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BARRELLED SPACES WITH A B-COMPLETE COMPLETION

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Let (E,η) be a barrelled locally convex space. The aim of this paper is to describe the barrelled topologies on E weaker than η. When the completion $\hat{E}_\eta$ of (E,η) is B-complete, various properties of the barrelled topologies weaker than η are proved. Some examples are given to illustrate the possible situations.

1. Although it will no longer be mentionned, all topologies considered in this paper are locally convex and Hausdorff. Given a vector space E (over $\mathbb{R}$ or $\mathbb{C}$) and a topology η on E, $\mathcal{U}_\eta$ denotes a base of absolutely convex neighborhoods of zero in E. The completion of (E,η) is denoted by $\hat{E}_\eta$ and its dual by E'_η. Given a dual pair $\langle E,F\rangle$, the corresponding Mackey, weak and strong topologies on E are denoted by $\mu(E,F)$, $\sigma(E,F)$, $\beta(E,F)$.

A subspace L of E'_η is <u>nearly closed</u> if $L\cap U^\circ$ is $\sigma(E',E)$-closed for each $U \in \mathcal{U}_\eta$. We denote by $\mathcal{L}_\eta$ the set of all nearly closed

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subspaces of E'_η and by $\mathcal{L}_{r\eta}$ the set of all nearly closed subspaces of E'_η which are dense for $\sigma(E'_\eta, E)$.

Let $\mathcal{M}_\eta$ be the set of all closed subspaces of $\hat{E}_\eta$ and $\mathcal{M}_{r\eta}$ the set of all $M \in \mathcal{M}_\eta$ such that $M \cap E = \{0\}$ (a subspace M of $\hat{E}_\eta$ will be called <u>transversal</u> to E if $M \cap E = \{0\}$). Finally $\mathcal{T}_\eta$ is the collection of all barrelled topologies on E weaker than η and $\mathcal{Q}_\eta$ is the collection of quotients of E, equipped with a barrelled topology weaker than the topology induced by E.

Equip $\mathcal{M}_\eta$ and $\mathcal{M}_{r\eta}$ with the usual order relation $M_1 \leq M_2 \iff M_1 \subset M_2$, $\mathcal{T}_\eta$ with the usual order relation $\xi_1 \leq \xi_2$ and $\mathcal{Q}_\eta$ with the order relation $(E/N_1, \xi_1) \leq (E/N_2, \xi_2) \iff N_2 \subset N_1$ and the canonical map $(E/N_2, \xi_2) \to (E/N_1, \xi_1)$ is continuous.

A topology η on E is called <u>minimal barrelled</u> if it is minimal among the barrelled topologies on E for the above mentionned order relation.

Although we did'nt find it explicitly in the literature, the following result is essentially known and is actually the basis of the Ptak's closed graph theorem and its generalizations.

PROPOSITION 1.1. Let (E,η) be barrelled :

(i) the map $j : \mathcal{Q}_\eta \to \mathcal{O}_\eta : L \to (E/L^\circ, \mu(E/L^\circ,L))$ is a bijection,

(ii) the map $j' : \mathcal{L}_{r\eta} \to \mathcal{T}_\eta : L \to \mu(E,L)$ is a bijection,

(iii) if $\hat{E}_\eta$ is B-complete, the maps $k : \mathcal{M}_\eta \to \mathcal{Q}_\eta : M \to (E/M\cap E, \mu(E/M\cap E, M^\circ))$ and $k' : \mathcal{M}_{r\eta} \to \mathcal{T}_\eta : M \to \mu(E,M^\circ)$ are bijections,

(iv) if $\hat{E}_\eta$ is B-complete, k' is a bijection between the closed algebraic complements of E in $\hat{E}_\eta$ and the minimal barrelled topologies on E weaker than η.

PROOF. (i) Let L be a subspace of E'_η. The space $(E/L^\circ, \mu(E/L^\circ,L))$ is a Hausdorff quotient of E. Since (E,η) is barrelled, the set $\{U^\circ\cap L : U \in \mathcal{U}_\eta\}$ is a fundamental family of bounded sets in $(L,\sigma(L,E/L^\circ))$. If $L \in \mathcal{L}_\eta$, each $U^\circ\cap L$ is closed hence compact in $(E'_\eta,\sigma(E',E))$. It is thus equicontinuous for $\mu(E/L^\circ,L)$. Hence $\mu(E/L^\circ,L)$ is a barrelled topology. Conversely, if $\mu(E/L^\circ,L)$ is barrelled, each $U^\circ\cap L$ is compact in $(L,\sigma(L,E/L^\circ))$. Since $\sigma(L,E/L^\circ) = \sigma(L,E)$, $U^\circ\cap L$ is thus closed in $(E'_\eta,\sigma(E'_\eta,E))$ and $L \in \mathcal{L}_\eta$.

