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A COUNTEREXAMPLE IN ELLIPTIC REGULARITY THEORY

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An example shows that bounded weak solutions of diagonal quasilinear elliptic systems with quadratic growth and satisfying a one-sided condition need not be Hölder continuous if the space dimension exceeds two.

INTRODUCTION

Consider the elliptic system

$$(1) \qquad Lu^{\ell} = f^{\ell}(x,u,\nabla u) \quad , \quad 1 \leq \ell \leq N,$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with a uniformly elliptic differential operator L of second order and a function f satisfying - apart from obvious regularity assumptions - the conditions

$$(2) \qquad |f(x,u,\nabla u)| \leq a|\nabla u|^2 \quad ,$$

$$(3) \qquad u \cdot f(x,u,\nabla u) \leq \lambda^* |\nabla u|^2 \quad ,$$

where

$$(4) \qquad \lambda^* < \lambda \ ,$$

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λ being the ellipticity constant of the operator L and the function a: $\Omega \to \mathbb{R}$ belonging to $L^\infty(\Omega)$, for example (cp.[4]).

If n = 2 any bounded weak solution $u \in (H_0^{1,2}(\Omega) \cap L^\infty(\Omega))^N$ of (1) is known to be Hölder continuous under the above assumptions (cp. [2], [10]). In the case of inhomogeneous boundary data and for n > 2 the question of regularity has remained open.

It was conjectured by Frehse [1] that bounded weak solutions of (1) will always be regular iff a Liouville-type theorem is true for the system studied, i.e. if any bounded weak solution of (1) on the whole space necessarily reduces to a constant. Indeed, such a theorem of Liouville type was shown to be true if n = 2 [8] whereas it was also demonstrated by Meier [8] that such a result will in general not hold if n > 2.

Moreover, it was shown by Ivert [6] that in general it is not possible to bound the Hölder norm of a solution of system (1) *a-priori* if n > 2.

Thus one is led to suspect that solutions of (1) will in general not be Hölder continuous under the above assumptions (2)-(4) in the case that n exceeds 2. That this suspicion is justified is shown by the example given below.

EXAMPLE

Let M be the function constructed by Meier in [8, ex. 3.4]. Choosing n = 3 $M \in C^2(\mathbb{R}^3;\mathbb{R}^2)$ has the following properties:

(5) i) There exists a vector $M_\infty \in \mathbb{R}^2$ and a constant c_0 such that for $s \in \{0,1,2\}$ and $|x| \geq c_0$

$$|\nabla^s(M(x) - M_\infty)| \leq c_0|x|^{-1-s} \quad .$$

Also for $|x| \geq c_0$

$$\Delta M(x) = 0 \ .$$

ii) There exists a positive constant c_1 such that for $|x| \leq c_0$

$$|\nabla M(x)| \geq c_1 .$$

iii) There exists a constant $a \in \mathbb{R}$ such that

$$|\Delta M| \leq a|\nabla M|^2 .$$

iv) $-\Delta M \leq 0$, $M \geq 0$, $-M\Delta M \leq 0$.

Choose constants p,q,r such that $p > 1$, $0 < q < 1$, $0 < 3r < 1$, $p^r q > 1$. Let e be a unit vector in $\mathbb{R}^3$ and define $Q_j = \sum_{i=0}^{j} q^i \cdot e$, $Q_\infty = \lim Q_j$. Moreover, letting $B_R(x) = \{y \,|\, |y-x| < R\}$ set

$$C_j = B_{p^{-rj}c_0}(Q_j) \; ; \; D_j = B_{p^{-j}c_0}(Q_j)$$

and let γ_j be the characteristic function of C_j, i.e. $\gamma_j(x) = 1$ for $x \in C_j$, $= 0$ else.

Define for some integer K to be yet determined

$$\phi(x) = M(x) - M_\infty \quad ,$$

$$\phi_j(x) = \phi(p^j(x-Q_j)) \; ,$$

$$\psi_j = \sum_{i=K}^{j} \phi_i \quad ; \quad \psi_\infty = \text{(pointwise) } \lim \psi_i \; ,$$

$$u = \psi_\infty - \inf \psi_\infty \text{ (componentwise)}.$$

LEMMA 1

The function u is well-defined. Moreover, for any $s \in \{0,1,2\}$ we have

$$\left|\nabla^s\left(\psi_\infty(x) - \psi_k(x) - \sum_{j>k} \gamma_j(x)\phi_j(x)\right)\right| \to 0$$

as $k \to \infty$, uniformly in $x \in \mathbb{R}^3 \setminus \bigcup_{j>k} \partial C_j$, $K \in \mathbb{N}$.

