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Autor: Chiantini, Luca

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Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

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ON FORM IDEALS AND ARTIN-REES CONDITION

Luca Chiantini

We look for conditions which make two ideals $\mathfrak{b}$ and $\mathfrak{b}'$ in a noetherian ring A have the same form ideal in an associated graded ring $G_A(\mathfrak{a})$. More precisely, when $\mathfrak{b} = (f_1,\dots,f_n)$ and $\mathfrak{b}' = (f_1',\dots,f_n')$ and $f_i - f_i' \in \mathfrak{a}^m$ $m \gg 0$, $\forall i$, we give a necessary and sufficient condition to have $\mathfrak{b}^* = (\mathfrak{b}')^*$, involving the first syzygies modules both of $(f_1,\dots,f_n)$ and $(f_1',\dots,f_n')$; our proof is based on the Artin-Rees lemma. Finally we show that, when the sequence $f_1,\dots,f_n$ is regular and $\mathfrak{a}^{r+q} \cap \mathfrak{b} = \mathfrak{a}^r(\mathfrak{a}^q \cap \mathfrak{b})$ for an integer q, then $f_i - f_i' \in \mathfrak{a}^{q+1}$ $\forall i$ implies $\mathfrak{b}^* = (\mathfrak{b}')^*$.

Introduction

Let A be a noetherian ring, $\mathfrak{a}$ and $\mathfrak{b} = (f_1,\dots,f_n)$ two ideals, then it is defined the form ideal $\mathfrak{b}^*$ as the ideal generated by all the initial forms of elements of $\mathfrak{b}$

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within the form ring $G_A(\alpha)$. If $f_1',\dots,f_n' \in A$ and moreover $f_i - f_i' \in \alpha^m$, $m \gg 0$, then the initial forms f_i^* and $(f_i')^*$ must coincide, so that we could expect the equality: $\mathfrak{b}^* = (f_1,\dots,f_n)^* = (f_1',\dots,f_n')^* = (\mathfrak{b}')^*$. In other words, a suitable approximation of the f_i's in the α-adic topology could be expected to give the same form ideal as the f_i's give (for instance, when α is a prime ideal and $\mathfrak{b} \subseteq \alpha$, then $\mathfrak{b}^*$ is just the ideal which defines the tangent cone of $\mathfrak{b}$ along α, thus if each f_i' is "very close" to f_i in the α-adic topology, $\mathfrak{b}$ and $\mathfrak{b}'$ could be expected to have the same tangent cone).

Unfortunately, this fact is definitely false (as our example (2.3) shows); such an equality fails because of the fact that $f_1^*,\dots,f_n^*$ do not generate $\mathfrak{b}^*$, unless some special conditions are fulfilled, like being a regular sequence (see [6], prop. (2.1)).

The above considerations give rise to the present paper, whose aim is the investigation of the following problem: when does the equality $(f_1,\dots,f_n)^* = (f_1',\dots,f_n')^*$ hold?

The research of an integer m such that, if $f_i - f_i' \in \alpha^m$ $\forall\, i = 1,\dots,n$ then $\mathfrak{b}^* = (\mathfrak{b}')^*$, leads us to the Artin-Rees lemma, which is the main tool in the paper; in fact, define the integer q to be an Artin-Rees number for $\mathfrak{b}$ if and only if, for all $r \geq 0$, $\alpha^{r+q} \cap \mathfrak{b} = \alpha^r(\alpha^q \cap \mathfrak{b})$, thus it is easy to see that, if $\forall i$ $f_i - f_i' \in \alpha^{q+1}$, then $\mathfrak{b}^* = (\mathfrak{b}')^*$ if and only if q is also an Artin-Rees number for $\mathfrak{b}'$.

Therefore we give a necessary and sufficient condition for q to be a common Artin-Rees number for $\mathfrak{b}$ and $\mathfrak{b}'$; such a condition strongly involves the behaviour of the first syzygies modules both of $\mathfrak{b}$ and of $\mathfrak{b}'$. There are remarkable cases of existence of a common Artin-Rees number q both for $\mathfrak{b}$ and $\mathfrak{b}'$: for instance, when the syzygies of $(f_1,\dots,f_n)$ are trivial; but it may happen (and we produce an example) that no q is good for our approximation, i.e. it is not true in general that, if $f_i - f_i' \in \alpha^m$, $m \gg 0$ then $\mathfrak{b}^* = (\mathfrak{b}')^*$. We are also able to show with an example that, even when $\mathfrak{b}$ and $\mathfrak{b}'$ have a common Artin-Rees number, it is not true that any Artin-Rees number for $\mathfrak{b}$ is good, being necessary to find a large enough q.

Therefore we are led to the sharper question: when does every Artin-Rees number q for $\mathfrak{b}$ work also for $\mathfrak{b}'$, provided $f_i - f_i' \in \alpha^{q+1}$ $\forall i$? We show that, if $f_1,\dots,f_n$ is a regular sequence (more generally, if it has only trivial syzygies, i.e. if $H_1K(f_1,\dots,f_n;A) = 0$) in an arbitrary noetherian ring, then the answer is positive. We remark that, to deal with form ideals of regular sequences (at least in a local ring) we need to prove the following fact: if $\forall i$ $f_i - f_i' \in \alpha^{q+1}$, q Artin-Rees number for $\mathfrak{b}$ and $f_1,\dots,f_n$ is a regular sequence, then also the sequence $f_1',\dots,f_n'$ is regular.

In section 0 we recall some notion which will be used in the paper; in section 1 we prove some more or less

well known fact about the Artin-Rees lemma and related topics; section 2 is mainly devoted to prove our necessary and sufficient condition to have $(\mathfrak{a}')^* = \mathfrak{a}^*$ when $\forall i$, $f_i - f_i' \in \mathfrak{a}^m$, $m \gg 0$; it contains also some fundamental example. Section 3 contains our results about the case of an ideal generated by a regular sequence.

0. Recalls and notations

All rings are commutative and noetherian.

