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DEFORMATIONS OF ALGEBRAS AND COHOMOLOGY OF FIXED POINT SETS

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In [13] it is shown that under certain conditions the cohomology algebra of the fixed point set of a space with group action is in an algebraic sense a deformation of the cohomology algebra of the space itself. Here we attempt to prove a converse of the above statement, i.e. we try to realize geometrically a given algebraic deformation of a (commutative) graded algebras as the cohomology algebra of the fixed point set of a suitable space with group action. The first part of this note in a sense reduces this realization problem in equivariant topology to a non-equivariant problem while the second part uses Sullivan's theory of minimal models to actually obtain a converse for S^1-actions, where cohomology is taken with rational coefficients.

1. Reduction to non-equivariant realization problems.

We recall some more or less standard notation (s. e.g. [2], [4]):

Let X be a compact space (a somewhat weaker finiteness assumption would suffice (s. [4] and the remark (1.3) below)) on which a group G acts; then $X \xrightarrow{i} X_G := X \times_G E_G \to B_G$ denotes the fibre bundle with fibre X associated to the universal principal G-bundle $G \to E_G \to B_G$, F denotes the fixed point set of the G-action on X, and H*(-) Čech cohomology with coefficients in a field K. The cohomology algebra $H^*(B_G)$ is abbreviated by R. As usual in P.A.Smith theory we allow the following pairs $(G,K) = (Z_p,Z_p)$, p prime (actually only the case p=2 is worked out in detail,

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the case p odd needs some modifications which are not given here) or (S^1,Q) for the first part of this note. The second part is restricted to $(G,K) = (S^1,Q)$.

Because of its importance for the following we recall the fundamental Borel et al. - localization theorem which can be stated as:

Localization Theorem (Borel et al.):

The R-algebra morphism $j_G^*: H^*(X_G) \to H^*(F_G) \cong H^*(F) \otimes R (= H^*(F) \otimes_K R)$ which is induced by the inclusion $j: F \to X$, becomes an isomorphism after localization with respect to the multiplicative set $\{t^n, n \in N\}$, where t denotes the polynomial generator of R.

Throughout this note we make the assumption that $\dim_K(\bigoplus_n H^n(X))$ is finite. Since a deformation of K-algebras does not change the dimension of the underlying K-vector space, the following proposition collects a number of statements which are equivalent to $\dim_K(\bigoplus_n H^n(X)) = \dim_K(\bigoplus_n H^n(F))$.

(1.1) PROPOSITION: The following conditions are equivalent:

(a) $\dim_K(\bigoplus_n H^n(X)) = \dim_K(\bigoplus_n H^n(F))$

(b) X is totally nonhomologous to zero in X_G (with respect to $H^*(-)$), i.e. $i^*: H^*(X_G) \to H^*(X)$ is surjective.

(c) G $(= \pi_1(B_G))$ acts trivially on $H^*(X)$ and the Serre spectral sequence of the fibration $X \to X_G \to B_G$ degenerates.

(d) G acts trivially on $H^*(X)$ and $H^*(X_G)$ is a free R-module (of rank $\dim_K(\bigoplus_n H^n(X))$)

(e) G acts trivially on $H^*(X)$ and $j_G^*: H^*(X_G) \to H^*(F_G) = H^*(F) \otimes_K R$ is injective.

(f) $i^*: H^*(X_G) \to H^*(X)$ induces an isomorphism $\overline{i^*}: H^*(X_G) \otimes_R K_0 \xrightarrow{\cong} H^*(X)$, where K_0 is K considered as an R-module via the augmentation $\varepsilon: R \to K$, $\varepsilon(t) = 0$.

Proof: That a), b) and c) are equivalent is shown in [4], VII (1.6). If (c) holds, then $H^*(X_G)$ as an R-module is isomorphic to the E_2-term of the Serre spectral sequence of the fibre bundle $X \to X_G \to B_G$, i.e. $H^*(X_G) \cong H^*(X) \otimes H^*(B_G) \cong H^*(X) \otimes R$ as R-modules, which gives (d).

If (d) holds, then $H^*(X_G)$ maps injectively into its localization $H^*(X_G)_\ell$. It therefore follows immediately from the localization theorem and the commutative diagram

$$\begin{array}{ccc} H^*(X_G) & \xrightarrow{j_G^*} & H^*(F) \otimes R \\ \downarrow & & \downarrow \\ H^*(X_G)_\ell & \xrightarrow{\cong} & H^*(F) \otimes R_\ell \end{array}$$

that j_G^* is injective, i.e. (e) holds.