(ii) Observe that, in (i), $L^\circ = \{0\}$ if and only if L is dense in $(E'_\eta,\sigma(E'_\eta,E))$.

(iii) The correspondance $l : \mathcal{M}_\eta \to \mathcal{L}_\eta : M \to M^\circ$ is a bijection. First, $l(\mathcal{M}_\eta) \subset \mathcal{L}_\eta$. Indeed, $M^\circ \cap U^\circ$ is always $\sigma(E'_\eta, \hat{E}_\eta)$-compact, hence $\sigma(E'_\eta, E)$-compact. Recall that E is B-complete if the nearly closed subspaces of E'_η are weakly closed. Since $\hat{E}_\eta$ is B-complete, for each $L \in \mathcal{L}_\eta$, $L = (L^\circ)^\circ$ (the first polar being taken in $\hat{E}_\eta$). Hence l is surjective. Since $M = M^{\circ\circ}$, it is also injective. Observe then that $k = j \circ l$. The conclusion follows from (i) and (ii).

(iv) It is trivial that, in case (iii), the maps k and k' reverse the order on $\mathcal{M}_\eta$, $\mathcal{Q}_\eta$ and $\mathcal{T}_\eta$: $M_1 \leq M_2 \iff kM_1 \geq kM_2$ or $k'M_1 \geq k'M_2$. Thus $\xi = kM$ is minimal barrelled if and only if M is maximal in $\mathcal{M}_{r\eta}$. It is obvious that $M \in \mathcal{M}_{r\eta}$ is maximal if and only if it is an algebraic complement of E in $\hat{E}_\eta$. Q.E.D.

REMARK. It is clear that $(\mathcal{M}_{r\eta}, <)$ has no maximum element; thus $(\mathcal{T}_\eta, <)$ has no minimum element.

COROLLARY 1.2. Let (E,η) be barrelled. Then η is minimal barrelled if and only if (E,η) is B_r-complete; (E,η) is B-complete if and only if, for every closed subspace M of E, E_η/M is minimal barrelled.

PROOF. By Prop. 1.1 (ii), η is minimal barrelled if and only if $\mathcal{L}_{r\eta} = \{E'_\eta\}$, which is the definition of B_r-completeness. For the second case, observe that E is B-complete if and only if E_η/M is B_r-complete for every closed subspace of E_η. Q.E.D.

COROLLARY 1.3. *Let* (E,η) *be barrelled and* $\hat{E}_\eta$ *be* B-*complete. Given* $\xi \leq \eta$, ξ *is minimal barrelled if and only if* (E,ξ) *is isomorphic to a quotient of* $\hat{E}_\eta$.

PROOF. By Prop. 1.1 (iv), if E_ξ is minimal, $\xi = \mu(E,M^\circ)$ for some closed algebraic complement of E in $\hat{E}_\eta$. The space $\hat{E}_\eta/M$ and E are naturally algebraically isomorphic. Moreover, they are equipped with the topologies $\mu(E,M^\circ)$ and $\mu(\hat{E}_\eta/M,M^\circ)$, which correspond to each other by this isomorphism. The converse follows from Cor. 1.2. Q.E.D.

COROLLARY 1.4. *Let* (E,η) *be barrelled and* $\hat{E}_\eta$ *be* B-*complete. For every* $\xi \leq \eta$, *the associated barrelled topology of* ξ *is* $\mu(E,L)$, *where* L *is the closure of* E'_ξ *in* $(E_\eta,\sigma(E'_\eta,\hat{E}_\eta))$.

PROOF. Indeed, the associated barrelled topology of ξ is $\mu(E,M^\circ)$ for the largest $M \in \mathcal{M}_{r\eta}$ such that $M^\circ \supset E'_\xi$. It is clear that $M = E'^\circ_\xi$ (polar in $\hat{E}_\eta$), hence the result. Q.E.D.