1) Let $f_1,\dots,f_n$ be elements of a ring A. We say that they form a "regular sequence" if:

i) the ideal $(f_1,\dots,f_n)$ is proper;

ii) f_1 is not a 0-divisor and $\forall i \geq 2$ f_i is not a 0-divisor mod$(f_1,\dots,f_{i-1})$ (0 itself is considered a 0-divisor).

The notion of "regular sequence" depends on the order of the f_i's, except for local rings (see [4], p. 102).

2) Let $f_1,\dots,f_n$ be elements of A; the "(first) syzygies module" S of $f_1,\dots,f_n$ is the submodule of the free A-module A^n generated by all n-tuples $(a_1,\dots,a_n)$ such that $\sum a_i f_i = 0$ (see [7], vol. II, p. 237).

Put $\delta_{i,j} = (0,..,0,\ \underset{j\text{-th place}}{-f_j},0,..,0,\ \underset{i\text{-th place}}{f_i},0,..,0)$;

then all the $\delta_{i,j}$'s belong to S; they are called "trivial syzygies" and span the "trivial syzygies module" T. The quotient S/T is just the first homology module for the Koszul complex $K(f_1,\dots,f_n;A)$. Then $H_1K(f_1,..,f_n;A)=0$

means that all the syzygies of $f_1,\dots,f_n$ are trivial.

If $f_1,\dots,f_n$ is a regular sequence or $(f_1,\dots,f_n)$ is the improper ideal, then $H_1K(f_1,\dots,f_n;A) = 0$; the converse is false in general, unless A is local.

If $\varphi : A \longrightarrow B$ is a flat morphism, then $H_1K(f_1,..,f_n;B) = H_1K(f_1,..,f_n;A) \otimes_A B$; then $H_1K(f_1,\dots,f_n;A) = 0$ implies $H_1K(f_1,\dots,f_n;B) = 0$ while the converse holds if φ is faithfully flat; moreover $H_1K(f_1,\dots,f_n;A) = 0$ if and only if, for every maximal ideal $\mathfrak{m}$ of A $H_1K(f_1,\dots,f_n;A_{\mathfrak{m}}) = 0$ (for all the facts quoted above, we refer to [5], ch. 8 or [4], § 18).

3) Let us consider the associated graded ring of A relative to an ideal $\mathfrak{a}$: $G_A(\mathfrak{a}) = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$; it is a graded A-algebra (see [7], vol. II, p. 248 for more details on the construction). If $x \in A$ and $x \in \mathfrak{a}^r - \mathfrak{a}^{r+1}$, we set : $x^* = x \bmod \mathfrak{a}^{r+1} \in G_A(\mathfrak{a})$ and call it "the initial form of x relative to $\mathfrak{a}$"; if $x \in \bigcap_{n=0}^{\infty} \mathfrak{a}^n$, put $x^* = 0$; note that, when $x^*y^* \neq 0$, then $x^*y^* = (xy)^*$. If $\mathfrak{b}$ is an ideal of A, let $\mathfrak{b}^*$ denote the ideal of $G_A(\mathfrak{a})$ generated by all the initial forms of elements of $\mathfrak{b}$; it is called "the form ideal of $\mathfrak{b}$ relative to $\mathfrak{a}$". $\mathfrak{b}^*$ is also the kernel of the canonical epimorphism $G_A(\mathfrak{a}) \longrightarrow G_{A/\mathfrak{b}}(\mathfrak{a}+\mathfrak{b}/\mathfrak{b})$. $\mathfrak{b}^*$ is a homogeneous ideal; we denote with $\mathfrak{b}^*_q$ its homogeneous part of degree q (see [3], ch. 2, § 2).

1. Some remarks on Artin-Rees lemma

We are going to look for conditions which make two ideals have the same form ideal. In this research we need a careful examination of the Artin-Rees lemma, whose consequences are used all through the paper. We collect here some facts on the lemma; many of them are more or less well known in the literature but they have never been explicitely written down.

ARTIN-REES LEMMA. Let A be a ring and $\alpha \subseteq A$ an ideal; if E is a finite A-module and $F \subseteq E$ is a submodule, then there is an integer q such that $\forall\, r \geqslant 0$

$$F \cap \alpha^{q+r}E = \alpha^{r}(\alpha^{q}E \cap F)$$ (see [7], vol. II, p. 255).

DEFINITION (1.1). Such a q is called "Artin-Rees number" (shortly: A.R.n., since we are going to use it all through the paper) for F relative to α.

Note that if q is an A.R.n. for F relative to α, the same holds for every $q' \geqslant q$.

The Artin-Rees numbers and the form ideal of an ideal are strictly related by the following:

PROPOSITION (1.2). Let α and $\mathfrak{b}$ be ideals of a ring A. Then q is an A.R.n. for $\mathfrak{b}$ relative to α if and only if $\mathfrak{b}^*$ is generated by forms of degree not greater than q in $G_A(\alpha)$.

Proof. Let $f_1,\dots,f_n \in \mathfrak{b}$ be elements such that $\mathfrak{b}^* = (f_1^*,\dots,f_n^*)$ with degree $f_i = d_i \leqslant q$. Then $\forall\, r \geqslant 0$

$$\alpha^{q+r} \cap \mathfrak{b} = \sum_{i=1}^{n} \alpha^{q+r-d_i} f_i = \alpha^r \left(\sum_{i=1}^{n} \alpha^{q-d_i} f_i\right) \subseteq \alpha^r (\alpha^q \cap \mathfrak{b})$$

thus $\alpha^{r+q} \cap \mathfrak{b} = \alpha^r (\alpha^q \cap \mathfrak{b})$.

Conversely, if q is an A.R.n. for $\mathfrak{b}$ then:

$$\mathfrak{b}^*_{r+q} = (\mathfrak{b} \cap \alpha^{r+q} + \alpha^{r+q+1}) / \alpha^{r+q+1} =$$

$$= (\alpha^r (\alpha^q \cap \mathfrak{b}) + \alpha^{r+q+1}) / \alpha^{r+q+1} =$$

$$= (\alpha^r / \alpha^{r+1})((\alpha^q \cap \mathfrak{b} + \alpha^{q+1}) / \alpha^{q+1}) = G_A(\alpha)_r \mathfrak{b}^*_q$$ and we are done.