If on the other hand (e) is assumed to hold, then $H^*(X_G)$ is a R-submodule of the free R-module $H^*(F) \otimes R$. Since R is a principal ideal domain, one gets (d).

It follows from the Eilenberg-Moore spectral sequence (s.e.g. [9], (4.7)) that (d) implies (f), and clearly (f) implies (b) since i* factors through $\overline{i^*}$, i.e.

$$i^*: H^*(X_G) \twoheadrightarrow H^*(X_G) \otimes_R K_0 \xrightarrow{\overline{i^*}} H^*(X).$$

To abbreviate notation, just for the purpose of this note, we make the following

(1.2) DEFINITION: A G-space is called h-simple (i.e. homologically simple) if one (and hence all) of the properties a)-f) in proposition (1.1) is (are) fulfilled.

Our aim is to realize geometrically a given injective morphism of graded R-algebras $\gamma: \mathcal{B} \to \mathcal{A} = A \otimes R$ (A graded K-algebra, $\mathcal{A}$ "extension" of A to an R-algebra using the usual multiplication for the tensor product), which becomes an isomorphism after localization (with respect to $\{t^n, n \in N\}$) by choosing a space F such that $H^*(F) \cong A$ and successively attaching "free G-cells" to F to obtain an equivariant relative CW-complex (X,F) in the sense of

[11] (compare [3]) such that the inclusion of the fixed point set $j: F \to X$ induces the map $\gamma: B \to A$ (up to isomorphism) when $H^*(-_G)$ is applied, i.e.

$$\gamma: B \cong H^*(X_G) \xrightarrow{j_G^*} H^*(F_G) \cong A.$$

(1.3) Remark: It is easy to see (imitating the usual proof) that the localization theorem for singular cohomology holds for a finite (or even finite dimensional) relative G-CW-complex (X,F) of the above form, i.e. for a G-space X which is obtained by attaching finitely many free G-cells (resp. free G-cells of bounded dimension) to a given space F on which G acts trivially.

We therefore investigate the situation which is induced in cohomology by attaching free G-cells $G\times D^n$ to a given G-space along the boundary $G\times S^{n-1}$. Let X be a G-space and $\alpha: S^{n-1} \to X$ a given (non-equivariant) map. There is a unique G-map $\tilde{\alpha}: G\times S^{n-1} \to X$ defined by $\tilde{\alpha}(g,s) = g\alpha(s)$, such that $\tilde{\alpha}|_{\{1\}\times S^{n-1}} = \alpha$. We obtain a G-space Y containing X, $X \xrightarrow{q} Y$, by attaching $G\times D^n$ via $\tilde{\alpha}$ to X, i.e. $Y := X \cup_{\tilde{\alpha}} (G\times D^n)$.

It follows that $Y_G = X_G \cup_{\tilde{\alpha}_G} (G\times D^n)_G \simeq X_G \cup_{i\circ\alpha} D^n$ because of the following commutative diagram:

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & G\times S^{n-1} & \xrightarrow{\tilde{\alpha}} & X \\ & \searrow^{\simeq} & \downarrow & & \downarrow i \\ & & (G\times S^{n-1})_G & \xrightarrow{\tilde{\alpha}_G} & X_G \end{array}$$

(the unnamed maps being canonical inclusions).

Clearly Y and X have the same fixed point set.

To calculate $H^*(Y_G)$ we have the long exact sequence:

$$\ldots \to \tilde{H}^{k-1}(S^{n-1}) \to H^k(Y_G) \xrightarrow{q_G^*} H^k(X_G) \xrightarrow{(i\alpha)^*} \tilde{H}^k(S^{n-1}) \to \ldots .$$

We assume for the following that $(G,K) = (S^1,\mathbb{Q})$ or (Z_2,Z_2).

