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CRITERIA FOR EBERLEIN COMPACTNESS IN SPACES OF CONTINUOUS FUNCTIONS

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Criteria for pointwise relative Eberlein compactness in spaces of continuous maps and in spaces of linear operators are given in terms of countable compactness, Stone-Čech extendability, and interchangeability of double limits.

Since their introduction by Amir and Lindenstrauss [1], the Eberlein compacta (E-compacta for short), i.e., those compact spaces that are homeomorphic with weakly compact sets in Banach spaces, have been investigated thoroughly, not only for their importance in Banach space theory and, in the convex case, for the features of their extremal structure, but also because of their remarkable topological properties that resemble those of (compact) metrizable spaces to some extent and of which we mention the following: Every E-compactum is closure-sequential, i.e., for each of its subsets the closure and the sequential closure are identical [20;p.313]; consequently, E-compacta are hereditarily sequential and sequentially compact. In every E-compactum K, there is a dense metrizable G_δ-subspace; this can be deduced from a theorem of Namioka [23;4.2] and, in turn, implies the following very

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useful metrizability criterion that had been obtained earlier by Rosenthal [27;p.230] by different means: K is metrizable if it satisfies the countable chain condition (CCC), i.e., if every family of pairwise disjoint non-empty open subsets of K is countable. These properties can be attributed to the fact that E-compacta admit homeomorphic representations, not just - as do all compacta - as pointwise compact sets of continuous, say real-valued, maps, but as compact sets of special continuous maps on special spaces. The main purpose of this paper is, conversely, to exhibit sufficient conditions on a given set of continuous maps guaranteeing its (relative) E-compactness with respect to the topology of pointwise convergence. These conditions will be mainly in terms of relative countable compactness, Stone-Čech extendability, and double limits, which have, in one form or another, been used in [2,4,6,8,10,14,24,25,28,29] and, certainly, elsewhere. Extensive use will be made of sequential arguments. By some of the results, theorems of Grothendieck [14] and Pryce [24] are sharpened and generalized. In particular, we will give some criteria for E-compactness of sets of linear operators in their weak or strong operator topologies. Special attention will be paid to the metrizability problem for compact sets of maps as far as it can be solved via E-compactness.

For a general discussion of E-compacta - in particular, their permanence properties, which we will utilize freely - the reader is referred to [21] and [32].

First, we fix some standing hypotheses and notation: Throughout, X,Y,Z are topological spaces (all spaces are assumed to be Hausdorff), Z regular, and $f: X \times Y \longrightarrow Z$ is a separately continuous map. For $x \in X$ the map $y \longmapsto f(x,y): Y \longrightarrow Z$ is denoted by x^*. Sets of maps, such as $B^* = \{x^* | x \in B\}$ for $B \subseteq X$, are always considered endowed with the topology of pointwise convergence, unless said otherwise. E,F,H are locally convex vector spaces, with F_k submetrizable. Here F_k denotes F with the finest locally convex vector topology that is compatible with the natural duality $\langle F,F'\rangle$ and agrees with the topology of F on compact subsets of F. And we recall that a space is (separably) submetrizable if it admits a continuous injection into some (separable) metrizable space. E_σ (E_τ) stands for E in its weak (Mackey) topology. (Note that F need not be submetrizable when F_k is. Such is the case, e.g., for $\ell^1(\Gamma)_\sigma$ and $(\mathbb{R}^\Gamma)'_\sigma$ if card $\Gamma >$ card $\mathbb{R}$.) Clearly, submetrizability of F_k is tantamount to 0 being a G_δ-point in F_k. Furthermore, F_o denotes F with some coarser locally convex vector topology that is finer than the weak topology. $\mathcal{L}(E,H)$ is the space of all continuous linear maps $E \to H$ (with the topology of pointwise convergence, as we agreed; that is, the strong operator topology); and $\mathcal{L}(E,H)_w$ stands for $\mathcal{L}(E,H)$ equipped with the weak operator topology. The scalar field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Write $C(X)$ instead of $C(X,\mathbb{K})$ and $\mathbf{I}$ for $[0,1]$. For $A \subseteq E$ the closed convex circled hull of A is denoted by $\overline{\text{conv ci}}\, A$. Recall that $B \subseteq X$ is relatively (countably) compact in X if every net (sequence) in B clusters in X; and B is relatively (strongly) sequentially compact in X if every sequence in B has a subsequence that converges (has compact metrizable closure) in X. As usual, 'relative' is omitted when B = X. We will call X nearly countably compact if it has a dense subset each sequence of which clusters in X. 'Relatively sequentially compact' is shortened to RS-compact, 'nearly countably compact' to NC-compact, etc.

1. Countable compactness versus Eberlein compactness

Throughout this section, we assume that Y_m is a sequence of RC-compact sets of Y whose union is dense.

LEMMA 1.1. Let S be a regular space, $g: S \longrightarrow T$ a continuous injection into a closure-sequential space, and B an RC-compact set in S. Then $\overline{B}$ is identical with the sequential closure $\overline{B}^S$ and is mapped under g homeomorphically onto a closed subspace of T. *)

Proof: In order to prove that the inverse h of $\overline{B} \longrightarrow g(\overline{B})$ is continuous, it suffices to show sequential continuity of h on each subspace $g(B) \cup \{t\}$ with $t \in g(\overline{B})$. Verifying this as well as the rest of the assertion is not difficult.

THEOREM 1.2. Let $\mathcal{K} \subseteq C(Y, F_\sigma)$ be NC-compact and $\varphi(Y_m)$ precompact in F for $\varphi \in \mathcal{K}$ and $m \in \mathbb{N}$. Then $\mathcal{K}$ is E-compact.

Proof: First, assume that Y is NC-compact and L a supremum norm bounded RC-compact subset of $C(Y)$. Then L is R-compact in $C(Y)$ [14;p.172]. (Actually, Grothendieck assumes Y to be C-compact; when inspecting his proof, however, one notices that NC-compactness of Y suffices.) Due to the fact that $C(Y)$ is completely regular, $\overline{L}$ is compact in $C(Y)$. Since the point evaluation map $\eta: Y \longrightarrow C(\overline{L})$ is continuous, $\eta(Y)$ is NC-compact in $C(\overline{L})$; whence, by the same token, $\overline{\eta(Y)}$ is compact. Point evaluation $\tau: \overline{L} \longrightarrow C(\overline{\eta(Y)})$ is a continuous injection. Thus, $\tau(\overline{L})$ being weakly compact in the Banach space $(C(\overline{\eta(Y)}), \| \|_\infty)$ [14;p.182], it follows that $\overline{L}$ is E-compact. - Consider now the general case: There exists a sequence U_n of closed convex circled 0-neighborhoods of F_k with $U_{n+1} \subseteq U_n$ and $\bigcap_n U_n = \{0\}$. Then the polars U_n° are compact in the weak*-dual F'_σ, and $\bigcup_n U_n^\circ$ is dense in F'_σ. Let $\alpha: \mathbb{K} \longrightarrow \mathbb{K}$ be a continuous bounded injection. Fix $n, m \in \mathbb{N}$. Let $\varphi \in \mathcal{K}$. The canonical map $\varphi(\overline{Y_m}) \longrightarrow \prod_i C(U_i^\circ)$ takes, by what has been said above, its values in some space $L = \prod_i L_i$, with

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*) The referee has pointed out that essentially the same lemma had been obtained by De Wilde in [5;1.7]

all L_i E-compact; as then L is E-compact [21;p.248], $\varphi(\overline{Y_m})$ is compact in F_σ in view of 1.1. Since $\varphi(\overline{Y_m})$ is still precompact in F [20;p.240,p.245], it is compact in F [20;p.385], hence is a subspace of F_k. By equicontinuity of $U_n^\circ \subseteq C(F_k)$, evaluation $U_n^\circ \times \varphi(\overline{Y_m}) \longrightarrow \mathbb{K}$ is continuous, which implies that $\varphi_{nm}: (v,y) \mapsto \alpha(v(\varphi(y))): U_n^\circ \times \overline{Y_m} \longrightarrow \mathbb{K}$ is continuous. As a consequence, $\varphi \mapsto \varphi_{nm}$ maps $\mathcal{K}$ continuously into $C(U_n^\circ \times \overline{Y_m})$. But $U_n^\circ \times \overline{Y_m}$ is NC-compact, and so the closure K_{nm} of $\{\varphi_{nm} \mid \varphi \in \mathcal{K}\}$ in $C(U_n^\circ \times \overline{Y_m})$ is E-compact. Via $\varphi \longmapsto (\varphi_{nm})_{(n,m)\in\mathbb{N}^2}$, $\mathcal{K}$ is now mapped continuously and injectively into $K = \prod_{n,m} K_{nm}$. Therefore, E-compactness of $\mathcal{K}$ follows from the fact that K is E-compact and from 1.1.