COROLLARY (1.3). Let $\mathfrak{b}$ and $\mathfrak{b}'$ be ideals such that $\mathfrak{b} \equiv \mathfrak{b}'$ mod α^{q+1} where q is an A.R.n. for $\mathfrak{b}$ relative to α. Then the following are equivalent:

a) q is an A.R.n. for $\mathfrak{b}'$ relative to α;

b) in $G_A(\alpha)$, $\mathfrak{b}^* = (\mathfrak{b}')^*$.

Proof. a) $\Rightarrow$ b); $\forall\, p \leq q$, $\mathfrak{b}^*_p = (\mathfrak{b} \cap \alpha^p + \alpha^{p+1}) / \alpha^{p+1} =$

$$= (\alpha^p \cap (\mathfrak{b} + \alpha^{p+1})) / \alpha^{p+1} = \alpha^p / \alpha^{p+1} \cap (\mathfrak{b} + \alpha^{p+1} / \alpha^{p+1}) =$$

$$= \alpha^p / \alpha^{p+1} \cap (\mathfrak{b}' + \alpha^{p+1} / \alpha^{p+1}) = (\mathfrak{b}')^*_p;$$ since by prop. (1.2) $\mathfrak{b}^*$ and $(\mathfrak{b}')^*$ are generated by forms of degree $\leq q$, we have $\mathfrak{b}^* = (\mathfrak{b}')^*$.

b) $\Rightarrow$ a); By assumptions, $(\mathfrak{b}')^*$ is generated by forms of degree $\leq q$, since $\mathfrak{b}^*$ is by prop. (1.2), hence by prop. (1.2) again, we are done.

PROPOSITION (1.4). Let F, F' be submodules of a finite A-module E. Then the following are equivalent:

a) $\forall r$, $F \cap \alpha^r E \subseteq F' + \alpha^{r+1} E$;

b) in $G_E(\alpha) = \bigoplus_{s=0}^{\infty} \alpha^s E / \alpha^{s+1} E$, we have $F^* \subseteq (F')^*$.

Proof. a) $\Rightarrow$ b); If $a \in (F \cap \alpha^r E) - \alpha^{r+1}E$ for some r then $a \in \alpha^{r+1}E + F'$ hence $a = a' + u$, $a' \in F'$, $u \in \alpha^{r+1}E$ thus $(a')^* = a^*$; if $a \in (\bigcap_{s=0}^{\infty} \alpha^s E) \cap F$ then $a^* = 0$; it follows $F^* \subseteq (F')^*$.

b) $\Rightarrow$ a); If $a \in F \cap \alpha^r E$, choose $a' \in F'$ with $a^* = (a')^*$; then $a - a' \in \alpha^{r+1}E$ and we are done.

REMARK (1.5). The following statements may be proved by straightforward computations, using [1], ch. 10, and [2], ch. II, p. 112. Let α, β be ideals of A;

a) let $J \subseteq \beta$ be another ideal and put $\bar{A} = A/J$, $\bar{\beta} = \beta/J$ $\bar{\alpha} = \alpha\bar{A} = (\alpha+J)/J$. Every A.R.n. for β relative to α is also an A.R.n. for $\bar{\beta}$ relative to $\bar{\alpha}$.

b) let $\varphi : A \longrightarrow B$ be a flat morphism. Every A.R.n. for β relative to α is also an A.R.n. for βB relative to αB.

c) q is an A.R.n. for β relative to α if and only if for every maximal ideal $\mathfrak{m}$ of A, q is an A.R.n. for $\beta A_{\mathfrak{m}}$ relative to $\alpha A_{\mathfrak{m}}$.

EXAMPLE (1.6). An A.R.n. for $\beta = (f_1, \ldots, f_n)$ relative to α need not be an A.R.n. for every ideal (f_i), neither in the case of a regular sequence $f_1, \ldots, f_n$ in a local ring A with $\alpha = \mathfrak{m}$ = maximal ideal of A.

Namely, put $A = K[\![X,Y,Z]\!]/(XY - Z^3) = K[\![x,y,z]\!]$; it is easy to check that A is a domain and $f_1 = x$, $f_2 = y^3$ is a regular sequence; moreover, by mean of [6], prop. (2.1), a direct computation shows that in $G_A(\alpha) = K[t_1, t_2, t_3]$ where $t_1 = x^*$, $t_2 = y^*$, $t_3 = z^*$, we have $\beta^* = (t_1, t_2^3, t_3^3)$,

therefore, by prop. (1.2), 3 is an A.R.n. for $\mathfrak{b}$. On the other hand 3 is not an A.R.n. for (f_2), indeed $x^3y^3 = z^9 \in$ $\in \mathfrak{m}^9 \cap (f_2)$ while $x^3y^3 \notin \mathfrak{m}^6(\mathfrak{m}^3 \cap (f_2))$ since $x^3 \notin \mathfrak{m}^6$.

However, if $f_1,..,f_n$ is a regular sequence in a local ring A and $\mathfrak{b}^* = (f_1^*,..,f_n^*)$, then by [6], th. 1.1 and prop. 2.6, an A.R.n. for $\mathfrak{b}$ is also an A.R.n. for every (f_i).