(1.4) PROPOSITION: *If* X *is an* h-*simple* G-*space and* $\alpha\colon S^{n-1} \to X$ *a given map, then* $Y := X \underset{\alpha}{\cup} G\times D^n$ *is* h-*simple if and only if* $\alpha^*\colon H^*(X) \to H^*(S^{n-1})$ *is surjective.*

Proof: By assumption, (1.1)(b) and (1.1)(e) $i^*\colon H^*(X_G) \to H^*(X)$ is surjective, $j_G^*\colon H^*(X_G) \to H^*(F_G)$ injective and G acts trivially on $H^*(X)$. Hence α^* surjective $\Leftrightarrow$ $(i\alpha)^*$ surjective $\Leftrightarrow$ q_G^* injective $\Leftrightarrow$ $(qj)_G^*$ injective $\Leftrightarrow$ Y h-simple (by (1.1)(e) and the fact that G acts trivially on $H^*(Y)$ if and only if α^* is surjective).

As an immediate consequence of this and of the localization theorem one has:

(1.5) COROLLARY: *If* X *is an* h-*simple* G-*space and* $\alpha\colon S^{n-1} \to X$ *a map such that* $\alpha^*\colon H^*(X) \to H^*(S^{n-1})$ *is surjective, then*:

(a) $Y := X \underset{\alpha}{\cup} G\times D^n$ *is an* h-*simple* G-*space*.

(b) $q_G^*\colon H^*(Y_G) \to H^*(X_G)$ *is an embedding of* R-*algebras which becomes an isomorphism after localization*.

(c) $\dim_K\left(H^*(X_G)/H^*(Y_G)\right) = 1$.

(d) $q^*\colon H^k(Y) \to H^k(X)$ *is surjective for* $k \neq n-1$, *and injective for* $k \neq n+1$ (*resp.* $k \neq n$) *if* $G = S^1$ (*resp.* $G = Z_2$). *In addition* $\dim_K(\ker q^*) = \dim_K(\operatorname{coker} q^*) = 1$.

Proof: Only part (d) is left to be proved. But (d) follows immediately from the Mayer-Vietoris sequence

$$\ldots \to \tilde{H}^{k-1}(G\times S^{n-1}) \to \tilde{H}^k(Y) \to \tilde{H}^k(X)\oplus\tilde{H}^k(G\times D^n) \to \tilde{H}^k(G\times S^{n-1}) \to \ldots$$

(1.6) COROLLARY: *Let* $B \overset{\gamma}{\to} A$ *be an embedding of graded* R-*algebras with* γ_ℓ *an isomorphism and* $\dim_K(A/B) = 1$. (Such a map will be called an elementary embedding of R-algebras for the purpose of this note.) *If there exist an* h-*simple* G-*space* X *such that* $H^*(X_G) \cong A$ *and a map* $\alpha\colon S^{n-1} \to X$ *such that* $\alpha^*\colon H^*(X) \cong A \underset{R}{\otimes} K_0 \to A/B \cong \tilde{H}(S^{n-1})$ *coincides with the map* $\bar{\pi}\colon A \underset{R}{\otimes} K_0 \to A/B$ *induced by the projection* $\pi\colon A \to A/B$, *then there exists an* h-*simple* G-*space* Y *and an inclusion*

$X \overset{q}{\to} Y$, <u>namely</u> $X \subset X \underset{\tilde\alpha}{\cup} G\times D^n = Y$ <u>such that</u> q <u>realizes</u> γ, <u>i.e.</u> $q_G^* = \gamma$ (up to isomorphism).

<u>Proof</u>: The above calculations show that $q_G^*: H^*(X_G) \to H^*(X_G)$ is just the kernel of $\alpha^* \circ i^*: H^*(X_G) \xrightarrow{i^*} H^*(X) = H^*(X_G) \underset{R}{\otimes} K_0 \xrightarrow{\alpha^*} \tilde{H}^*(S^{n-1})$, whereas $\gamma: B \to A$ is the kernel of $\pi: A \to A_{/B}$ which allows the factorization $\pi: A \to A \underset{R}{\otimes} K_0 \overset{\bar\pi}{\to} A_{/B}$, since for degree reasons $A_{/B}$ is a trivial R-module (i.e.: $R^+ \cdot (A_{/B}) = 0$, where R^+ are the elements of R which have degree > 0.)