The fact that a barrelled space (E,η) has

a B-complete completion is inherited by all weaker barrelled topologies on E.

PROPOSITION 1.5. Suppose that E_η is barrelled and that $\hat{E}_\eta$ is B-complete. Then, for every barrelled $\xi \leq \eta$, $\hat{E}_\xi$ is B-complete.

PROOF. Observe first that $\sigma(E'_\xi,\hat{E}_\xi)$ is finer than $\sigma(E'_\xi,\hat{E}_\eta)$ ([6], p.158).

Suppose that L is nearly closed in $(E'_\xi,\sigma(E'_\xi,\hat{E}_\xi))$. For every $U \in \mathcal{U}_\eta$, $U^\circ \cap E'_\xi$ is equicontinuous for ξ. Thus $L \cap U^\circ$ is closed and even compact in $(E'_\xi,\sigma(E'_\xi,\hat{E}_\xi))$. By the above remark, it is thus also compact in $(E'_\xi,\sigma(E'_\xi,\hat{E}_\eta))$, and it is closed in $(E'_\eta,\sigma(E'_\eta,\hat{E}_\eta))$. Since $\hat{E}_\eta$ is B-complete, it follows that L is closed in $(E'_\eta,\sigma(E'_\eta,\hat{E}_\eta))$ hence, using again the remark, in $(E'_\xi,\sigma(E'_\xi,\hat{E}_\xi))$. Q.E.D.

It is well-known that every normed or metrizable space (E,η) has a B-complete completion. If (E,η) is barrelled, its normed or metrizable character is again inherited by the elements of τ_η. This follows from the next slightly more general result.

PROPOSITION 1.6. If (E,η) is metrizable (resp. normed), every weaker barrelled topology ξ on E is metrizable (resp. normed). If (E,η) is moreover barrelled, $\beta(E,E'_\xi)$ is metrizable

(resp. normed) for every $\xi \leq \eta$.

PROOF. Suppose that (E,η) is metrizable (resp. normed). Let $\xi \leq \eta$ be barrelled. If U_n $(n \in \mathbb{N})$ is a base of neighborhoods of zero in (E,η), the subsets $U_n^\circ \cap E'_\xi$ are bounded for $\sigma(E'_\xi,E)$, hence equicontinuous for ξ. Conversely, every equicontinuous subset of E'_ξ is contained in some $U_n^\circ \cap E'_\xi$. Thus E'_ξ has a fundamental sequence of equicontinuous subsets and (E,ξ) is metrizable. If (E,η) is normed, replacing U_n by $n^{-1}U$, where U is the unit ball of (E,η), we conclude that (E,ξ) is normed.

Suppose now that (E,η) is moreover barrelled. Then $(E'_\eta,\sigma(E'_\eta,E))$ has a fundamental sequence of bounded sets (or a generating bounded set). So does $(E'_\xi,\sigma(E'_\xi,E))$ for every $\xi \leq \eta$, hence $\beta(E,E'_\xi)$ is metrizable or normed. Q.E.D.

Let η and ξ be two locally convex topologies on E, such that $\xi \leq \eta$. We say that η satisfies the filter condition with respect to ξ if every Cauchy filter for η which converges to zero for ξ also converges to zero for ξ. This condition holds if and only if $\hat{E}_\eta$ can be embedded in $\hat{E}_\xi$. In dual form, it means that E'_ξ is dense in $(E'_\eta,\sigma(E'_\eta,\hat{E}_\eta))$.

It is known [3] that the associated barrelled topology to η satisfies the filter condition with respect to η. The next proposition

provides a partial converse to this result.

PROPOSITION 1.7. Suppose that (E,η) is barrelled and that $\hat{E}_\eta$ is B-complete. Let $\xi \leq \eta$. The following are equivalent :

(i) τ is the barrelled topology associated to ξ,

(ii) τ is barrelled, satisfies the filter condition with respect to ξ and $\xi \leq \tau \leq \eta$,

(iii) τ is the finest topology weaker than η which satisfies the filter condition with respect to ξ.