2. On the equality of the form ideals of two ideals; the general case

In our research of conditions which make two ideals have the same form ideal, corol. (1.4) of § 1 suggests to investigate in the following direction: let $\mathfrak{b} = (f_1,..,f_n)$ and α be ideals of a ring A and let q be an A.R.n. for $\mathfrak{b}$ relative to α. If $f_1',..,f_n'$ are elements of A such that $\forall i\ f_i - f_i' \in \alpha^{q+1}$, when is q also an A.R.n. for $\mathfrak{b}' =$ $= (f_1',..,f_n')$ so that $\mathfrak{b}^* = (\mathfrak{b}')^*$? More precisely, we give:

DEFINITION (2.1). Let $f_1,..,f_n$ be elements of A. We say that the n-tuple $(f_1',..,f_n') \in A^n$ is an " α-adic m-approximation" of $(f_1,..,f_n)$ if $\forall i\ f_i - f_i' \in \alpha^{m+1}$. We say also, by abuse, that the ideal $\mathfrak{b}'$ is an " α-adic m-approximation" of $\mathfrak{b} = (f_1,..,f_n)$ if $\mathfrak{b}'$ is generated by elements $f_1',..,f_n'$ which form an α-adic m-approximation of $f_1,..,f_n$.

Then we may ask: if q is an A.R.n. for $\mathfrak{b}$ and $\mathfrak{b}'$ is an α-adic q-approximation of $\mathfrak{b}$, when is q also an A.R.n; for $\mathfrak{b}'$?

The natural answer: any A.R.n. q for $\mathfrak{b}$ works, is easily

belied by the following:

EXAMPLE (2.2). Put $R = K[X,Y][Z]$, $A = R/(Z^4(1-YX^3)) =$ $= K[x,y][z]$; $(Z^4(1-YX^3))$ is a homogeneous ideal for the grading of R given by Z, hence also A is graded by z. Put $\mathfrak{b} = (x^2z^2, xyz^2)$, $\mathfrak{a} = (z)$; then one may check directly that 2 is an A.R.n. for $\mathfrak{b}$ relative to $\mathfrak{a}$. Now an easy computation shows that in $G_A(\mathfrak{a}) = K[t_1,t_2][t_3]$ graded by t_3, where $t_1 = x^*$, $t_2 = y^*$, $t_3 = z^*$, we have: $\mathfrak{b}^* =$ $= ((x^2z^2)^*, (xyz^2)^*) = (t_1^2t_3^2, t_1t_2t_3^2)$.

Put $\mathfrak{b}' = (x^2z^2+z^3, xyz^2+z^3)$; then $\mathfrak{b}'$ is an $\mathfrak{a}$-adic 2-approximation of $\mathfrak{b}$, but $\mathfrak{b}^* \neq (\mathfrak{b}')^*$ indeed $(\mathfrak{b}')^*$ contains the element $(y(x^2z^2 + z^3) - x(xyz^2 + z^3))^* = ((y-x)z^3)^* = (t_2 - t_1)t_3^3$ which does not belong to $\mathfrak{b}^*$, as one can easily see.

Nevertheless, if $m \geq 3$, since $z^4 \in \mathfrak{b}$, if $\mathfrak{b}'$ is an $\mathfrak{a}$-adic m-approximation of $\mathfrak{b}$ then $\mathfrak{b}' \subseteq \mathfrak{b}$ hence $(\mathfrak{b}')^* \subseteq \mathfrak{b}^*$ while it is clear that $t_1^2t_2^2$ and $t_1t_2t_3^2$ belong to $(\mathfrak{b}')^*$, hence $(\mathfrak{b}')^* \supseteq \mathfrak{b}^*$. It follows that, for $m \geq 3$, for every $\mathfrak{a}$-adic m-approximation $\mathfrak{b}'$ of $\mathfrak{b}$ we have $\mathfrak{b}^* = (\mathfrak{b}')^*$.

The example suggests that the following could be true: for $m \gg 0$, if $\mathfrak{b}'$ is an $\mathfrak{a}$-adic m-approximation of $\mathfrak{b}$, then $\mathfrak{b}^* = (\mathfrak{b}')^*$. But next example shows that this is false:

EXAMPLE (2.3). Put $A = K[[X,Y,Z]]/(XY) = K[[x,y,z]]$, $\mathfrak{a} = (z)$, $\mathfrak{b} = (x)$; it is easy to see that 0 is an A.R.n. for $\mathfrak{b}$ relative to $\mathfrak{a}$; we have $G_A(\mathfrak{a}) = K[[T_1,T_2]][T_3]/(T_1T_2) =$ $= K[[t_1,t_2]][t_3]$ and $\mathfrak{b}^* = (t_1)$.

Choose $f' \in A$ such that $f' - x \in \alpha$, i.e. $f' = x + a_1(x,y)z + a_2(x,y)z^2 + \ldots$; then (f') is an α-adic 0-approximation of $\mathcal{Z}$; suppose $(x)^* = (f')^* = (t_1)$, then a straightforward computation shows that $a_i(x,y) \in (x)$ $\forall i$, so $f' = x(1 + a_1'(x,y)z + a_2'(x,y)z^2 + \ldots) = xu$ with u unit. It follows that for all $m \geq 0$, for an α-adic m-approximation $\mathcal{Z}' = (f')$ of $\mathcal{Z}$, $\mathcal{Z}^* = (\mathcal{Z}')^*$ if and only if $\mathcal{Z} = \mathcal{Z}'$.

If we put $\forall n$ $f_n = x + z^n$, then in the α-adic topology, $x = \lim\{f_n\}$ hence we can choose f_n with $f_n - x \in \alpha^n$, n as large as we want, but never $(x) = (f_n)$ hence never $(x)^* = (f_n)^*$.

Further, since A is complete in its α-adic topology, if $\mathcal{Z}'$ is an ideal with $\mathcal{Z}^* = (\mathcal{Z}')^* = (t_1)$, then by [7], vol. II th. 7, p. 279, $\mathcal{Z}'$ is generated by any of its elements f' such that $(f')^* = t_1 = x^*$ (i.e. such that $f' - x \in \alpha$). It follows by the previous argument that for every ideal $\mathcal{Z}'$ $\mathcal{Z}^* = (\mathcal{Z}')^*$ if and only if $\mathcal{Z} = \mathcal{Z}'$; thus this example provides an ideal which is characterized by its form ideal.