PROOF

First observe that for large k given any $x \in \mathbb{R}^3$ there is at most one index $j > k$ such that $\gamma_j(x) \neq 0$. Indeed,

assuming $i > j$ to be two such numbers we have

$$|Q_i - Q_j| \geq q^{j+1} .$$

Also since $x \in C_i \cap C_j$ we obtain that

$$|Q_i - Q_j| \leq c_o(p^{-ri} + p^{-rj}) \leq 2c_o p^{-rj}$$

whence we conclude that for all $j \in \mathbb{N}$

$$(p^r q)^j \leq \text{const.}$$

contrary to the assumption $p^r q > 1$.

By (5.i) for any $m > k$ we obtain the estimate

$$\begin{aligned} &\left|\nabla^s\left(\psi_m(x) - \psi_k(x) - \sum_{j=k+1}^{m} \gamma_j(x)\phi_j(x)\right)\right| \\ &\leq \sum_{j=k+1}^{m} \sup\{|\nabla^s\phi_j(x)| \,|\, x \notin C_j\} \\ &\leq c_o \sum_{j>k} p^{sj}\left(p^{(1-r)j}c_o\right)^{-1-s} \\ &\leq \text{const. } p^{-k((1+s)(1-r)-s)} . \end{aligned}$$

From this estimate and the fact that $C_j \cap C_i = \phi$ for $i > j > k$, k large, the pointwise convergence of the sequence ψ_j follows. Since the limit function ψ_∞ also is seen to be uniformly bounded in $\mathbb{R}^3$ u is well-defined.

Passing to the limit $m \to \infty$ in the above estimate we also obtain the second assertion of the lemma.

qed.

LEMMA 2

For any $x \in D_j$,

$$\left|\nabla\left(\psi_\infty(x) - \phi_j(x)\right)\right| / |\nabla\phi_j(x)| \to 0$$

as $j \to \infty$, uniformly in $x \in D_j$.

PROOF

By (5.ii) we have $|\nabla\phi_j(x)| \geq p^j c_1$ on D_j. Let k be an integral number, $k \in \{\frac{g-1}{2}, j/2\}$. By Lemma 1 and since $|\nabla\psi_k| \leq$ $\leq$ const. p^k we obtain

$$\left|\nabla\Big(\psi_\infty(x) - \phi_j(x)\Big)\right| \leq \left|\nabla\Big(\psi_\infty(x) - \psi_k(x) - \phi_j(x)\Big)\right|$$
$$+ |\nabla\psi_k(x)| \leq \varepsilon + \text{const. } p^k \leq \text{const. } p^{-k}|\nabla\phi_j|.$$

qed.

Choose K sufficiently large such that in Lemma 2 the left-hand side may be estimated by 1/2 for i > K and such that for i > j > K $C_i \cap C_j = \phi$.

LEMMA 3

The function u: $\mathbb{R}^3 \to \mathbb{R}^2$ satisfies the following:

i) $u \in L^\infty(\mathbb{R}^3;\mathbb{R}^2)$; $|\nabla u| \in L^2(\mathbb{R}^3)$

ii) $|\Delta u| \leq 4a|\nabla u|^2$

iii) $-u\Delta u \leq 0$

iv) u is not continuous .

PROOF

i) By Lemma 1 $u \in L^\infty$; moreover $|\nabla u| \in L^2(\mathbb{R}^3)$ since $|\nabla M| \in L^2(\mathbb{R}^3)$ and

$$\|\nabla u\|_2 \leq \sum_{j=0}^{\infty} \|\nabla\phi_j\|_2 \leq \sum_{j=0}^{\infty} p^{-j/2}\|\nabla M\|_2 < \infty.$$

ii) By our choice of K the sets D_j, j > K, are disjoint. For $x \in D_j$ we thus have using (5.iii)

$$|\Delta u(x)| = |\Delta\phi_j(x)| \leq a|\nabla\phi_j(x)|^2$$
$$\leq 4a|\nabla u(x)|^2 \quad .$$

(Note that Lemma 2 implies

$$|\nabla\phi_j(x)| \leq \left|\nabla\Big(\psi_\infty(x) - \phi_j(x)\Big)\right| + |\nabla\psi_\infty(x)|$$

$$\leq 2^{-1}|\nabla\phi_j(x)| + |\nabla\psi_\infty(x)|$$

which implies the last inequality in the above estimate, since $\nabla\psi_\infty \equiv \nabla u$.)