<u>(1.7) Examples</u>: a) The case n=1 needs a little extra attention. One could of course avoid complications by assuming all spaces and algebras to be connected. But this is not adequate since clearly connected G-spaces may have non-connected fixed point sets. Since the cohomology algebra of a topological space has a canonical splitting into a product of connected algebras (namely the cohomology algebras of the components) we assume (on the algebraic side) that all graded algebras considered here are products of connected graded algebras. A morphism between such algebras respects the splitting in the following sense:
$\gamma: B = \overset{s}{\underset{j=1}{\times}} B_j \to A = \overset{r}{\underset{i=1}{\times}} A_i$ decomposes into $\gamma = \overset{s}{\underset{j=1}{\times}} \gamma_j$, $\gamma_j: B_j \to \underset{i \in I_j}{\times} A_i$, with $\{I_j, j=1,\dots,s\}$ a suitable partition of the set $\{1,\dots,r\}$ into disjoint subsets. (In general $I_j = \emptyset$ may occur for some indices j in which case $\underset{i\in\emptyset}{\times} A_i := 0$.)
If such a morphism $\gamma: B \to A$ is an embedding of R-algebras with $A_{/B} \cong \tilde{H}(S^0)$ and $A \cong H^*(X_G)$ for an h-simple G-space X, then $\bar\pi: A \underset{R}{\otimes} K_0 \to A_{/B}$ can always be realized by a map $\alpha: S^0 \to X$.

b) Let $\gamma: B \to A$ be given with $A \cong H^*(X_G)$, X an h-simple G-space. Assume in addition: X simply connected finite CW-complex, $H^k(X) = 0$ for $0<k<n$, $n\geq 2$ and $A_{/B} \cong \tilde{H}^*(S^n)$. Then by the Hurewicz theorem $\bar{\pi}: A \underset{R}{\otimes} K_0 \cong H^*(X) \to \tilde{H}^*(S^n)$ can be realized by a map $\alpha: S^n \to X$, and by (1.6) $X \to X \underset{\alpha}{\cup} G\times D^{n+1} = Y$ realizes γ. It follows from the Mayer-Vietoris sequence in cohomology that $H^k(Y) = 0$ for $0<k<n$ and $\dim_K H^n(Y) = \dim_K(H^n(X))-1$.
Clearly Y is also simply connected.

<u>(1.8) PROPOSITION</u>: <u>Let</u> $\gamma: B\to A$ <u>be an embedding of</u> R-<u>algebras, where</u> A <u>and</u> B <u>are products of connected algebras</u>.
<u>If</u> $\dim_K(A_{/B}) = r<\infty$ <u>and</u> $\gamma_\ell: B_\ell \to A_\ell$ <u>is an isomorphism then</u> γ <u>can be decomposed into a finite sequence of elementary embeddings</u> (s.(1.6)), $\gamma = \gamma_1 \circ \ldots \circ \gamma_r$, <u>such that</u> $|\bar{q}_i|\leq|\bar{q}_j|$ <u>for</u> $i<j$, <u>where</u> $\bar{q}_i$ <u>denotes the generator of coker</u> γ_i.

<u>Proof</u>: Let $Q := {}^A/_B$. Choose a basis $\{q_1,\ldots,q_r\}$ of Q as a K-vector space such that $|q_i|\leq|q_j|$ for $i<j$, ($|\ |$ denotes the degree of an element). Define $C_1:=\ker \pi_1: A \xrightarrow{\pi} Q \xrightarrow{\rho_1} Q_1$, where $\rho_1: Q \xrightarrow{\rho_1} Q_1$ is the projection of Q onto the 1-dimensional (graded) K-vectorspace Q_1 generated by $\bar{q}_1$, $\rho_1(q_1) = \bar{q}_1$, $\rho_1(q_i) = 0$, $i\neq 1$. If $|q_1|>0$ then Q_1 can be given the trivial R-algebra structure (without unit), and for degree reasons $\rho_1: A \to Q_1$ is a morphism of R-algebras. It follows that $\gamma_1: C_1 \to A$ is an elementary embedding. If $|q_1| = 0$ the assumed splitting of γ (compare (1.7)a)) gives the desired result also. By construction $B \to C_1$ is an embedding of R-algebras which fulfils the hypothesis of the proposition except that $\dim_K({}^{C_1}/_B) = r-1$. The proof is finished by induction.

Clearly the above decomposition of γ is not uniquely determined, but the corollary (1.11) of the following proposition makes up for this to some extent.

(1.9) PROPOSITION: Let X be a G-CW-complex (of finite type in case $K = \mathbb{Z}_2$) which is obtained from its fixed point set F by attaching free G-cells of dimension > 0. Assume that X is h-simple and all components of X and F are simply connected. Then there exists a G-CW-complex Y with the same properties but such that all equivariant relative n-skeletons $(Y,F)^n$ are h-simple, and there exists a G-map $f\colon (Y,F) \to (X,F)$ such that $f|_F = \mathrm{id}$, and $f_G\colon Y_G \to X_G$ (and hence $f\colon Y \to X$, too) is a K-homotopy equivalence.