Hence we now give:

THEOREM (2.4). Let α, $\mathcal{Z} = (f_1, \ldots, f_n)$ and $\mathcal{Z}' = (f_1', \ldots, f_n')$ be ideals of a ring A and let q be an A.R.n. for $\mathcal{Z}$ relative to α; suppose $\forall i$ $f_i - f_i' \in \alpha^{q+1}$. Let S and S' be the (first) syzygies modules of $f_1, \ldots, f_n$ and $f_1', \ldots, f_n'$ respectively; then the following are equivalent:

i) q is an A.R.n. for $\mathcal{Z}'$ relative to α;

ii) $\mathcal{Z}^* = (\mathcal{Z}')^*$ in $G_A(\alpha)$;

iii) $\forall r\ S \cap \alpha^r A^n \subseteq S' + \alpha^{r+1} A^n$;

iv) in $G_{A^n}(\alpha)$, $S^* = (S')^*$.

Proof. i)$\Leftrightarrow$ii); Since our assumptions imply $\mathcal{Z} \equiv \mathcal{Z}'$ mod α^{q+1}, this is just corol. (1.3).

i)$\Rightarrow$iii); If $(a_1,\dots,a_n) \in S \cap \alpha^r A^n$ then $\forall i\ a_i \in \alpha^r$ and $\sum a_i f_i = 0$ hence $\sum a_i f_i' = \sum a_i (f_i' - f_i) \in \alpha^{r+q+1} \cap \mathcal{Z}' =$ $= \alpha^{r+1}(\alpha^q \cap \mathcal{Z}')$ by i), thus $\sum a_i f_i' = \sum_j h_j (\sum_i u_{i,j} f_i') =$ $= \sum_i (\sum_j h_j u_{i,j}) f_i'$ where $\forall j\ h_j \in \alpha^{r+1}$ so $b_i =$ $= \sum_j h_j u_{i,j} \in \alpha^{r+1}$. Then $\sum (a_i - b_i) f_i' = 0$ and $(a_1,..,a_n) = ((a_1 - b_1) + b_1, \dots, (a_n - b_n) + b_n) \in S' + \alpha^{r+1} A^n$.

iii)$\Rightarrow$ i); We only need to show that $\alpha^{q+r} \cap \mathcal{Z}' \subseteq \alpha^r (\alpha^q \cap \mathcal{Z}')$ $\forall r$. We prove it in several steps.

STEP I. Make induction on r; the case r=0 is trivial; for the general case, pick an element $\sum a_i f_i' \in \alpha^{r+q}$; we must prove that $\sum a_i f_i' \in \alpha^r (\alpha^q \cap \mathcal{Z}')$.

STEP II. First we may assume that $\forall i\ a_i \in \alpha^{r-1}$, in fact $\sum a_i f_i' \in \alpha^{q+r-1} \cap \mathcal{Z}' = \alpha^{r-1}(\alpha^q \cap \mathcal{Z}')$ by induction, so that $\sum a_i f_i' = \sum h_j g_j$ where $\forall j\ h_j \in \alpha^{r-1}$ and $g_j = \sum c_{i,j} f_i' \in \alpha^q \cap \mathcal{Z}'$.

STEP III. We may assume that $(a_1,..,a_n) \in S \cap \alpha^{r-1} A^n$. Namely we have $\sum a_i (f_i - f_i') \in \alpha^{r-1} \alpha^{q+1} = \alpha^{q+r}$ hence $\sum a_i f_i \in \alpha^{q+r} \cap \mathcal{Z} = \alpha^r (\alpha^q \cap \mathcal{Z})$ i.e. $\sum a_i f_i =$ $= \sum_j h_j (\sum_i u_{i,j} f_i) = \sum_i (\sum_j h_j u_{i,j}) f_i$ where $h_j \in \alpha^r\ \forall j$ hence $b_i = \sum_j h_j u_{i,j} \in \alpha^r\ \forall i$. Now we see that: $\sum (a_i - b_i) f_i' - \sum a_i f_i' \in \alpha^r (\alpha^q \cap \mathcal{Z}')$ so that we may replace a_i by $(a_i - b_i)$ $\forall i$ and this gives also $\sum a_i f_i = 0$,

i.e. the required condition. We have now:

$$\sum a_i f_i' \in \alpha^{r+q};\ (a_1,\dots,a_n) \in S \cap \alpha^{r-1}A^n. \quad (*)$$

STEP IV. By iii), we have $(a_1,\dots,a_n) \in S' + \alpha^r A^n$, hence $(a_1,\dots,a_n) = (b_1+u_1,\dots,b_n+u_n)$ with $b_i \in \alpha^r$ $\forall i$ and $(u_1,\dots,u_n) \in S'$ so that $\sum a_i f_i' = \sum b_i f_i'$ and replacing the a_i's with the b_i's we may suppose $\forall i\ a_i \in \alpha^r$.

STEP V. Just as in step III, we see that now $\sum a_i f_i \in$ $\in \alpha^{r+q} \cap \mathfrak{b} = \alpha^r(\alpha^q \cap \mathfrak{b})$, hence $\sum a_i f_i = \sum_j k_j (\sum_i v_{i,j} f_i =$ $= \sum_i (\sum_j k_j v_{i,j}) f_i$ where $\forall j\ k_j \in \alpha^r$ hence $c_i =$ $= \sum_j k_j v_{i,j} \in \alpha^r$, so that $\sum_i (a_i - c_i) f_i' =$ $= \sum_i (a_i - c_i)(f_i' - f_i) \in \alpha^{q+r+1}$. If we replace a_i by $a_i - c_i$ $\forall i$, we have the situation:

$$\sum a_i f_i' \in \alpha^{q+r+1},\quad (a_1,\dots,a_n) \in S \cap \alpha^r A^n$$

which is similar to situation $(*)$ but with the exponents of α increased by 1.