For $x \notin \bigcup_j D_j$

$$\Delta u(x) = 0.$$

This proves ii).

iii) By (5.iv) and since $D_i \cap D_j = \phi$ for $i > j > K$ $\Delta u \geq 0$. By our definition of u also $u \geq 0$ which implies iii).

iv) Choose $x,y \in C_o$, $\varepsilon > 0$ such that $|\phi_o(x) - \phi_o(y)| > \varepsilon$. Let $x_i = p^{-i}x + Q_i$ $y_i = p^{-i}y + Q_i$. Then $|x_i - y_i| \to 0$ $(i \to \infty)$, whereas by Lemma 1 and the continuity of ψ_k

$$\begin{aligned}|u(x_i)-u(y_i)| &\geq |\phi_i(x_i)-\phi_i(y_i)| - |\psi_\infty(x_i)-\psi_k(x_i)-\phi_i(x_i)| \\ &\quad - |\psi_\infty(y_i)-\psi_k(y_i)-\phi_i(y_i)| - |\psi_k(x_i)-\psi_k(y_i)| \\ &> \varepsilon/2\end{aligned}$$

if first k and then i is chosen sufficiently large. Thus u is not continuous.

This concludes the proof and the construction. □

REMARK

The above example can easily be modified to show that bounded weak solutions of diagonal systems with zero boundary data need not be smooth if $n > 2$ and if one imposes only the conditions (4),

$$(2') \qquad |f(x,u,\nabla u)| \leq a|\nabla u|^2 + b ,$$

and

$$(3') \qquad u \cdot f(x,u,\nabla u) \leq \lambda^*|\nabla u|^2 + b^*$$

where $b, b^* \in \mathbb{R}$. Indeed one obtains a counterexample from the function u constructed above by multiplying u with a smooth cut-off function $\tau \in C_0^\infty(\mathbb{R}^3)$, satisfying $0 \leq \tau \leq 1$, $\tau \equiv 1$ on $\bigcup_{j>K} C_j$.

REFERENCES

As anyone familiar with problem (1) will easily notice the list of references we give below is far from complete. In addition to the publications cited above we have included some further references relevant for the investigation of Liouville type properties of elliptic systems and studies of possible connections between Liouville type theorems and regularity results. For a vast survey of articles of a more general scope we refer the reader to [2].

[1] FREHSE, J.: Essential selfadjointness of singular elliptic operators, Bol. Soc. Bras. Mat. 8.2, 87-107 (1977)

[2] FREHSE, J.: On two-dimensional quasi-linear elliptic systems, manuscripta math. 28, 21-49 (1979)

[3] GIAQUINTA, M. and NECAS, J.: On the regularity of weak solutions to nonlinear elliptic systems via Liouville's type property, Comment. Math. Univ. Carolinea 20, 111-121 (1979)

[4] HILDEBRANDT, S. and WIDMAN, K.-O.: On the Hölder-continuity of weak solutions of quasilinear elliptic systems of second order, Ann. Sc. Norm. Sup. Pisa (4), 4, 145-178 (1977)

[5] HILDEBRANDT, S. and WIDMAN, K.-O.: Sätze vom Liouville-schen Typ für quasilineare Gleichungen und Systeme, Nachr. Akad. Wiss. Göttingen, 41-61 (1979)

[6] IVERT, P.A.: On quasilinear elliptic systems of diagonal form, Math. Zeitschrift 170, 283-286 (1980)

[7] KAWOHL, B.: On Liouville-theorems, continuity, and Hölder continuity of weak solutions to some quasi-linear elliptic systems. Preprint

[8] MEIER, M.: Liouville theorems for nonlinear elliptic equations and systems. manuscripta math. 29, 207-228 (1979)

[9] WIEGNER, M.: Das Existenz- und Regularitätsproblem bei Systemen nichtlinearer elliptischer Differentialgleichungen, Habilitationsschrift, Bochum (1977)

[10] WIEGNER, M.: On two-dimensional elliptic systems with a one-sided condition. Preprint

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