Proof: Roughly the proof consists of omitting those equivariant n-cells which do not contribute to the (co)-homology of X, i.e. generate elements in the (co)-homology of the n-skeleton which do not survive in X, together with the (n+1)-cells which kill these additional elements.

Since X is h-simple each component of X is a G-CW-complex itself and we therefore can assume without restriction that X is connected. Put $(Y,F)^0 := (X,F)^0 = F$, $(Y,F)^1 :=$ a minimal connected sub-G-CW-complex of $(X,F)^1$ (containing F) and $(Y,F)^2 = (Y,F)^1 \underset{\tilde{\alpha}_\nu}{\cup} G\times D^2$, where the $\alpha_\nu\colon S^1 \to (Y,F)^1$ form a "basis" of the (non-abelian) free group $\pi_1((Y,F)^1)$ (which actually is trivial in case $G = S^1$). Since the composition $S^1 \to (Y,F)^1 \to X$ is nullhomotopic by assumption one can extend the inclusion $(Y,F)^1 \to (X,F)^2$ to a G-map f^2: $(Y,F)^2 \to (X,F)^2$ such that $(f^2_G)_*\colon H_i((Y,F)^2_G) \to H_i((X,F)^2_G)$ is isomorphic for $i = 0,1$, and injective for $i = 2$. The equivariant skeletons $(Y,F)^i$, $i = 0,1,2$ are h-simple and $(Y,F)^2$ is simply connected. Assume now by induction hypothesis that one has already constructed the equivariant n-skeleton $(Y,F)^n$ and a G-map $f^n\colon (Y,F)^n \to (X,F)^n$ such that $(Y,F)^n$ is h-simple and simply connected, and $(f^n_G)_*\colon H_i((Y,F)^n_G) \to H_i((X,F)^n_G)$ is isomorphic for $i<n$ and injective for $i=n$ $(n\geq 2)$. Let $(X,F)^{n+1}$ be obtained from $(X,F)^n$ by attaching a disjoint union $\coprod_{\nu\in V} G\times D^{n+1}_\nu$ of free G-cells along the maps $\alpha_\nu\colon S^n_\nu \to (X,F)^n$ (extended to G-maps

$\tilde{\alpha}_\nu \colon G\times S^n_\nu \to (X,F)^n)$. Choose a subset $V' \subset V$ such that the composition

$$H_n\Big(\coprod_{\nu\in V'} S^n_\nu\Big) \xrightarrow{(\alpha_\nu)_*} H_n((X,F)^n) \to H_n((X,F)^n_G/_{(f^n_G)_*(H_n((Y,F)^n_G))}$$

is an isomorphism, which is possible because the following composition is surjective

$$H_n(F_G) \overset{\cong}{\to} H_n((Y,F)^n_G) \to H_n((X,F)^n_G) \to H_n((X,F)^{n+1}_G) \overset{\cong}{\to} H_n(X_G).$$

Define $\overline{(X,F)}^n := (X,F)^n \cup_{\alpha_\nu} \Big(\coprod_{\nu\in V'} G\times D^{n+1}_\nu\Big)$; then the composition $H_i((Y,F)^n_G) \to H_i((X,F)^n_G) \to H_i(\overline{(X,F)}^n_G)$ is isomorphic for $i\leq n$, in fact for all i, since $H_i(F_G) \overset{\cong}{\to} H_i((Y,F)^n_G) \overset{\cong}{\to} H_i((X,F)^n_G) \overset{\cong}{\to} H_i(\overline{(X,F)}^n_G)$ for $i>n$ because $(X,F)^n$ and $(Y,F)^n$ are relative equivariant n-skeletons. It follows from the (generalized) Whitehead theorem that $\pi_n((Y,F)^n_G) \to \pi_n(\overline{(X,F)}^n_G)$ and $\pi_n((Y,F)^n) \to \pi_n(\overline{(X,F)}^n)$ are isomorphic modulo torsion in case $(G,K) = (S^1,Q)$ (resp. q-torsion, q prime to 2, in case $(G,K) = (Z_2,Z_2)$. Here we use the "finite type" assumption to conclude that $\pi_n(\overline{(X,F)}^n,(Y,F)^n)$ is q-torsion from the fact that its 2-completion is zero.) We therefore can find maps $\beta_\nu \colon S^n_\nu \to (Y,F)^n$ such that $\gamma\alpha_\nu m_\nu \simeq \gamma f^n\beta_\nu \colon S^n_\nu \to \overline{(X,F)}^n$, where $\gamma\colon (X,F)^n \to \overline{(X,F)}^n$ is the canonical inclusion and $m_\nu \colon S^n_\nu \to S^n_\nu$ is a map of degree $m_\nu \in Z$, $m_\nu \neq 0$ in case $(G,K) = (S^1,Q)$ (resp. $m_\nu \neq 0$ and prime to 2, i.e. m_ν odd, in case $(G,K) = (Z_2, Z_2)$), i.e. m_ν is a K-homotopy equivalence. Defining $(Y,F)^{n+1} := (Y,F)^n \cup_{(\tilde{\beta}_\nu)} (\coprod_{\nu\in V''} G\times D^{n+1}_\nu)$, where $V'' \subset (V\setminus V')$ is a subset such that the composition