STEP VI. Repeating step IV and step V a suitable number of times, we finally have an element $\sum a_i f_i'$ such that $\forall i$ $a_i \in \alpha^{q+r}$ and it is clear that such an element belongs to $\alpha^r(\alpha^q \cap \mathfrak{b}')$.

iv)$\Rightarrow$iii); This is just an application of prop. (1.4).

iii)$\Rightarrow$iv); By prop. (1.4) we have $S^* \subseteq (S')^*$, but iii)$\Rightarrow$i) and i) says that we may interchange the roles of $\mathfrak{b}$ and $\mathfrak{b}'$ hence we have also $S' \cap \alpha^r A^n \subseteq S + \alpha^{r+1}A^n$ $\forall r$ and by prop. (1.4) again, $(S')^* \subseteq S^*$, thus iv) holds.

REMARK (2.5). Let q' be an A.R.n. for S relative to α;

then if condition iii) works for all $r \leqslant q'$, it works also $\forall r$; namely in this case, if $r > q'$, $S \cap \alpha^r A^n = \alpha^{r-q'}(\alpha^{q'} A^n \cap S) \subseteq \alpha^{r-q'}(\alpha^{q'+1} A^n + S') \subseteq S' + \alpha^{r-1} A^n$. Of course it may be uneasy to find an A.R.n. q' for S.

COROLLARY (2.6). If $S \subseteq S'$ then condition iii) holds, hence $\mathcal{b}^* = (\mathcal{b}')^*$.

The general phylosophy indeed is that, for $m \gg 0$, an α-adic m-approximation $\mathcal{b}'$ of $\mathcal{b}$ satisfies condition iii) (hence has the same form ideal of $\mathcal{b}$) when S is small with respect to S'.

Let us examinate more closely the case of a principal ideal $\mathcal{b} = (f)$ which will be often used in section 3; here condition iii) becomes:

iii)' $\forall r \ \mathrm{Ann}(f) \cap \alpha^r \subseteq \alpha^{r+1} + \mathrm{Ann}(f')$

COROLLARY (2.7). Let f and f' be elements of A such that $f - f' \in \alpha^{q+1}$ where q is an A.R.n. for (f) relative to α. Then condition iii)' holds when:

(a) $\mathrm{Ann}(f) \subseteq \mathrm{Ann}(f') + \bigcap_{s=0}^{\infty} \alpha^s$ hence in particular when:

(b) $\mathrm{Ann}(f) \subseteq \mathrm{Ann}(f')$

(c) f is not a 0-divisor.

In all those cases we have $(f)^* = (f')^*$.

REMARK (2.8). It is easy to check that many facts quoted in example (2.2) and example (2.3) may be proved in an easier way using th. (2.4).

3. The case of a regular sequence

In this section we examinate more closely the case of a regular sequence; more precisely, if $\mathcal{b} = (f_1,..,f_n)$ with $f_1,...,f_n$ regular sequence, using th. (2.4) one can show with a simple argument that every α-adic m-approximation of $\mathcal{b}$, for $m \gg 0$, has in $G_A(\alpha)$ the same form ideal of $\mathcal{b}$. However we want to sharpen this result: really we want to prove that when the sequence $f_1,..,f_n$ is regular, every α-adic q-approximation, with q arbitrary A.R.n. for $\mathcal{b}$, works. More generally we prove that this is true when all the syzygies of $\mathcal{b}$ are trivial.

We start with the local case. The problem is strictly connected with the fact that, under some assumption, approximating "very closely" a regular sequence in the α-adic topology, we get a new regular sequence.

PROPOSITION (3.1). Let α be an ideal of a ring A such that $\bigcap_{s=0}^{\infty} \alpha^s = (0)$ and let $f \in A$ be a non 0-divisor; let q be an A.R.n. for (f) relative to α and let f' be such that $f - f' \in \alpha^{q+1}$. Then f' is not a 0-divisor.

Proof. If $af' = 0$ and $a \neq 0$ then $a \in \alpha^r - \alpha^{r+1}$ for some r and $af = a(f - f') \in \alpha^{q+r+1} \cap (f) = \alpha^{r+1}(\alpha^q \cap (f)) \subseteq \alpha^{r+1} f$ which implies $a \in \alpha^{r+1}$, absurd.

REMARK (3.2). Note that, by corol. (2.7), the assumptions of prop. (3.1) imply $(f)^* = (f')^*$ in $G_A(\alpha)$.

EXAMPLE (3.3). The hypothesis $\bigcap_{s=0}^{\infty} \alpha^s = (0)$ is necessary in

prop. (3.1), indeed put $A = K[X,Y,Z]/(YZ, Z-ZX) = K[x,y,z]$, $\alpha = (x,y,z)$ and consider $f = y+z$ and $f' = y$; f is not a 0-divisor and $f' - f = z \in \bigcap_{s=0}^{\infty} \alpha^s$ since $z=zx=zx^2\ldots$ hence $f' - f \in \alpha^{q+1}$ for an A.R.n. q of (f) (for all such numbers) but y is a 0-divisor.

COROLLARY (3.4). Let $f_1,..,f_n$ be a regular sequence in a local ring $(A, \mathfrak{m})$ and let q be an A.R.n. for $\mathcal{b} = (f_1,..,f_n)$ relative to a proper ideal α. If f' is an element such that $f' - f_n \in \alpha^{q+1}$ then $f_1,\ldots,f_{n-1},f'$ is also a regular sequence.

Proof. $\bar{A} = A/(f_1,..,f_{n-1})$ is a local ring; put $\bar{\alpha} = \alpha\bar{A}$, then $\bigcap_{s=0}^{\infty} \bar{\alpha}^s = (0)$, $\bar{f}_n = f_n \mod(f_1,..,f_{n-1})$ is not a 0-divisor and, by remark (1.5)a), q is an A.R.n. for $(\bar{f}_n)$ relative to $\bar{\alpha}$, hence by prop. (3.1) $\bar{f}' = f' \mod(f_1,..,f_{n-1})$ is not a 0-divisor and we are done.