<u>COROLLARY</u> 1.3. $\overline{\mathcal{K}}$ <u>is E-compact in</u> C(Y,F), <u>provided</u> $\mathcal{K}$ <u>is</u> RC-<u>compact in</u> C(Y,F).

<u>COROLLARY</u> 1.4. <u>If</u> X <u>is</u> NC-<u>compact and</u> Z <u>submetrizable, then</u> X* <u>is</u> E-<u>compact</u>.

<u>Proof</u>. Z admits a continuous injection into some Hilbert space H. If ι denotes the induced continuous injection $C(Y,Z) \longrightarrow C(Y,H)$, then $\iota(X^*)$ is E-compact according to 1.2. Now use 1.1.

<u>COROLLARY</u> 1.5. <u>Let</u> X <u>be</u> NC-<u>compact</u>, Z <u>a Banach space</u>, f <u>bounded</u>, <u>and</u> μ <u>a</u> $\mathbb{K}$-<u>valued Borel measure on</u> Y. <u>Then</u> $x \longmapsto \int f(x,y)\, d\mu(y) : X \longrightarrow Z$ <u>is continuous</u>.

<u>Proof</u>. Every x* is μ-measurable [9;p.148] because it takes its values in the separable subspace $\overline{\bigcup_m x^*(Y_m)}$ of Z. Since μ is bounded [9;p.127], X* consists of μ-integrable maps. By Lebesgue's Theorem and the fact that X* is sequential, $x^* \longmapsto \int x^*\, d\mu$ is continuous on X*.

<u>PROPOSITION</u> 1.6. <u>Let</u> K <u>be an</u> E-<u>compact set in</u> H <u>for which</u> $\overline{\text{conv}}$ K <u>is sequentially complete or Mackey complete</u>. <u>Then</u> $\overline{\text{conv ci}}$ K <u>is</u> E-<u>compact</u>.

<u>Proof</u>. The case of complex scalars is readily reduced to

that of real scalars. So let $\mathbb{K} = \mathbb{R}$. First, suppose that $\overline{\text{conv}}\ K$ is complete in H_τ. Since the completion G of H_τ is Mackey [20;p.262], the closed convex circled hull C of K in G is weakly compact [20;p.325]. Thus, the restriction map from G'_τ into the Banach space $C(K)$ is continuous, whence the adjoint $\rho: C(K)'_\sigma \to (G'_\sigma)'_\sigma$ is continuous. Since continuous images of E-compacta are E-compact [3] and the norm unit ball U of $C(K)'$ is weak*-E-compact [21;p.249], C is E-compact, being the image of U under $\varepsilon\rho$, where $\varepsilon: (G'_\sigma)'_\sigma \to G_\sigma$ is the canonical map. Thus, $\overline{\text{conv}}\ K$ is E-compact in H_σ. As now $\overline{\text{conv}}\ K$ is precompact in H, it is E-compact in H [20; p.385]. - On the other hand, suppose that $\overline{\text{conv}}\ K$ is sequentially complete in H. Let then L be the closed convex hull of K in the completion $\tilde{H}$. Being complete in $(\tilde{H})_\tau$ [20;p.210], L is E-compact. But $\overline{\text{conv}}\ K = L$ by sequential closedness of $\overline{\text{conv}}\ K$ in L. - So in either case, $\overline{\text{conv}}\ K$ is E-compact. By [9;p.415], there is a continuous surjection $\mathbb{I} \times \overline{\text{conv}}\ K \times \overline{\text{conv}}\ (-K) \longrightarrow \overline{\text{conv}}\ (K \cup (-K))$. The latter set, however, is nothing but $\overline{\text{conv ci}}\ K$.

COROLLARY 1.7. <u>Hypotheses of 1.2. Moreover, let $\mathcal{S}$ be a set of RC-compact sets of Y so that all $\mathcal{K}(S)$, with $S \in \mathcal{S}$, are bounded in F and a map $Y \to \mathbb{R}$ is continuous when continuous on members of $\mathcal{S}$. Let $\overline{\text{conv}}\ \mathcal{K}(y)$ be Mackey complete or sequentially complete in $F_\circ$ for $y \in Y$. Then $\overline{\text{conv ci}}\ \mathcal{K}$ is E-compact in $C(Y,F_\circ)$ and consists of separably valued maps.</u>

<u>Proof</u>. According to 1.6, 1.2, and Tychonoff's Theorem, $\overline{\text{conv ci}}\ \mathcal{K}$ is E-compact in $F_\bullet^Y$. It must be shown that $\overline{\text{conv ci}}\ \mathcal{K} \subseteq C(Y,F_\circ)$. In view of our hypotheses on $\mathcal{S}$, we may assume to this end that Y is NC-compact and that $\mathcal{K}(Y)$ is bounded in F. Let $\varphi \in \overline{\text{conv ci}}\ \mathcal{K}$. Fix $b \in Y$, and let U be any closed convex circled 0-neighborhood in $F_\circ$. The canonical map $\Gamma: C(Y,F_\circ) \to C(U^\circ \times Y)$ being continuous, $\Gamma(\mathcal{K})$ is compact in $C(U^\circ \times Y)$, hence weakly compact with respect to the $\|\ \|_\infty$-topology (cf. proof 1.2).

Consequently [20;p.325], $\overline{\text{conv ci}}\ \Gamma(\mathcal{K})$ is compact in $C(U^\circ \times Y)$. It follows that $(v,y) \mapsto v(\varphi(y))$: $U^\circ \times Y \to \mathbb{K}$ is continuous. Hence there exists a neighborhood N of b such that $|v(\varphi(y))-v(\varphi(b))| \leq 1$ for all $(v,y) \in U^\circ \times N$; that is, $\varphi(y)-\varphi(b) \in U$. - Given $\varphi \in \overline{\text{conv ci}}\ \mathcal{K}$, choose a sequence φ_n in conv ci $\mathcal{K}$ converging to φ. Since $\mathcal{K}$ consists of separably valued maps, every φ_n has its image in some separable subspace S_n of F_o. Thus, φ takes its values in the separable space $\overline{\bigcup_n S_n}$.

We now point out a few situations in operator theory where 1.2 becomes applicable:

THEOREM 1.8. *Suppose E is a Banach space C(S), with S E-compact, and F is, in addition, complete and weakly sequentially complete (e.g., F of type $L^1(\mu)$). Let $\mathcal{K} \subseteq \mathcal{L}(E,F_o)$ with $\mathcal{K}(y)$ RC-compact in F_o for $y \in E$. Then $\overline{\text{conv ci}}\ \mathcal{K}$ is E-compact in $\mathcal{L}(E,F_o)$.*

Proof: Since F is Mackey complete, $\overline{\text{conv ci}}\ \mathcal{K}(y)$ is compact in F_o for every $y \in E$ [20;p.314,p.325]. Thus, $\overline{\text{conv ci}}\ \mathcal{K}$ is compact in F_o^E. However, due to the barrelledness of E, $\overline{\text{conv ci}}\ \mathcal{K} \subseteq \mathcal{L}(E,F_o)$. It follows from [15;Th.6, Th.1] that every $T \in \mathcal{L}(E,F)$ transforms weakly compact subsets of E into S-compact subsets of F. And by [21;p.249], there exists a total weakly compact set Y in E. Then restriction ϱ : $\overline{\text{conv ci}}\ \mathcal{K} \longrightarrow C(Y,F_o)$ is continuous and injective. But $\varrho(\overline{\text{conv ci}}\ \mathcal{K})$ is E-compact according to 1.2.