$$H_n(\coprod_{\nu\in V''} S^n_\nu) \xrightarrow{(\alpha_\nu)_*} H_n((X,F)^n) \xrightarrow{\gamma_*} H_n(\overline{(X,F)}^n) \to H_n(\overline{(X,F)}^n_G)$$

is injective and $\operatorname{im}\Big(H_n(\coprod_{\nu\in V''} S^n_\nu) \to H_n(\overline{(X,F)}^n_G)\Big) = \operatorname{im}\Big(H_n(\coprod_{\nu\in V} S^n_\nu) \to H_n(\overline{(X,F)}^n_G)\Big)$, one gets a G-map $f^{n+1}\colon (Y,F)^{n+1} \to (X,F)^{n+1}$ which extends f^n such that $(f^{n+1}_G)_*\colon H_i((Y,F)^{n+1}_G) \to H_i((X,F)^{n+1}_G)$ is isomorphic for $i\leq n$ and injective for $i = n+1$.

One finally gets a G-map of simply connected, h-simple G-spaces $f\colon Y \to X$ such that $(f_G)_*\colon H_i(Y_G) \to H_i(X_G)$ is an isomorphism for all i, hence f_G and f are K-homotopy equivalences.

(1.10) Remark: The assumptions in (1.9) can be weakened.
a) The condition on the dimension of attached cells to be greater than zero is not really necessary. One can replace a connected G-CW-complex with non-empty fixed point set by a G-homotopy equivalent one which has no free G-cells of dimension zero.
b) If $G = Z_2$ then any G-action is semifree and any G-CW-complex (X,F) can be obtained by attaching free G-cells to the fixed point set F. For $G = S^1$ this is not the case. But if X has only finitely many orbit types (e.g. (X,F) finite) then one can choose a finite cyclic subgroup H containing all finite isotropy groups which occur in X. The group ${}^G/H \cong S^1$ then acts semifreely on $X_{/H}$ with fixed point set F, and the projection $(X,F) \to (X_{/H},F)$ induces an isomorphism $H_*(X_G) \to H_*((X_{/H})_G)$.
c) Certainly the assumption on the vanishing of fundamental groups can be weakened but we will not go into this here.

(1.11) COROLLARY: If $\gamma\colon B \to A$ can be realized by a G-CW-complex (X,F) which fulfils the assumption of (1.9), then any decomposition $\gamma = \gamma_r \cdot \ldots \cdot \gamma_1$ of γ which fulfils the dimension condition in (1.8) can be realized.

Proof: By (1.9) one can replace (X,F) by a G-CW-complex (Y,F) such that all equivariant skeletons of (Y,F) are h-simple. The inclusions of the equivariant skeletons induce a decomposition of γ such that the given one is a refinement of this "skeletal" decomposition. After rearranging the attaching maps without changing the equivariant K-homotopy type one can realize the refinement by attaching the free G-cells one at a time in an order which corresponds to the given decomposition.

Similar to the proof of (1.9) one can show the following

(1.12) Remark: If X is a simply connected CW-complex (of finite type in case $K=Z_p$) such that $\dim_K(\bigoplus_i H_i(X))<\infty$ then there exists a finite, simply connected CW-complex Y and a map $f: Y \to X$ which is a K-homotopy equivalence.

