778
02468233	02463778	02467283	02462378	02466783	02483727	02483677	02482367
11236778	11236788	11237778	11237788	11111177	11111777	11111777	11112378
02483177	02487223	02481237	02486723	03603622	03602229	03602779	03607266
11112778	11111177	11111778	11111277	11112678	11112778	11123477	11123778
03607276	03636917	03636574	03669718	03667834	03667784	03672618	03672136
11112777	11112778	11111277	11111277	11112237	11112277	11111278	11111277
03707244	03707468	03719147	03719198	03714783	03714748	03735091	03735074
11112778	11112777	11112277	11112278	11122778	11111277	11111778	11112778
03797461	03791244	03754724	03754738	03747688	03031679	03062248	03067478
11112277	11112377	11112777	11117777	11112267	11112277	11122377	11123677
03016729	03017246	03017247	03022244	03114783	03114778	03147807	03148357
11123777	11112778	11125678	11125778	11126778	11126788	11127778	11127788
03148235	03361474	03614838	03614788	03613478	03618334	03612348	03617834
11122678	11122778	11124678	11124778	11126778	11127778	11127788	11234778
03647838	03647788	03648373	03648233	03643578	03642358	03642836	03483256
11237788	11111227	11111277	11111278	11112227	11112277	11112677	11112777
03480236	04826184	04071246	04071628	04127451	04261818	04261468	04261346
11122678	11122778	11124778	11127788	11123678	11123778	11123778	11127778
04246155	04246015	04614367	04611347	06726834	06726784	07748363	07744683
11123778	11223678	11223778	11227778	11234678	11234778	11236778	11237788
07837461	02461505	02461455	02467145	02483616	02483566	02486131	02480636
11237778	11245678	11245778	11246778	11246778	11246788	11247778	11247788
02483561	03472616	03472566	03471256	03472066	03476112	03470126	03475612
12345678	12345778	12346788	12346778	12347788	12347778		
04837262	04837122	04832667	04836172	04831267	04835672		

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