COROLLARY 1.9. *Let E be as in 1.8, F a reflexive Fréchet space, and $\mathcal{K} \subseteq \mathcal{L}(E,F)$ pointwise bounded. Then $\overline{\text{conv ci}}\ \mathcal{K}$ is E-compact in $\mathcal{L}(E,F)_w$.*

THEOREM 1.10. *Suppose E has a sequence of weakly RC-compact sets whose union is total. Let $\mathcal{K} \subseteq \mathcal{L}(E,H)$ be RC-compact.*

(1) *$\overline{\mathcal{K}}$ is E-compact if H_τ admits a continuous linear injection into some $L^\infty(\mu)$, with μ σ-finite.*

(2) *If* H_τ *is submetrizable and all* $T \in \mathcal{K}$ *are separably valued,* $\mathcal{K}$ *is RSS-compact in* $\mathcal{L}(E,H)$.

Proof. (1): Via 1.1, one reduces to the case where E contains a total weakly RC-compact subset Y and $H = L^\infty(\mu)_\sigma$. In $L^1(\mu)$, there exists a total weakly compact subset V [21;p.240]. It now suffices to show that, given $T \in \mathcal{L}(E, L^\infty(\mu)_\sigma)$, the map $(\varphi, y) \mapsto \int \varphi \cdot Ty \, d\mu$: $V \times \bar{Y} \to \mathbb{K}$ is continuous, where $\bar{Y}$ denotes the weak closure (cf. proof 1.2). Since $V \times T(\bar{Y})$ is E-compact, it is good enough to this end to have $(\varphi, \psi) \longmapsto \int \varphi\psi \, d\mu$: $V \times T(\bar{Y}) \to \mathbb{K}$ sequentially continuous. However, if $(\varphi_n, \psi_n) \longrightarrow (\varphi, \psi)$ in $V \times T(\bar{Y})$, then $\int \varphi_n \psi_n d\mu \to \int \varphi\psi d\mu$ indeed [15;p.138,p.139].
(2): It must be shown that $\bar{\mathcal{K}}$ is compact metrizable in $\mathcal{L}(E,H)$ if $\mathcal{K}$ is, in addition, countable. In this situation, we may assume that H_σ is separable. A union of countably many compact sets being dense in H'_σ, a sequence is actually dense in H'_σ. Thus, H_σ is submetrizable; and so 1.1 and 1.4 now do the rest.

PROPOSITION 1.11. *Let* (S, Σ, μ) *be a* σ*-finite positive measure space,* F *a Banach space, and* $\mathfrak{M}$ *a uniformly bounded set of* μ*-measurable maps* $S \to F$ *each sequence* h_n *of which has a subsequence* h_{n_k} *such that* $\lim_k h_{n_k}(s)$ *exists in* F_σ *for* μ*-almost all* s. *Let* $\mathcal{K}$ *be all operators* $g \mapsto \int gh \, d\mu$ *with* $h \in \mathfrak{M}$. *Then* $\mathcal{K}$ *is E-compact in* $\mathcal{L}(L^1(\mu), F_\sigma)$ *and consists of representable operators.*

Proof. Choose a total weakly compact set Y in $E = L^1(\mu)$. Let $\tilde{\mathfrak{M}}$ denote the set of all bounded maps $h: S \to F$ for which there exists a sequence h_n in $\mathfrak{M}$ with $\lim_n h_n(s) = h(s)$ in F_σ μ-almost everywhere. Any such h is μ-measurable, as can be seen by employing Pettis' Theorem [9;p.149]; and the operator T_h: $g \longmapsto \int gh \, d\mu$: $E \to F$ maps Y onto an RS-compact subset of F [9; p.510]. Now $\{T_h | h \in \tilde{\mathfrak{M}}\}$ is NC-compact in $\mathcal{L}(E, F_\sigma)$. Namely, let h_n be any sequence in $\mathfrak{M}$. Then $\lim_k h_{n_k}(s)$

$= h(s)$ in $F_\bullet$ for a suitable subsequence h_{n_k} of h_n, some $h \in \tilde{\mathfrak{M}}$, and μ-almost all s. Fix $g \in E$. If B is any Banach space and $v \in \mathcal{L}(F_\bullet, B)$ arbitrary, then $\lim_k \int g \cdot v h_{n_k} \, d\mu = \int g \cdot vh \, d\mu$ in B by Lebesgue's Theorem, i.e., $\lim_k v(\int g h_{n_k} d\mu) = v(\int gh \, d\mu)$ [9;p.153]. Consequently, $T_{h_{n_k}} \longrightarrow T_h$ in $\mathcal{L}(E, F_\circ)$. Using essentially the same arguments, one obtains that $\mathcal{K}$ is dense in $\{T_h | h \in \tilde{\mathfrak{M}}\}$. In order to conclude that $\{T_h | h \in \tilde{\mathfrak{M}}\}$ is E-compact, it now suffices to utilize 1.1 and 1.2.

COROLLARY 1.12. <u>Suppose in 1.11 $\mathfrak{M}$ is even RS-compact in $F_\bullet^S$ and $\mu(\{s | h(s) \neq k(s)\}) > 0$ for $h \neq k$ in $\overline{\mathfrak{M}}^s$. Then $\overline{\text{conv ci}}\ \mathfrak{M}$ is E-compact in $F_\bullet^S$ and consists of μ-measurable maps.</u>

<u>Proof</u>. Working with $\overline{\mathfrak{M}}^s$, the sequential closure of $\mathfrak{M}$ in $F_\bullet^S$, instead of $\tilde{\mathfrak{M}}$ in proof 1.11, one gets E-compactness of $\overline{\mathcal{K}} = \{T_h | h \in \overline{\mathfrak{M}}^s\}$ in $\mathcal{L}(E, F_\circ)$. Let $h, k \in \overline{\mathfrak{M}}^s$ with $T_h = T_k$. Then $\int_A h \, d\mu = \int_A k \, d\mu$ for every $A \in \Sigma$ with $\mu(A) < \infty$, implying that $h(s) = k(s)$ μ-almost everywhere in S (cf. [7;p.47]). Thus, $h = k$. In order to establish continuity of $T_h \mapsto h: \overline{\mathcal{K}} \rightarrow \overline{\mathfrak{M}}^s$, it is good enough to verify that, if $T_{h_n} \longrightarrow T_h$ in $\overline{\mathcal{K}}$ with $h \in \overline{\mathfrak{M}}^s$ and $h_n \in \mathfrak{M}$, then h_n clusters at h in $\overline{\mathfrak{M}}^s$; because $\overline{\mathcal{K}}$ is hereditarily sequential. But $h_{n_\ell} \longrightarrow k$ in $\overline{\mathfrak{M}}^s$ for a suitable subsequence h_{n_ℓ} of h_n and some $k \in \overline{\mathfrak{M}}^s$. As in proof 1.11, one obtains that $T_{h_{n_\ell}} \longrightarrow T_k$ in $\overline{\mathcal{K}}$, whence $h = k$. Consequently [3], $\overline{\mathfrak{M}}^s$ is E-compact, being the continuous image of $\overline{\mathcal{K}}$. But then $\overline{\mathfrak{M}}^s = \overline{\mathfrak{M}}$. Since now $\overline{\text{conv ci}}\ \mathfrak{M}(s)$ is compact in $F_\circ$ for every $s \in S$ [20;p.325], it follows by means of 1.6 that $\overline{\text{conv ci}}\ \mathfrak{M}$ is E-compact in $F_\bullet^S$. Thus, every h in $\overline{\text{conv ci}}\ \mathfrak{M}$ is the sequential limit in $F_\bullet^S$ of maps from conv ci $\mathfrak{M}$; whence $h: S \longrightarrow F$ is μ-measurable.

THEOREM 1.13. <u>Suppose Z is submetrizable, X the support of a σ-finite positive Borel measure μ, and $\mathcal{K}$ RS-compact in $C(X,Z)$. Then $\overline{\mathcal{K}}$ is E-compact.</u>

Proof. We may assume that Z is a Hilbert space and then that $\mathcal{K}$ is uniformly bounded (cf. proof 1.4). Let $(Q_\lambda)_{\lambda\in\Lambda}$ be any family of pairwise disjoint non-empty open subsets of X. If B_n is a sequence of Borel sets with $\bigcup_n B_n = X$ and $\mu(B_n) < \infty$, then, for reasons of summability, every $\Lambda_n = \{\lambda \in \Lambda \mid \mu(B_n \cap Q_\lambda) > 0\}$ is countable; consequently, $\Lambda = \bigcup_n \Lambda_n$ is countable. Thus, X satisfies CCC. So every $\varphi \in C(X,Z)$ is separably valued, hence μ-measurable. The sequential closure $\overline{\mathcal{K}}^s$ in Z^X is contained in $C(X,Z)$ and is, therefore, E-compact in view of 1.12.