Proof: Without restriction we can assume that X is connected and $X^0 = X^1 = \{*\}$. Put $Y^0 = X^0$ and $Y^1 = X^1$. Assume by induction hypothesis (which is fulfilled for n = 0,1) that Y^n is simply connected and finite, and $f^n: Y^n \to X^n$ has been constructed such that $f^n_*: H_i(Y^n) \to H_i(X^n)$ is isomorphic for i<n and the composition $H_n(Y^n) \to H_n(X^n) \to H_n(X)$ is an isomorphism. Let X^{n+1} be obtained from X^n by attaching $\coprod_{\nu\in V} D^{n+1}_\nu$ along $\alpha_\nu: S^n_\nu \to X^n$. Choose a subset $V'\subset V$ such that $H_n\Big(\coprod_{\nu\in V'} S^n_\nu\Big) \xrightarrow{(\alpha_\nu)_*} H_n(X)$ is injective and $\mathrm{im}\Big(H_n(\coprod_{\nu\in V'} S^n_\nu) \to H_n(X^n)\Big) = \mathrm{im}\Big(H_n(\coprod_{\nu\in V} S^n_\nu) \to H_n(X)\Big)$. Define $\tilde{X}^n :=$ $X^n \cup_{\alpha_\nu} (\coprod_{\nu\in V'} D^{n+1}_\nu)$, then the composition $H_i(Y^n) \to H_i(X^n) \to H_i(\tilde{X}^n)$ is isomorphic for all i. We therefore can find maps $\beta_\nu: S^n_\nu \to Y^n$ such that $\gamma\, \alpha_\nu m_\nu \simeq \gamma\beta_\nu: S^n_\nu \to X^n \overset{\gamma}{\to} \tilde{X}^n$, $m_\nu: S^n_\nu \to S^n_\nu$ as in the proof of (1.9). Defining $Y^{n+1} := Y^n \cup_{\beta_\nu} (\coprod_{\nu\in V''} D^{n+1}_\nu)$, where $V'' \subset (V\setminus V')$ is a finite subset such that the composition $H_n(\coprod_{\nu\in V''} S^n_\nu) \xrightarrow{\partial^{-1}} H_{n+1}(X^{n+1}) \to H_{n+1}(X)$ is an isomorphism (∂ being the boundary operator in the long exact sequence $\to H_{n+1}(X^{n+1}) \overset{\partial}{\underset{\cong}{\to}} H_n(\coprod_{\nu\in(V\setminus V')} S^n_\nu) \overset{\varrho}{\to} H_n(\tilde{X}^n) \to \dots$) one gets a map $f^{n+1}: Y^{n+1} \to X^{n+1}$ which extends f^n such that $f^{n+1}_*: H_i(Y^{n+1}) \to H_i(X^{n+1})$ is isomorphic for $i\leq n$ and the composition $H_{n+1}(Y^{n+1}) \to H_{n+1}(X^{n+1}) \to H_{n+1}(X)$ is an iso-

morphism. Since $\dim_K(\oplus H_i(X))<\infty$ there exists a (big enough) integer m such that for $Y = Y^m$ and $f = f^m$ one has $f_*: H_i(Y) \to H_i(X)$ is isomorphic for all i. Hence $f: Y \to X$ is a K-homotopy equivalence.

(1.13) Remark: The preceding results reduce the problem of realizing a given embedding $\gamma: B \to A$ by a relative G-CW-complex (X,F), where X and each component of the fixed point set F is simply connected, to the problem of realizing certain morphisms in cohomology as induced by elements of homotopy groups.

(1.14) Remark: Given a space F. There exists an h-simple G-space X with fixed point set F such that the cup-product structure in $\tilde{H}^*(X)$ is trivial. This corresponds to the fact, that any algebra can be obtain by deforming the algebra which has trivial product structure (s.e.g.[5], 2.2). If S^2 is equipped with the (orthogonal) G-action which keeps the poles fixed acting freely on the equator, then X can be defined as the smash product $X = F \wedge S^2$ with the "diagonal" action. Since X is a suspension the cup-product structure is trivial. (In the simply connected case one could also apply (1.7) several times to obtain such a space X.)