PROOF. (i) $\Rightarrow$ (ii). Denote by τ_o the barrelled topology associated to ξ. Since $\tau_o \leq \eta$, by Prop. 1.5, $\hat{E}'_{\tau_o}$ is B-complete. By Cor. 1.4, E'_ξ is thus dense in $(E'_{\tau_o},\sigma(E'_{\tau_o},\hat{E}_{\tau_o}))$ hence τ_o verifies the filter condition with respect to ξ.

(ii) $\Rightarrow$ (i). Suppose that τ verifies (ii). Then $\hat{E}_\tau$ is B-complete (Prop. 1.5) and, by the filter condition, E'_ξ is dense in $(E'_\tau,\sigma(E'_\tau,\hat{E}_\tau))$. By Cor. 1.4, τ is thus the barrelled topology associated to ξ.

(iii) $\Rightarrow$ (i). Since the supremum of topologies satisfying the filter condition with respect to ξ satisfies this condition itself,

the topology τ defined by (iii) exists. It is moreover barrelled. Indeed, the topology τ' for which the barrels of (E,τ) form a base of neighborhoods of zero still verifies the filter condition with respect to ξ and is weaker than η, hence it coincides with τ. Thus τ verifies (ii) and $\tau = \tau_0$. Q.E.D.

2. We shall give now some examples illustrating the results above.

With the notations of Prop. 1.1, the description of $\mathcal{M}_{r\eta}$ is not easy in general. Of course, every finite dimensional subspace $M \subset \hat{E}_\eta$ transversal to E is closed. If E has finite codimension in $\hat{E}_\eta$, E admits thus always a minimal barrelled topology $\xi \leq \eta$. This, combined with Cor. 1.3, gives back the result of [5] that every finite codimensional subspace of a Banach space has a weaker complete norm. In the same paper, S.A. Saxon and A. Wilansky ask whether the same is true for a subspace of countable codimension. Such a subspace is known to be barrelled [7]. Since a Frechet space admits no closed subspace of countable dimension, by Prop. 1.1 (iv), the answer is no.

If E has uncountable codimension in $\hat{E}_\eta$, there is no evidence that there exists a closed subspace M of infinite dimension in $\hat{E}_\eta$ transversal to E or, equivalently, that there

exists a weaker barrelled topology ξ on E such that E'_ξ has infinite codimension in E'_η. We give two examples where this is true.

EXAMPLE 2.1. An infinite dimensional closed subspace of $\hat{E}_\eta$, transversal to E.

Let $\mathbb{R}^d$ be an uncountable product of copies of $\mathbb{R}$. Denote by E the subspace of $\mathbb{R}^d$ consisting of all elements with at most countably many non zero components. The space E equipped with the topology η included by $\mathbb{R}^d$ is barrelled ([4], p.214) ant its completion $\mathbb{R}^d$ is B-complete. Indeed, the dual of $\mathbb{R}^d$ is the locally convex direct sum $\oplus^d\mathbb{R}$ and every hyperplane, hence every subspace of $\oplus^d\mathbb{R}$ is closed for $\sigma(\oplus^d\mathbb{R},\mathbb{R}^d)$.

Consider a partition $\{I_n : n \in \mathbb{N}\}$ of the set of indices I such that card $I_n > \aleph_o$ for each $n \in \mathbb{N}$ and define L as the set of all elements of $\mathbb{R}^d$ which are constant on each I_n. It is clearly closed, infinite dimensional and such that $L \cap E = \{0\}$.

The following example concerns normed or metrizable spaces.

EXAMPLE 2.2. Normed or metrizable barrelled spaces (E,η) with a minimal barrelled topology ξ such that E'_ξ has infinite codimension in E'_η.