In the sequel, A denotes a commutative complex Banach algebra with identity e, $\|e\| = 1$, γ the Gel'fand transformation of A, and U,V open connected sets of A. Recall [18;p.115] that $g: U \to V$ is (L)-analytic if for every $u \in U$ there exists an $a \in A$ with the property that, given $\varepsilon > 0$, there is some $\delta > 0$ such that $\|g(x)-g(u)-a(x-u)\| \leq \varepsilon \|x-u\|$ for $x \in U$ with $\|x-u\| \leq \delta$. Let $LA(U,V)$ denote the set of all such maps, given the subspace topology of $C(U,A_\sigma)$.

THEOREM 1.14.

(1) Every RC-compact set of $LA(U,V)$ has E-compact closure.

(2) Suppose γ is a homeomorphism onto its image, V is a ball, and $g \in LA(V,A)$. If g is bounded on weakly convergent sequences or if $V = A$, then composition $h \mapsto g\circ h: LA(U,V) \to LA(U,A)$ is continuous on compact sets.

Proof. (1): Fix some $u \in U$ and choose a convergent sequence $u_k \to u$ in U in such a way that all $u_k - u$ are invertible (e.g., $u_k = u + r\cdot k^{-1}e$, where $r > 0$ is so that $\{x \in A \mid \|x-u\| \leq r\} \subseteq U$). Every $h \in LA(U,V)$ admits a power series expansion $h(x) = \sum_{n=0}^{\infty} a_n(x-y)^n$ in every ball $\{x \in A \mid \|x-y\| < \rho\}$ that lies in U [18;p.770]. A straightforward modification of the proof in the classical case $A = \mathbb{C}$, therefore, yields that any two (L)-analytic maps agreeing on $\{u_k | k \in \mathbb{N}\}$ agree on all of U. As a consequence, given $\mathcal{K} \subseteq LA(U,V)$ RC-compact, $\overline{\mathcal{K}}$ admits a continuous injection into some product of countably many E-compacta.

(2): Let $\mathcal{K} \subseteq LA(U,V)$ compact. In order to show continuity of $h \mapsto g \circ h : \mathcal{K} \longrightarrow LA(U,A)$, it suffices to check sequential continuity. So let $v_k \longrightarrow v$ weakly in V, and verify that $g(v_k) \longrightarrow g(v)$ in A_σ : By hypothesis, V = $\{x \in A | \, \|x-w\| < r\}$ for suitable $w \in V$ and $0 < r \leq \infty$. Let $g(x) = \sum_{n=0}^{\infty} a_n (x-w)^n$ be the power series expansion of g about w in V. It follows from the proof of [18; 3.19.1] that $\limsup_{n \to \infty} \|a_n\|^{1/n} \leq r^{-1}$. Let S denote the structure space of A. Fix $\tau \in S$, that is, $0 \neq \tau \in A'$ multiplicative. Then the complex power series $p(z) = \sum_{n=0}^{\infty} \tau(a_n)(z-\tau(w))^n$ converges for every z in D = $\{z \in \mathbb{C} | \, |z-\tau(w)| < r\}$ because $\|\tau\| = 1$. Thus, we obtain $p(\tau(v_k)) \longrightarrow p(\tau(v))$ in $\mathbb{C}$, since $\tau(v_k) \longrightarrow \tau(v)$ in D; in other words, $\tau(g(v_k)) \longrightarrow \tau(g(v))$. Consequently, $\gamma(g(v_k)) \longrightarrow \gamma(g(v))$ pointwise on S. Now, $g(v_k)$ is bounded in A; this is part of the hypothesis if $V \neq A$ and follows for V = A from the fact that the power series representing g converges absolutely. Therefore, $\gamma(g(v_k)) \longrightarrow \gamma(g(v))$ weakly in the Banach space C(S). But $A_\sigma \longrightarrow \gamma(A)_\sigma$ is a topological isomorphism as well, which implies that $g(v_k) \longrightarrow g(v)$ in A_σ.

REMARKS AND EXAMPLES 1.15. 1. 1.2 cannot be improved to yield E-compactness of $\overline{\mathcal{K}}$ if $\mathcal{K}$ is only RC-compact in $C(Y,F_\circ)$: Let F be a non-separable Hilbert space, $F_\circ = F_\sigma$, Y the unit ball of F with the weak topology, and $\mathcal{K}$ the restrictions to Y of all projections on F with finite-dimensional range. $\overline{\mathcal{K}}$ is compact in $C(Y,F_\circ)$, but is not even closure-sequential because the identity $Y \longrightarrow F_\circ$ belongs to $\overline{\mathcal{K}}$ (cf. proof 1.7).

2. Examples show that neither of the conditions on $\mathcal{S}$ in 1.7 can be omitted. If they are omitted, every pointwise limit of maps from conv ci $\mathcal{K}$ is still of

Baire class 1, i.e., the pointwise limit of a sequence from $C(Y,F_o)$.

3\. 1.8 collapses for non-E-compact S: Let $F = \mathbb{K}$ and $\mathcal{K}$ all Dirac measures on S. - The same example demonstrates that 1.10 is false for arbitrary E. - For a discussion of Banach spaces with total weakly compact sets see [21].

4\. If E and $\mathcal{K}$ are as in 1.10, $\overline{\mathcal{K}}$ is E-compact when H_τ is separable metrizable. This follows by means of 1.10.(1) because H_τ can then be embedded into $(\ell^\infty)^N$.

5\. In 1.10.(1), $L^\infty(\mu)$ may be replaced by any Banach space C(S), S compact, if S is the support of a σ-finite Borel measure, but not if S is arbitrary: Let G be a compact non-metrizable group, S the unit ball of $L^2(G)$ in its weak topology, and $\mathcal{K} \subseteq \mathcal{L}(C(S),C(S))$ all right translation operators. $\mathcal{K}$ is homeomorphic with G, and G satisfies CCC (cf. proof 1.13).

6\. For Banach spaces H = C(S), S a compact CCC-space, 1.10.(2) holds: For then compact sets of H_σ are separable [27;4.5] (cf. 2.7).

7\. 1.10.(2) fails to hold for arbitrary operators: Let G be as in 5 and $\mathcal{K} \subseteq \mathcal{L}(L^2(G),L^2(G))$ all right translation operators. Compact groups are known to be dyadic; and a dyadic space can be shown to be metrizable if it is SS-compact.

8\. Even if $\mathcal{K}$ in 1.11 consists of compact operators: if $\mathcal{M}$ is not RS-compact modulo μ-null sets, $\overline{\mathcal{K}}$ may be E-compact without consisting entirely of representable operators: Let $E = L^1([0,1])$, $F = c_0$, and $\mathcal{M} = \{h_m\}$, where $h_m(t) = (\sin 2\pi t, \ldots, \sin 2^m \pi t, 0, 0, \ldots)$. Then T_{h_m} converges in $\mathcal{L}(E,F)$ to the operator $g \longmapsto (\int_0^1 g(t) \sin 2^n \pi t \, dt)_{n=1,2,\ldots}$, which is not representable [7;p.60].

9\. In case $F_o = F$, the uniform boundedness condition in 1.12 is dispensable: consider the maps $s \mapsto (1+\|h(s)\|)^{-1} h(s)$ instead of $\mathcal{M}$.

10\. 1.13 becomes false if X is assumed to just satisfy CCC or certain of its stronger relatives: Let T be compact, S-compact, non-E-compact, $X = C(T)$, and $\mathcal{K}$ all Dirac measures on T. Then X is a (K)-space, i.e., every uncountable family of non-empty open sets has an uncountable subfamily any two members of which meet; namely, $\mathbb{K}^T$ is a (K)-space [22], and X is dense in $\mathbb{K}^T$.