LEMMA (3.5). Let $f_1,\ldots,f_n$ be a regular sequence in a ring A and let q be an A.R.n. for $\mathcal{b} = (f_1,..,f_n)$ relative to an ideal α. Let f' be an element such that $f_n - f' \in \alpha^{q+1}$. Then $\forall r\ (f_1,..,f_{n-1}) \cap \alpha^{r+q} \subseteq \alpha^r(\alpha^q \cap (f_1,..,f_{n-1},f'))$.

Proof. Put $\mathcal{b}' = (f_1,..,f_{n-1},f')$ and pick an A.R.n. q' for $\mathcal{b}'$. If $q' \leq q$ we are done, hence suppose $q' > q$. Let $w = a_1f_1+\ldots+a_{n-1}f_{n-1} \in \alpha^{r+q}$, then we must prove $w \in \alpha^r(\alpha^q \cap \mathcal{b}')$.

By assumptions, $w \in \alpha^r(\alpha^q \cap \mathcal{b})$ hence write $w = \sum_j h_j(\sum_i u_{i,j}f_i)$ where $\forall j\ h_j \in \alpha^r$ and $\sum_i u_{i,j}f_i \in \alpha^q$ hence $\forall j\ d_j = u_{1,j}f_1+\ldots+u_{n-1,j}f_{n-1}+u_{n,j}f' \in \alpha^q \cap \mathcal{b}'$ thus

$g = \sum_j h_j d_j \in \alpha^r(\alpha^q \cap \mathcal{b}')$ and we may restrict us to prove the assertion for $w' = w-g = (\sum_j h_j u_{n,j})(f_n - f') \in$ $\in \alpha^r \alpha^{q+1} = \alpha^{r+q+1}$; moreover $(\sum_j h_j u_{n,j}) f_n \in (f_1,..,f_{n-1})$ since w does, hence by the regularity of the sequence $f_1,\ldots,f_n$, $(\sum_j h_j u_{n,j}) \in (f_1,\ldots,f_{n-1})$ so that $w' \in (f_1,\ldots,f_{n-1}) \cap \alpha^{q+r+1}$.

Repeating the argument, it is clear that in finitely many steps we may restrict us to an element $w_0 \in$ $\in (f_1,\ldots,f_{n-1}) \cap \alpha^{q'+r}$ and clearly $w_0 \in \alpha^r(\alpha^{q'} \cap \mathcal{b}') \subseteq$ $\subseteq \alpha^r(\alpha^q \cap \mathcal{b})$.

Now we are ready to prove:

THEOREM (3.6). Let $(A, \mathfrak{m})$ be a local ring, $f_1,\ldots,f_n$ a regular sequence, α a proper ideal, q an A.R.n. for $\mathcal{b} = (f_1,..,f_n)$ relative to α. If $\forall i\ f_i - f_i' \in \alpha^{q+1}$ then:
(a) $f_1',\ldots,f_n'$ is a regular sequence;
(b) q is an A.R.n. for $\mathcal{b}' = (f_1',..,f_n')$ relative to α.

Proof. Put $\bar{A} = A/(f_1,\ldots,f_{n-1})$, $\bar{\alpha} = \alpha\bar{A}$ and denote with a bar the residues $\bmod(f_1,\ldots,f_{n-1})$. Then $\bar{f}_n$ is a non 0-divisor in $\bar{A}$ and, by remark (1.5)a), q is also an A.R.n. for $(\bar{f}_n)$ relative to α. It follows from corol. (2.7) that q is an A.R.n. for $(\bar{f}')$; now, if $w \in \alpha^{q+r} \cap \mathcal{b}'$, then $\bar{w} \in (\bar{f}') \cap \bar{\alpha}^{r+q} = \bar{\alpha}^r(\bar{\alpha}^q \cap (\bar{f}'))$ hence $\bar{w} = \sum \bar{h}_j \bar{u}_j \bar{f}'_n$ where $\bar{h}_j \in \bar{\alpha}^r$, $\bar{u}_j \bar{f}'_n \in \bar{\alpha}^q$ $\forall j$. Then choose $\forall j\ h_j \in \alpha^r$ lifting for $\bar{h}_j$ and $c_j = u_{1,j} f_1 + \ldots + u_{n-1,j} f_{n-1} + u_{n,j} f'_n \in \alpha^q$ lifting for $\bar{u}_j \bar{f}'_n$ such that $w = \sum h_j c_j + k$ where $k \in (f_1,..,f_{n-1})$ is a suitable element. Since $w-k \in \alpha^{q+r}$ then also $k \in \alpha^{q+r}$

hence, by lemma (3.5), $k \in \alpha^r(\alpha^q \cap (f_1,\ldots,f_{n-1},f'_n))$ so $w \in \alpha^r(\alpha^q \cap (f_1,\ldots,f_{n-1},f'_n))$.

This proves that q is an A.R.n. for $(f_1,\ldots,f_{n-1},f'_n)$ moreover, by corol. (3.4), $f_1,..,f_{n-1},f'_n$ is still a regular sequence, hence also $f'_n,f_1,\ldots,f_{n-1}$ is a regular sequence, since the ring is local, and we may repeat the argument to show that q is an A.R.n. for the ideal $(f'_n,f_1,..,f_{n-2},f'_{n-1})$ and that this sequence is regular and so on: in finitely many steps we are done.

REMARK (3.7). Note that, using the argument of th. (3.6), (a) and (b) must be proved together and at the same time. We do not know a separate proof for them.

COROLLARY (3.8). Let $\mathfrak{b} = (f_1,..,f_n)$ be an ideal of a ring A such that $H_1K(f_1,..,f_n;A) = 0$ (e.g. the sequence $f_1,..,f_n$ is regular); let q be an A.R.n. for $\mathfrak{b}$ relative to an ideal α. If $\mathfrak{b}' = (f'_1,..,f'_n)$ is an α-adic q-approximation of $\mathfrak{b}$ then q is an A.R.n. for $\mathfrak{b}'$ relative to α (hence $\mathfrak{b}^* = (\mathfrak{b}')^*$ in $G_A(\alpha)$).