Denote by E a Banach space with norm η

and by L a dense subspace of codimension one in E. Denote by M an algebraic complement of L. Consider the space $l_1(E)$ of all absolutely summable sequences in E, equipped with the usual norm $\|x_n(n \in \mathbb{N})\| = \sum_{n=1}^{\infty} \eta(x_n)$. It is known that, if E is metrizable and barrelled, $l_1(E)$ is barrelled (see [2], I, p.65 and III, ex. 1, p.338). Thus $l_1(L)$ is a barrelled subspace of $l_1(E)$. It is clearly dense in $l_1(E)$. It has a weaker complete norm : if ξ is the weaker complete norm of L, equip $l_1(L)$ with the norm of $l_1(L_\xi)$. Moreover $l_1(L_\xi)'$ has infinite codimension in $l_1(E)'$: if $f \in E'_\eta \setminus L'_\xi$, the linear forms $f_i(x_n\ (n \in \mathbb{N})) = f(x_i)$ belong to a cobase of $l_1(L_\xi)'$ in $l_1(E)'$. Another example can be obtained as follows. The subspace $l_1(M)$ is a Banach space, hence it is closed in $l_1(E)$. Moreover $l_1(M) \cap l_1(L) = \{0\}$. Choose then a subspace H of $l_1(E)$ such that $l_1(M) \cap H = \{0\}$, $l_1(M)+H = l_1(E)$ and $l_1(L) \subset H$. The space H is dense and barrelled in $l_1(E)$ and, by Prop. 1.1 (iv), it admits a minimal barrelled weaker topology. The same arguments remain valid if (E,η) is a Frechet space and if L is dense and barrelled in (E,η) and has a closed algebraic complement M.

We do not know whether, for every normed barrelled space (E,η) whose codimension in $\hat{E}_\eta$ is uncountable, there always exists a minimal barrelled topology (hence a weaker complete

norm), or even a weaker barrelled norm ξ such that E'_ξ has infinite codimension in E'_η (or equivalently, a closed subspace of infinite dimension in $\hat{E}_\eta$, transversal to E).

On the contrary, it is easy to show that there exist normed but non barrelled topologies weaker than η. The next result is slightly more general.

PROPOSITION 2.3. If (E,η) is an infinite dimensional normed space, there exists a weaker normed topology ξ on E such that (E,η) has a base of neighborhoods of zero which are closed for ξ.

PROOF. Denote by U the closed unit ball of (E,η), by F a subspace of infinite countable dimension in E and by F' the dual of (F,η). Since $(F',\sigma(F',F))$ is separable and metrizable, $A = (U\cap F)^\circ$ equipped with the topology induced by $\sigma(F',F)$ admits a dense countable subset $\{f_n : n \in \mathbb{N}\}$, the linear hull of which is dense in $(F',\sigma(F',F))$. Denote by B the closure of the absolutely convex hull of $\{n^{-1}f_n : n \in \mathbb{N}\}$ in F'_A (i.e. F' equipped with the gauge norm of A). Since A is a Banach disk, B is compact in F'_A hence in $(F',\sigma(F',F))$.

Let V be the polar of B in F. The linear hull of B is dense in $(F',\sigma(F',F))$, hence V defines a normed topology λ on F. The dual of

(F,λ) can be identified to the linear hull of B in F'. The subset $U \cap F$ is closed in (F,λ). Indeed, it is the polar of A, hence of $\{f_n : n \in \mathbb{N}\} \subset F'_\lambda$.

The disk U+V defines a norm ξ on E. Indeed, if $nx \in U+V$ for each $n \in \mathbb{N}$, we have $x = y_n+z_n$ with $y_n \in n^{-1}U$, $z_n \in n^{-1}V$. The sequence $z_n = x-y_n$ is a Cauchy sequence in (F,η) and it converges to zero in (F,λ). Since $U \cap F$ is closed in (F,λ), $z_n \to 0$ in (F,η), hence $y_n+z_n \to 0$ in (E,η) and $x = 0$.

The norm ξ is strictly weaker than η. Indeed, if $U+V \subset \theta U$ for some $\theta > 0$, $V \subset \theta U \cap F$ hence $\theta B \supset A$. However B is compact in F'_A, hence F'_A would be finite dimensional. This contradicts the fact that F has infinite dimension.

We are left to prove that η has a base of neighborhoods of zero which are closed for ξ. It is sufficient to prove that the closure $\bar{U}$ of U in (E,ξ) is contained in θU for some $\theta > 0$. For every $\varepsilon > 0$, $\bar{U} \subset U + \varepsilon(U+V) \subset (1+\varepsilon)U + \varepsilon V$. Thus $\bar{U} \subset (1+\varepsilon)U + (\varepsilon V) \cap 2(1+\varepsilon)\bar{U} \subset (1+\varepsilon)U + 2(1+\varepsilon)(V \cap \bar{U})$. However, $V \cap \bar{U} \subset V \cap ((1+\varepsilon')U + \varepsilon' V) \subset (1+\varepsilon')(V \cap U) + \varepsilon' V \subset 2(V \cap U) + \varepsilon' V$ for all $\varepsilon' \in]0,1[$. Thus $V \cap \bar{U} \subset 2\overline{V \cap U}$ (the closure of $V \cap U$ is taken in (F,λ)). But $V \cap U$ is closed in (F,λ). Thus $V \cap U \subset 2(V \cap U) \subset 2U$ and finally $\bar{U} \subset 5(1+\varepsilon)U$. Q.E.D.