11. I do not know whether 1.13 remains true if $\mathcal{K}$ is only RC-compact. It can be shown that the answer is affirmative if X, in addition, has a dense K-analytic subspace.

12. The boundedness condition imposed on g in 1.14.(2) is essential: Take $U = V$ the open unit ball in $A = (C(\mathbb{I}), \| \|_\infty)$ and $g: v \mapsto \exp((e-v)^{-1})$. Let $h_k \in LA(U,V)$ be the constant map with value x_k, where x_k is such that $x_k(0) = 0$, $x_k(s) = 0$ for $s \geq (k-1)^{-1}$, and $x_k(k^{-1}) = 1-k^{-1}$. Then $h_k \to 0$ in $LA(U,V)$, but $g \circ h_k \not\to g \circ 0$ in $LA(U,A)$. - Composition with g is generally not continuous on all of $LA(U,V)$; such is the case, e.g., for $U = V = A = \ell^\infty$ and $g: x \mapsto x^2$: For every finite set ϕ of complex regular Borel measures on $\beta\mathbb{N}$ choose some $n_\phi \in \mathbb{N}$ with $\Sigma_{\mu \in \phi} |\mu| (\{n_\phi\}) \leq n_\phi^{-1}$; and let $x_\phi(n) = n_\phi$, for $n = n_\phi$, and 0 otherwise. Then $x_\phi \to 0$ in ℓ^∞_σ, but $x_\phi^2 \not\to 0$.

2. Eberlein compactness via Stone-Čech extendability and the double limit condition

Throughout this section, C, D denote dense subsets of X, Y, respectively. X, Y, Z are assumed to be completely regular. $\overline{Z}^s$ denotes the sequential closure of Z in βZ. For $g \in C(Y,Z)$ let $g^\beta: \beta Y \to \beta Z$ be the continuous extension. Write x^β instead of $x^{*\beta}$ when $x \in X$. If $\mathcal{K} \subseteq C(Y,Z)$, then $\mathcal{K}^\beta$ stands for $\{g^\beta | g \in \mathcal{K}\}$.

We will call f Stone-Čech extendable or an SČ-map if it admits a separately continuous extension $\beta X \times \beta Y \to \beta Z$; such an extension then is uniquely determined and is denoted by f_β. In this case, $\overline{X^\beta}$ is compact in

$C(\beta Y, \beta Z)$; in fact, $\overline{X^\beta} = (\beta X)^*$. In particular, X^β is homeomorphic with X^* via $x^* \mapsto x^\beta$. Thus, if $\varepsilon: \beta X \longrightarrow \overline{X^\beta}$ is the continuous extension of $x \mapsto x^\beta$, then $f_\beta(\hat{x}, \hat{y}) = \varepsilon(\hat{x})(\hat{y})$ for $(\hat{x}, \hat{y}) \in \beta X \times \beta Y$.

Resuming ideas of Eberlein and Grothendieck, we will also consider the following conditions:

DLC: There is no sequence (c_n, d_n) in $C \times D$ such that $\lim_n \lim_m f(c_n, d_m)$ and $\lim_m \lim_n f(c_n, d_m)$ exist in Z, but are different.

DCPC: Every double sequence $f(c_n, d_m)$, with $(c_n, d_m) \in C \times D$, has a double cluster point z, i.e., for every neighborhood U of z in Z there are $n_1 < n_2 < \ldots.$ and $m_1 < m_2 < \ldots.$ such that $\{m \mid f(c_{n_i}, d_m) \in U\}$ and $\{n \mid f(c_n, d_{m_i}) \in U\}$ are infinite for all i.

ROC: All sets $f(C \times \{d\})$, with $d \in D$, are RC-compact in Z.

LOC: $f(\{c\} \times D)$ is RC-compact in Z for every $c \in C$.

We now list some criteria for f to be Stone-Čech extendable. X_u (X_{pu}) denotes X with the finest (finest precompact) uniformity compatible with the topology of X.

LEMMA 2.1 Under each of the following conditions, f is an SČ-map:

(1) f satisfies DCPC.

(2) f satisfies DLC; and for every sequence (c_n, d_n) in $C \times D$ the sets $\bigcup_{n \in \mathbb{N}} \overline{\{f(c_n, d_m) \mid m \in \mathbb{N}\}}^s$ and $\bigcup_{m \in \mathbb{N}} \overline{\{f(c_n, d_m) \mid n \in \mathbb{N}\}}^s$ are RS-compact in Z.

(3) f is continuous and $X \times Y$ pseudocompact.

(4) X^* is a uniformly equicontinuous set of maps $Y_{pu} \longrightarrow Z_{pu}$.

(5) Y is compact and $\varphi f: X_u \times Y \longrightarrow \mathbb{I}$ uniformly continuous for $\varphi \in C(Z, \mathbb{I})$.

(6) There is a net f^α of SČ-maps $X \times Y \longrightarrow Z$ converging to f uniformly on countable subsets of $C \times D$ with respect to Z_{pu}.

(7) <u>There exists</u> $\mathcal{F} \subseteq C(Z,\mathbb{I})$ <u>such that</u>

(a) <u>for any two disjoint zero-sets</u> N, M <u>of</u> Z <u>there is some</u> $\varphi \in \mathcal{F}$ <u>with</u> $\overline{\varphi(N)} \cap \overline{\varphi(M)} = \emptyset$

(b) φf <u>is an</u> $S\check{C}$<u>-map for</u> $\varphi \in \mathcal{F}$.

<u>Proof</u>. (1): Evaluation $C^{\beta} \times \beta Y \to \beta Z$ satisfies DCPC on $C^{\beta} \times D$. Therefore [14;p.172], the closure $\overline{C^{\beta}}$ in $\beta Z^{\beta Y}$ is contained in $C(\beta Y, \beta Z)$. By compactness of $\overline{C^{\beta}}$, one gets $\overline{X^{\beta}} = \overline{C^{\beta}}$ in $C(\beta Y, \beta Z)$. But then $f_{\beta}: (\hat{x}, \hat{y}) \mapsto \varepsilon(\hat{x})(\hat{y}): \beta X \times \beta Y \to \beta Z$ is separately continuous.
(2): It suffices to verify (1). In case Z is metrizable, $C = X \subseteq C(Y,Z)$, f evaluation, and $f(C \times D)$ R-compact in Z, this has been done by Grothendieck [14;p.174]. (In this case, the second condition in (2) is automatically satisfied, of course.) By following Grothendieck's arguments, utilizing a suitably refined diagonal process, one obtains that, for any sequences c_n in C and d_m in D, there are subsequences c_{n_i} and d_{m_j}, respectively, such that $\lim_i \lim_j f(c_{n_i}, d_{m_j})$ and $\lim_j \lim_i f(c_{n_i}, d_{m_j})$ exist in Z; whence it follows by means of DLC that $f(c_n, d_m)$ has a double cluster point. - (For $C = X$, $D = Y$, and Z a compact subspace of $\mathbb{R}$, a combinatorial proof of the existence of f_{β} if DLC holds has been given by Pták [25;p.573].)
(3): If $X \times Y$ is pseudocompact, $\beta X \times \beta Y = \beta(X \times Y)$ [13].
(4): X^{β} is R-compact in $C(\beta Y, \beta Z)$ since X^{β} is a uniformly equicontinuous set of maps $\beta Y \to \beta Z$.
(5): follows from (4) because the uniformity of Z_{pu} is the finest on Z rendering uniformly continuous all $\varphi \in C(Z, \mathbb{I})$.
(6): Let (c_n, d_n) be a sequence in $C \times D$ and then $A = \{c_n | n \in \mathbb{N}\}$, $B = \{d_n | n \in \mathbb{N}\}$; let $i: A \to X$ and $j: B \to Y$ denote the inclusions. Then $g: (\hat{a}, \hat{b}) \mapsto \lim_{\alpha} f^{\alpha}_{\beta}(i^{\beta}(\hat{a}), j^{\beta}(\hat{b})): \beta A \times \beta B \to \beta Z$ is a well-defined separately continuous map. If $(\hat{a}, \hat{b})$ is a cluster point of (c_n, d_n) in $\beta A \times \beta B$, then $g(\hat{a}, \hat{b})$ is a double cluster point of $f(c_n, d_m)$ in βZ.
(7): If $\mathcal{B}$ denotes the set of those $\phi \in C(\beta Z, \mathbb{R})$ for which $(\phi|_Z)f$ has a separately continuous extension

$\beta X \times \beta Y \to \mathbb{R}$, then $\mathcal{F}^\beta \subseteq \mathcal{B}$ by means of (b). Moreover, $\mathcal{F}^\beta$ separates points of βZ due to (a). Since $\mathcal{B}$ is a uniformly closed subalgebra of $C(\beta Z, \mathbb{R})$ containing the constant maps, it is all of $C(\beta Z, \mathbb{R})$. Now $f_\beta : (\hat{x}, \hat{y}) \mapsto ((\varphi f)_\beta(\hat{x}, \hat{y}))_{\varphi \in C(Z, \mathbb{I})}$ is a separately continuous extension of f taking its values in βZ.