Proof. Let $\mathfrak{m}$ be a maximal ideal of A;

I case. $\alpha A_{\mathfrak{m}} = A_{\mathfrak{m}}$. Every non negetive integer is an A.R.n. for all ideals of $A_{\mathfrak{m}}$ relative to $\alpha A_{\mathfrak{m}}$.

II case. $\alpha \subseteq \mathfrak{m}$, $\mathfrak{b} \not\subseteq \mathfrak{m}$; then $1 = \sum w_i f_i$ with $w_i \in A_{\mathfrak{m}}$ thus $\sum w_i f'_1 = \sum w_i (f'_i - f_i) + \sum w_i f_i \in 1 + \mathfrak{m}A_{\mathfrak{m}}$, then $\mathfrak{b}'A_{\mathfrak{m}}$ is the improper ideal and every non negative integer is an A.R.n. for it.

III case. $\alpha \subseteq \mathfrak{m}$, $\mathfrak{b} \subseteq \mathfrak{m}$; $\mathfrak{b}A_{\mathfrak{m}}$ has only trivial sy-

zygies hence the sequence $f_1,..,f_n$ is regular in $A_{\mathfrak{m}}$, moreover, by remark (1.5)c), q is an A.R.n. for $\mathfrak{b}A_{\mathfrak{m}}$; it follows by th. (3.6) that q is an A.R.n. for $\mathfrak{b}'A_{\mathfrak{m}}$.

Then for every maximal ideal $\mathfrak{m}$ of A, q is an A.R.n. for $\mathfrak{b}'A_{\mathfrak{m}}$ relative to $\mathfrak{a}A_{\mathfrak{m}}$, hence by remark (1.5)c) we are done.

COROLLARY (3.9). Let $\mathfrak{a}$, $\mathfrak{b} = (f_1,..,f_n)$ be ideals of A such that in the $\mathfrak{a}$-adic completion B of A, $H_1K(f_1,..,f_n;B)$ vanishes. If q is an A.R.n. for $\mathfrak{b}$ relative to $\mathfrak{a}$, then every $\mathfrak{a}$-adic q-approximation $\mathfrak{b}'$ of $\mathfrak{b}$ has the same form ideal of $\mathfrak{b}$ in $G_A(\mathfrak{a})$.

Proof. The morphism $A \longrightarrow B$ is flat, hence by remark (1.5)b) q is also an A.R.n. for $\mathfrak{b}B$ relative to $\mathfrak{a}B$ and $\mathfrak{b}'B$ is an $\mathfrak{a}B$-adic q-approximation of $\mathfrak{b}B$, then by corol. (3.8) in $G_B(\mathfrak{a}B)$ $(\mathfrak{b}'B)^* = (\mathfrak{b}B)^*$, but by [3], ch. II, p. 198, $\mathfrak{b}^* = (\mathfrak{b}B)^*$ and $(\mathfrak{b}')^* = (\mathfrak{b}'B)$ in $G_A(\mathfrak{a}) \simeq G_B(\mathfrak{a}B)$.

REMARK (3.10). If $H_1K(f_1,..,f_n;A) = 0$, since the morphism $A \longrightarrow B$ is flat (but not necessarily faithfully flat) then $H_1K(f_1,..,f_n;B) = 0$ (while the converse does not hold in general). Then corol. (3.9) shows that we can weaken the hypothesis of corol. (3.8).

Just as in the proof of corol. (3.8), we can show:

PROPOSITION (3.11). With the assumptions of corol. (3.8) if moreover $\mathfrak{a}$ is contained in the Jacobson radical of A, then all the syzygies of $\mathfrak{b}'$ are trivial.

EXAMPLE (3.12). The previous proposition does not hold in general if $\alpha \nsubseteq \mathrm{Rad}(A)$. Namely put $A = K[x,y]$, $\alpha = (x,y)$, $\mathfrak{b} = (y+1, xy)$; $\alpha \nsubseteq \mathrm{Rad}(A)$ and $y+1$, xy is a regular sequence. In $G_A(\alpha)$, $\mathfrak{b}^*$ is the improper ideal, hence $q=0$ is an A.R.n. for $\mathfrak{b}$ relative to α; $\mathfrak{b}' = (y+1, xy+y)$ is an α-adic 0-approximation of $\mathfrak{b}$, and clearly $(\mathfrak{b}')^* = \mathfrak{b}^* = (1)$ but $(x, -1)$ is a non trivial syzygy of $\mathfrak{b}'$.

We show an application of our statements to polynomial and power series rings.

EXAMPLE (3.13). Let K be a field and put $A = K[x_1,..,x_r]$, $B = K[[x_1,...,x_r]]$; then B is the completion of A in the topology of $\alpha = (x_1,..,x_r)$. Let $\mathfrak{b}$ be an ideal of B generated by a regular sequence $f_1,..,f_n$; then by th. (3.6), if $f_1',..,f_n'$ are polynomials such that $\forall i\ f_i' - f_i \in \alpha^{q+1}$ then they span in A an ideal $\mathfrak{b}'$ such that $\mathfrak{b}^* = (\mathfrak{b}'B)^*$; but note that, by [3], ch. II, p. 198, $(\mathfrak{b}')^* = (\mathfrak{b}'B)^*$ in $G_A(\alpha) \simeq G_B(\alpha B)$.

It follows that simply cutting off from every series f_i all the terms of degree $> q$, we get polynomials which span an ideal $\mathfrak{b}' \subseteq A$ such that $\mathfrak{b}^* = (\mathfrak{b}')^*$, even if, perhaps, $\mathfrak{b}^* \neq (f_1^*,..,f_n^*)$ (see [6], § 2).

Moreover, by prop. (3.11), the sequence $f_1',..f_n'$ is also regular in B, hence it is regular in $A_\alpha = K[x_1,..,x_r]_{(x_1,..,x_r)}$ since the morphism $A_\alpha \longrightarrow B$ is faithfully flat.

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Luca CHIANTINI
Istituto Matematico del Politecnico
Corso Duca degli Abruzzi 24
10129 TORINO
(ITALIA)

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