A space E equipped with two normed topologies satisfying the assumptions of Prop.2.3

is also called a two-normed space.

In [5], Saxon and Wilansky construct a non barrelled norm ξ finer than a given barrelled norm τ on E. We prove here that it is always possible to insert ξ between τ and a given finer norm on E.

PROPOSITION 2.4. Let η and τ be two normed topologies on E such that $\tau \leq \eta$ and that (E,τ) is barrelled. If E' has infinite codimension in E'_η, there always exists a non barrelled normed topology ξ on E such that $\tau < \xi < \eta$ and that E'_τ has infinite codimension in E'_ξ and E'_ξ infinite codimension in E'_η. If (E,τ) is separable, we can choose ξ such that (E,ξ) is separable.

PROOF. Denote by U the unit ball of (E,η) and set $A = U^\circ \cap E'$. Choose a sequence $f_n \in n^{-1}U^\circ$ in a cobase of E'_τ in E'_η. Since (E,τ) is barrelled, A is compact in $(E'_\tau,\sigma(E'_\tau,E))$, hence in $(E'_\eta,\sigma(E',E))$. The absolutely convex hull of A and $\{f_1,\ldots,f_{n-1}\}$ is thus closed and does not contain f_n. Therefore there exists $x_n \in E$ such that $f_n(x_n) > 1$, $|f_i(x_n)| \leq 1$ for all $i < n$ and $x_n \in A^\circ$. Set $B = \{x_n : n \in \mathbb{N}\}^\circ \cap U^\circ$ and equip E with the norm ξ defined by B°. Since $E'_\xi \supset E'_\tau$, it is indeed a norm. It is clearly weaker than η. It is finer than τ because $B \supset A$. Moreover E'_τ has infinite codimension in E'_ξ. The space (E,ξ) is not barrelled, because $\{nf_n : n \in \mathbb{N}\}$ is

bounded in $(E'_\xi,\sigma(E'_\xi,E))$ but not absorbed by B.

Finally E'_ξ has infinite codimension in E'_η. Indeed, we have just seen that $U^\circ \cap E'_\xi$ is not equicontinuous for ξ. If E'_ξ had finite (and even countable) codimension in the Banach space E'_η with the gauge norm of U°, it would be barrelled for the induced norm [7]. But then, by the closed graph theorem, this norm would be equivalent to the gauge of B and $U^\circ \cap E'_\xi$ would be equicontinuous.

Suppose now that (E,τ) is separable. We construct a norm η' such that $\tau \leq \eta' \leq \eta$, that (E,η') is separable and that E'_τ has infinite codimension in $E'_{\eta'}$. The conclusion follows by applying the first part of the proof to τ and η'.

Choose f_n $(n \in \mathbb{N})$ as above. Denote by M the linear hull of $\{f_n : n \in \mathbb{N}\}$, by F the linear hull of $E' \cup M$ and by V the unit ball of (E,τ). It is clear that the closed absolutely convex hulls K_n of $V^\circ \cup \{f_1,\ldots,f_n\}$ $(n \in \mathbb{N})$ form a fundamental sequence of absolutely convex compact subsets in $(F,\sigma(F,E))$. Therefore $\mu(E,F) = \sup(\tau,\sigma(E,M))$ and $(E,\mu(E,F))$ is metrizable and separable.

The unit ball U of (E,η) is thus bounded and separable in $(E,\mu(E,F))$. Since $(E,\mu(E,F))$ is metrizable, there exists a bounded disk D in $(E,\mu(E,F))$ such that the gauge norm of D coincides in U with $\mu(E,F)$ (see for instance [1], p.33). We may assume that D contains U.

Hence the gauge η' of D is a new norm on E, weaker than η. Since U is absorbent in E, (E,η') is separable. Moreover $\eta' \geq \mu(E,F)$, hence E'_τ has infinite codimension in $E'_{\eta'}$. Q.E.D.

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