We will say that X satisfies the <u>joint continuity condition</u> (JCC) if, for every separately continuous map $\varphi : X \times Y \to Z$, with Y compact and Z metrizable, there is a dense G_δ-set G in X so that φ is continuous at all points of $G \times Y$. By a theorem of Namioka [23;1.2], X satisfies JCC, e.g., if X is a G_δ-subspace of some locally compact space. [For locally compact X a simpler proof is now available [30]. X can also be shown to satisfy JCC, e.g., if a) X is Baire and contains a sequence of subspaces satisfying JCC and CCC whose union is dense, or if b) X is closed-hereditarily Baire and has a dense subspace of countable tightness. (Proofs will appear elsewhere.)]

<u>PROPOSITION</u> 2.2. <u>Let</u> f <u>be an</u> SČ-<u>map satisfying</u> ROC. <u>Suppose</u> Z <u>admits a continuous injection into an</u> E-<u>compactum and</u> $\overline{Z}^s = Z$ (cf. 2.8). <u>Let</u> $\mathcal{K} \subseteq \overline{X^*}$.

(1) $\overline{\mathcal{K}}$ <u>is</u> E-<u>compact if</u> $\mathcal{K}(Y)$ <u>is a</u> CCC-<u>subspace of</u> Z.

(2) <u>If</u> g(Y) <u>is a</u> CCC-<u>space for</u> $g \in \mathcal{K}$, <u>then</u> $\mathcal{K}$ <u>is</u> R-<u>compact and</u> RSS-<u>compact in</u> C(Y,Z).

(3) $\overline{\mathcal{K}}$ <u>is compact metrizable, provided</u> $\mathcal{K} \times Y$ <u>satisfies</u> CCC.

<u>Proof</u>. (1): Let $\gamma : \beta Z \to K$ be a continuous map into an E-compactum that is injective on Z. Then $\Gamma : \varphi \mapsto \gamma\varphi : C(\beta Y, \beta Z) \to C(\beta Y, K)$ is homeomorphic on $\overline{\mathcal{K}^\beta}$, being injective on the larger compact space $\overline{X^\beta}$. Namely, let $\varphi, \psi \in \overline{X^\beta}$ with $\gamma\varphi = \gamma\psi$, and let $d \in D$. Now, C^β is dense in $\overline{X^\beta}$, whence $f(C \times \{d\})$ is dense in $\overline{X^\beta}(d)$. But $f(C \times \{d\})$ is R-compact in Z in view of 1.1. Thus, $\varphi(d)$, $\psi(d) \in Z$, hence $\varphi(d) = \psi(d)$. - Since $\mathcal{K}(Y)$ is dense in $\overline{\mathcal{K}^\beta}(\beta Y)$, the latter is a CCC-subspace of βZ. Consequently, $\Gamma(\overline{\mathcal{K}^\beta})(\beta Y)$ is a CCC-subspace of K, and as

such it is metrizable. According to 1.4, $\Gamma(\overline{\mathcal{K}^\beta})$ is E-compact; hence so is $\overline{\mathcal{K}^\beta}$. Let now $\varphi \in \overline{\mathcal{K}^\beta}$. Then $g_n^\beta \to \varphi$ in $C(\beta Y, \beta Z)$ for some sequence g_n in $\mathcal{K}$. For every $y \in Y$ then $g_n(y) \to \varphi(y)$ in βZ, whence $\varphi(y) \in Z$ because of $\overline{Z}^S = Z$. Thus, $\varphi \mapsto \varphi|_Y$ injects $\overline{\mathcal{K}^\beta}$ continuously into $C(Y,Z)$, which implies that $\overline{\mathcal{K}}$ is homeomorphic with $\overline{\mathcal{K}^\beta}$.

(2): If $\mathcal{E} \subseteq \mathcal{K}$ is countable, $\overline{\mathcal{E}}$ is E-compact by (1), hence is metrizable. For every $y \in Y$ then $\overline{\mathcal{K}^\beta}(y) \subseteq Z$, because $\mathcal{K}(y)$ is RS-compact in Z. Thus, $\varphi \mapsto \varphi|_Y$ maps $\overline{\mathcal{K}^\beta}$ continuously onto $\overline{\mathcal{K}}$.

(3): In view of (1) and the Rosenthal criterion, it is sufficient to show that the closure $\overline{\mathcal{K}(Y)}$ in βZ satisfies CCC. So let $(Q_\lambda)_{\lambda \in \Lambda}$ be any family of open subsets of βZ for which all $Q_\lambda \cap \overline{\mathcal{K}(Y)}$ are non-empty and pairwise disjoint. For every λ choose some $h_\lambda \in C(\beta Z)$ with $\emptyset \neq h_\lambda^{-1}(\mathbb{K} \setminus \{0\}) \cap \overline{\mathcal{K}(Y)} \subseteq Q_\lambda$. Then $k_\lambda : (\varphi, \hat{y}) \mapsto h_\lambda(\varphi(\hat{y})) : \overline{\mathcal{K}^\beta} \times \beta Y \longrightarrow \mathbb{K}$ is separately continuous. Due to the fact that $\overline{\mathcal{K}^\beta}$ satisfies JCC, there exists a dense subset G_λ of $\overline{\mathcal{K}^\beta}$ such that k_λ is continuous at all points of $G_\lambda \times \beta Y$. Because $k_\lambda(G_\lambda \times \beta Y)$ is dense in $h_\lambda(\overline{\mathcal{K}(Y)})$, there is a non-empty open subset $U_\lambda \times V_\lambda$ of $\overline{\mathcal{K}^\beta} \times \beta Y$ with $0 \notin k_\lambda(U_\lambda \times V_\lambda)$. The sets $((U_\lambda \cap \mathcal{K}^\beta)|_Y) \times (V_\lambda \cap Y)$ now being pairwise disjoint and non-empty and $\mathcal{K} \times Y$ satisfying CCC, it follows that Λ is countable.

<u>THEOREM</u> 2.3. <u>Let</u> Z <u>and</u> f <u>be as in</u> 2.2. <u>Then</u> $\overline{X^*}$ <u>is</u> E-<u>compact</u>, <u>provided</u> Y^* <u>is separable</u>.

<u>Proof</u>. Consider the maps $\pi: y \mapsto y^*: Y \to Y^*$ and $x^\circ: y^* \mapsto f(x,y): Y^* \to Z$ for $x \in X$. Given a sequence (x_n, y_n) in $X \times Y$, let $(\hat{x}, \hat{y})$ be a cluster point in $\beta X \times \beta Y$; then $f_\beta(\hat{x}, \hat{y})$ is a double cluster point in βZ of $x_n^{\circ\beta}(y_m^*)$. Thus, evaluation $X^{\circ\beta} \times \beta Y^* \to \beta Z$ satisfies DCPC on $X^{\circ\beta} \times Y^*$, where $X^\circ = \{x^\circ | x \in X\}$. As a consequence (cf. proof 2.1.(1)), $X^{\circ\beta}$ has compact closure in $C(\beta Y^*, \beta Z)$. Therefore, $\psi \mapsto \psi \pi^\beta$ establishes a homeomorphism between $\overline{X^{\circ\beta}}$ and $\overline{X^\beta}$. Looking back to proof 2.2.(1), we see that X^β is RS-compact in $\overline{X^\beta}$. Hence X° is RS-

compact in its closure $\mathcal{K}$ in $C(Y^*,Z)$. Since $\mathcal{K}$ admits a continuous injection into the E-compactum $K^{\mathbb{N}}$, it is E-compact by virtue of 1.1. Observe now that $\mathcal{K}$ is homeomorphic with $\overline{X^*}$.

COROLLARY 2.4. *Suppose* Z *is* E-*compact*, Y^* *separable, and* f *satisfies* DLC. *Then* $\overline{X^*}$ *is* E-*compact*.

THEOREM 2.5. *Suppose* Z *is submetrizable*, $\overline{Z}^s = Z$, *and one of the following equivalent conditions holds*:

(1) f *is an* SČ-*map satisfying* LOC *and* ROC.

(2) f *satisfies* DCPC.

(3) f *admits a separately continuous extension* $\beta X \times \beta Y \longrightarrow Z$.

Then $\overline{X^*}$ *and* $\overline{Y^*}$ *are* E-*compact*.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear. Suppose that (1) holds. Let $\gamma: Z \longrightarrow M$ be a continuous injection, with M metrizable. We may assume that M is a subspace of the unit sphere S of some Hilbert space H. As is well-known, the topologies of H and H_σ coincide on S. The unit ball K of H as a subspace of H_σ is, therefore, an E-compactum containing M as a subspace. For every $c \in C$ then $f(\{c\} \times Y)$ is R-compact in Z, hence separable. Let c_n be a sequence in C. By proof 2.2.(1), some subsequence $c_{n_k}^\beta$ of c_n^β converges in $C(\beta Y, \beta Z)$, say to φ. For every $\hat{y} \in \beta Y$ the sequence $c_{n_k}^\beta(\hat{y})$ in Z converges to $\varphi(\hat{y})$ in βZ, whence $\varphi(\hat{y}) \in Z$. Consequently, C^β is RS-compact in $C(\beta Y, Z)$. By virtue of 1.4, the closure $\mathcal{K}$ of C^β in $C(\beta Y, Z)$ is an E-compactum. It follows that $\overline{X^*}$ is E-compact, since $\varphi \mapsto \varphi|_Y: \mathcal{K} \longrightarrow C(Y,Z)$ is a continuous injection mapping C^β onto C^*. For reasons of symmetry, $\overline{Y^*}$ is E-compact as well. Finally, in order to verify (3), it suffices to show that $f_\beta(\beta X \times \beta Y) \subseteq Z$. This, however, is a consequence of the fact that C^β is dense in $\overline{X^\beta}$ and R-compact in $C(\beta Y, Z)$.

COROLLARY 2.6. *If* Z *is compact metrizable and* f *satisfies* DLC, *then* $\overline{X^*}$ *and* $\overline{Y^*}$ *are* E-*compact*.

COROLLARY 2.7. *Hypotheses of 2.5. If* X *or* Y *satisfies* CCC, *then* $\overline{X^*}$ *and* $\overline{Y^*}$ *are metrizable,* $f(X \times Y)$ *is separable, and* f *is Baire measurable.*

Proof: Suppose that X satisfies CCC. Then so does $\overline{X^*}$; whence $\overline{X^*}$ is metrizable in view of 2.5. Now Y^* is in a canonical way homeomorphic with a certain subspace Y^{**} of $C(X^*,Z)$; but the latter is R-compact in $C(X^*,Z)$ by 2.5 since evaluation $Y^{**} \times X^* \longrightarrow Z$ satisfies DCPC. This implies that Y^{**} is separable; whence $\overline{Y^*}$ is separable and, consequently, metrizable as well. Separability of $f(X \times Y)$ is now immediate from the fact that X^* and Y^* are separable and that $f^*\colon (x^*,y^*) \longmapsto f(x,y)\colon X^* \times Y^* \longrightarrow Z$ is separately continuous. - Baire measurability of f means that, for any Baire set B of Z (i.e., member of the σ-algebra generated by the zero-sets of Z), $f^{-1}(B)$ is a Baire set of $X \times Y$. It therefore suffices that φf be of Baire class 1 for given $\varphi \in C(Z, \mathbb{R})$. According to [16;p.327], however, the separately continuous map φf^* is of Baire class 1 because $X^* \times Y^*$ is separable metrizable.

The condition $\overline{Z}^s = Z$, which had been part of some of the preceding results, is actually tantamount to a weak form of normality of Z:

LEMMA 2.8. Z *is sequentially closed in* βZ, *provided any two disjoint closed sets of* Z, *one of which is countable discrete, have disjoint neighborhoods in* Z. *The converse is generally false.*

Proof. Assume that a sequence z_n in Z with distinct terms converges to some $\hat{z} \in \beta Z \setminus Z$. Then $A = \{z_{2n+1} \mid n \in \mathbb{N}\}$ and $B = \{z_{2n} \mid n \in \mathbb{N}\}$ are disjoint closed discrete sets in Z, whence $\varphi(A) = \{0\}$ and $\varphi(B) = \{1\}$ for some $\varphi \in C(Z, \mathbb{I})$ [12;3L4]. Thus, $0 = \varphi^\beta(\hat{z}) = 1$, a contradiction. - In order to construct a space Z that is sequentially closed in βZ, but does not satisfy the condition above, let $\mathbb{D} = \{0,1\}$, $\Lambda = \mathbb{I} \cup [2,3]$, and $S \subseteq \mathbb{D}^\Lambda$ the set of all maps vanishing off countable subsets of Λ. As is well-known, some sequence z_n in $\mathbf{D}^{I}$ is without convergent subsequences. Pick a sequence μ_n in $[2,3]$ with distinct terms, and extend each z_n to $z_n \in \mathbf{D}^{\Lambda}$ by defining $z_n(\mu_n) = 0$ and $z_n(\mu) = 1$ for $\mu \neq \mu_n$. Then $Z = S \cup \{z_n \mid n \in \mathbb{N}\}$ is pseudocompact, whence $\beta Z = \mathbf{D}^{\Lambda}$ [11;p.187]. If Z would satisfy the normality condition, it would be C-compact [12;3L5]; but $\{z_n \mid n \in \mathbb{N}\}$ is closed discrete in Z.

COROLLARY 2.9. *Suppose* Z *is metrizable and* f *satisfies* DCPC. *Then there are first category subsets* R,S *of* X,Y, *respectively, such that* f *is continuous at all points of* $(X \setminus R) \times Y \cup X \times (Y \setminus S)$.

Proof: Since βX is a JCC-space, there exists a dense G_δ-subset G in βX such that $f_\beta : \beta X \times \beta Y \longrightarrow Z$ is continuous at all points of $G \times \beta Y$. Then $R = X \setminus G$ is a union of countably many nowhere dense subsets of X. - Existence of S analogously.

Recall that Y is *pseudo-$\aleph_1$-compact* if every locally finite family of non-empty open subsets of Y is countable. It is not difficult to show that Y is pseudo-$\aleph_1$-compact, e.g., if it is Lindelöf, or pseudocompact, or a CCC-space.

COROLLARY 2.10. *Let* f *be an* $\check{S}C$*-map satisfying* ROC, Z *submetrizable, and* $\overline{Z}^{s} = Z$.

(1) *If* Y *is pseudo-$\aleph_1$-compact,* $\overline{X^*}$ *is compact and* SS-*compact*.

(2) $\overline{X^*}$ *is closure-sequential if* Y^n *is Lindelöf for all* $n \in \mathbb{N}$.

(3) $\overline{X^*}$ *is compact metrizable, provided* Y *satisfies* JCC *and* CCC.

(4) $\overline{X^*}$ *is* E-*compact if* Z *is separably submetrizable*.

Proof. Let $\gamma : \beta Z \longrightarrow K$ be a continuous map, with K an E-compactum and $\gamma(Z)$ metrizable, that is injective on Z. The conditions in 2.2 pertaining to CCC had actually been set up to ensure separability of $\gamma(\mathcal{K}(Y))$ and of the $\gamma g(Y)$ $(g \in \mathcal{K})$, respectively. Thus, (4) is immediate from 2.2.(1). As for (1), it suffices to show that $\gamma g(Y)$ is separable for given $g \in \overline{X^*}$: Due to the metrizability of $\gamma g(Y)$, there exists a sequence $((U^n_\lambda)_{\lambda \in \Lambda_n})_n$ of locally finite families of non-empty open subsets of $\gamma g(Y)$ such that $\bigcup_n \{U^n_\lambda | \lambda \in \Lambda_n\}$ is a base for the topology of $\gamma g(Y)$ [19;p.127]. Being a continuous image of Y, then $\gamma g(Y)$ is pseudo-$\aleph_1$-compact as well; whence all Λ_n are countable.

(2): Let $\mathcal{K} \subseteq \overline{X^*}$ and $g \in \overline{\mathcal{K}}$. In view of (1), it is good enough to get $g \in \overline{\mathcal{A}}$ for some countable $\mathcal{A} \subseteq \mathcal{K}$. Because $\overline{X^*}$ is compact, there is no loss of generality in assuming Z metrizable. For σ-compact Y and $Z = \mathbb{K}$ the desired conclusion $g \in \overline{\mathcal{A}}$ is known to be true (cf. [20; p.312]); and a modification of the standard argument takes care of the general situation here.

(3): Consider the subspace $\tilde{Y} = \{\tilde{y} | y \in Y\}$ of $C(\overline{X^*}, \gamma(Z))$, where $\tilde{y}(g) = \gamma g(y)$. Since $\overline{X^*}$ is compact by virtue of (1), there exists a dense G_δ-subset R of Y such that $(y,g) \mapsto \tilde{y}(g)$: $Y \times \overline{X^*} \to \gamma(Z)$ is continuous at all points of $R \times \overline{X^*}$. If now $\mathcal{C}$ stands for $C(\overline{X^*},\gamma(Z))$, equipped with the topology of uniform convergence with respect to some compatible metric of $\gamma(Z)$, it follows that $\rho : y \mapsto \tilde{y}$: $R \to \mathcal{C}$ is continuous. Being a CCC-subspace of the metrizable space $\mathcal{C}$, then $\rho(R)$ is separable in $\mathcal{C}$ and, therefore, separable in $C(\overline{X^*},\gamma(Z))$. Consequently, $\tilde{Y}$ is separable. But $\overline{X^*}$ can be injected continuously into $C(\tilde{Y},\gamma(Z))$ in a canonical fashion.

THEOREM 2.11. Let Y_m be a sequence of subsets of E whose union is total, V_n a sequence of equicontinuous sets of H' whose union is weak*-total, and $\mathcal{K} \subseteq \mathcal{L}(E,H)_W$ RC-compact as well as uniformly bounded on each Y_m. Suppose for n,m and sequences (T_i, v_i, y_i) in $\mathcal{K} \times V_n \times Y_m$ $\lim_i \lim_j v_i(T_j y_i) = \lim_j \lim_i v_i(T_j y_i)$ whenever these limits exist. Then $\mathcal{K}$ is E-compact.

Proof: Fix n,m. Then there exists a compact subspace

Z_{nm} of $\mathbb{K}$ such that the separately continuous $\mathbb{K}$-valued map $(T,(v,y)) \mapsto v(Ty)$ on $\mathcal{K} \times (V_n \times Y_m)$ takes its values in Z_{nm}. According to 2.6, then the set of all maps $(v,y) \mapsto v(Ty)$: $V_n \times Y_m \to Z_{nm}$, with $T \in \mathcal{K}$, has E-compact closure K_{nm} in $C(V_n \times Y_m, Z_{nm})$. The proof can now be finished like that of 1.2.

COROLLARY 2.12. *Let E be a Hilbert space, B a base, and $\mathcal{K} \subseteq \mathcal{L}(E,E)$ a pointwise bounded set of compact operators. Suppose for every sequence (T_n, v_n, b_n) in $\mathcal{K} \times B^2$, with $T_n \neq T_m$, $v_n \neq v_m$, $b_n \neq b_m$ if $n \neq m$, $\lim_m \lim_n \langle T_n b_m, v_m \rangle = 0$ whenever the limits exist. Then $\overline{\text{conv ci}}\ \mathcal{K}$ is E-compact in $\mathcal{L}(E,E)_w$.*

REMARKS AND EXAMPLES 2.13. 1. Via 2.1.(7), the problem of the SČ-extendability of f can be reduced to the corresponding one for (sufficiently many) associated scalar maps. For such maps a number of criteria has been given by Pták [25].

2. 2.2.(1) breaks down if Z cannot be injected into an E-compactum: Take $X = Y = Z$ the Helly space [19;p.164] and f: $(x,y) \mapsto \frac{1}{2}(x+y)$.

3. The hypotheses in 2.2 referring to CCC are essential: Let X be the group $\mathbf{D}^{\mathbb{R}}$, $Y = Z$ the unit ball of $L^2(X)$ in its weak topology, and f translation.

4. In 2.2.(2), $\mathcal{K}$ need not be compact if Z cannot be injected into an E-compactum, even if $X = Y = Z$ is an SS-compact closure-sequential CCC-space: Take the space S of proof 2.8 and f multiplication.

5. If Y is C-compact and Z is as in 2.5, then (1)-(3) and DLC are also necessary in order that $\overline{X^*}$ be compact. The same cannot be said of the metrizability and compactness required of $\overline{f(X \times Y)}$ in [14;p.175,Cor.2] : Let X be the circle group. Let $Y \subseteq C(X)$ consist of a convergent sequence plus limit point, convergence not being uniform. Consider $Z = \{\tilde{x}(\varphi) \mid x \in X,\ \varphi \in Y\}$, where $\tilde{x}(\varphi)(h) = \varphi(xh)$, and f: $(x,\varphi) \mapsto \tilde{x}(\varphi)$. If Z were metrizable or compact, the group action $X \times Z \to Z$ would

be continuous (cf. [17]), hence Y's topology finer than the topology of uniform convergence [19;p.223].

6. If DLC is dropped in 2.4 or 2.6, $\overline{X^*}$ may be compact, separable, and first countable, without being E-compact: Take $Z = \mathbb{I}$, $Y = [0,1]$ discrete, X all non-decreasing maps in $C(\mathbb{I},\mathbb{I})$, and f evaluation; then $\overline{X^*}$ is the Helly space.

7. From [31;p.484,Prop.6] it follows that, for every locally compact Abelian group G, $\alpha(G)$ is sequentially closed in G^a, where $\alpha: G \to G^a$ is the canonical map from G into its Bohr compactification; therefore, $\alpha(G)$ is sequentially closed in $\beta\alpha(G)$. We will give an example showing that, for non-locally compact G, this is no longer true. Since $\beta G = G^a$ for this G, hereby it is also shown that Z is generally not sequentially closed in βZ (contrary to what has been claimed in [26]): If S is as in proof 2.8 and G the subgroup of $\mathbb{D}^{\mathbb{R}}$ generated by S and the characteristic maps of $]-\infty, n]$ $(n \in \mathbb{N})$, then $G^a = \mathbb{D}^{\mathbb{R}}$. Also $\beta G = \mathbb{D}^{\mathbb{R}}$ [11;p.187]. But G is not sequentially closed in $\mathbb{D}^{\mathbb{R}}$ because the map $t \mapsto 1: \mathbb{R} \to \mathbb{D}$ does not belong to G.

8. 2.9 collapses utterly if DLC is violated: There are countable groups Y of homeomorphisms of $\mathbb{I}^2$ for which evaluation $\mathbb{I}^2 \times Y \to \mathbb{I}^2$ is even sequentially discontinuous at all points [17;p.162].

9. Up to homeomorphy, every E-compactum occurs as $\mathcal{K}$ in 2.12. In fact, if K is weakly compact in a Banach space X, there are $E, B, \mathcal{K}$ as in 2.12 and an affine homeomorphism $\alpha: \overline{\text{conv ci}}\ K \to \overline{\text{conv ci}}\ \mathcal{K}$ with $\alpha(K) = \mathcal{K}$: By [21;2.4], it suffices to consider $X = c_0(\Gamma)$. Let then $E = \ell^2(\Gamma)$ and B the usual base. For $g \in c_0(\Gamma)$ define $T_g \in \mathcal{L}(E,E)$ by $T_g \varphi = g\varphi$, and let $\mathcal{K} = \{T_g \mid g \in K\}$. Then $\alpha: g \mapsto T_g$ is as desired.

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