The connection between embeddings of R-algebras and deformation of K-algebras is given by the following two propositions:

(1.15) PROPOSITION: *If $\gamma: B \to A = A \otimes R$ is an embedding of R-algebras such that γ_ℓ is an isomorphism, then A is a deformation of the K-algebra $B = B \otimes_R K_0$ in the sense of Gerstenhaber (s. [6], compare [13]).*

Proof: Since $\gamma_\ell: B_\ell \to A_\ell$ is an isomorphism one has $B \otimes_R K_1 = B_\ell \otimes_{R_\ell} K_1 \overset{\cong}{\to} A_\ell \otimes_{R_\ell} K_1 = A \otimes_R K_1$ where K_1 is K considered as an R-(resp. R_ℓ-)algebra via $\eta: R = K[t] \to K, \eta(t) = 1$ (compare [12]). As an R-module B is isomorphic to $B \otimes R$ since it is a (free) submodule of the free R-module $A \otimes R$ and $\operatorname{rank}_R B = \operatorname{rank}_K B$. Hence B being isomorphic to $B \otimes R$ with "twisted" multiplication can be considered a one-parameter family of deformations of the algebra B which, evaluated at t=1, gives the algebra A (compare [13]).

To get some kind of a converse of (1.12) in our situation one has to use the additional structure which comes from filtrations and gradings of the algebras involved. If X is an h-simple G-space then $B = H^*(X_G)$ has a finite filtration $0 \subset F_0(B) \subset \ldots \subset F_k(B) \subset F_{k+1}(B) \subset \ldots$ by (graded) R-modules such that the cup-product maps $F_i(B) \otimes F_j(B)$ to $F_{i+j}(B)$ and the associated graded, in fact bi-graded, R-algebra is isomorphic to $B \otimes R$ (with R-algebra structure given by the usual tensor product) as a bi-graded algebra. Tensoring with the R-module K_1 gives an induced filtration $0 \subset F_0(A) \subset \ldots F_{k-1}(A) \subset F_k(A)$ of $A = B \underset{R}{\otimes} K_1$,

$F_k(A) = \operatorname{im}(F_k(B) \underset{R}{\otimes} K_1 \to B \underset{R}{\otimes} K_1)$. Since $F_k(B)/F_{k-1}(B)$ is a free R-module $(\underset{k}{\oplus} F(B)/F_{k-1}(B) \cong B \otimes R)$,

$F_{k-1}(B) \underset{R}{\otimes} K_1 \to F_k(B) \underset{R}{\otimes} K_1$ is again an inclusion and one has $\underset{k}{\oplus} F_k(A)/F_{k-1}(A) \cong (B \otimes R) \underset{R}{\otimes} K_1 \cong B$ as graded algebras.

On the other hand $A \cong H^*(F)$ has a grading as the cohomology algebra of the fixed point set F of X. If we denote the elements of A of filtration k and degree r by $F_k^r(A)$ we have the properties:

(i) $F_i^r(A) \otimes F_j^s(A) \to F_{i+j}^{r+s}(A)$ (cup-product)

(ii) $F_k^r(A) = 0$ if $k < r$,

where (i) follows from the multiplicative properties of the filtration $F_*(B)$ (and the cup-product) and (ii) from the fact that $B = H^*(X_G) \to A = H^*(F) \otimes R$ is filtration preserving where the filtration of A is given by $F_k(A) = \left(\bigoplus_{i=o}^{k} H^i(F)\right) \otimes R$, which implies (by tensoring with the R-module K_1) that $F_k(A)$ is contained in $\bigoplus_{i=o}^{k} H^i(F)$.

We therefore get that the embedding $H^*(X_G) \to H^*(F) \otimes R$ not only implies that $H^*(F)$ is a deformation of $H^*(X)$ but also gives a filtration on $H^*(F)$ which fulfils (i) and (ii) above such that the associated graded algebra is isomorphic to $H^*(X)$.

If $B = (B^*[t], g_t)$ is a one-parameter family of deformations of the graded commutative algebra B^* one can define a filtration on B by $F_k(B) = \left(\bigoplus_{i=o}^{k} B^k\right)[t]$ which fulfils $F_i(B) \otimes F_j(B) \xrightarrow{g_t} F_{i+j}(B)$. If $A = B_1 = B \otimes_R K_1$ we therefore get an induced filtration $F_0(A) \subset \ldots \subset F_{k-1}(A) \subset F_k(A) \subset \ldots$ on A from which one can recover the grading of B since by the same argument as above we get $\bigoplus_k F_k(A)/F_{k-1}(A) \cong B$ as graded algebras. But since K_1 is not a graded R-module (η: $K[t]=R \to K, \eta(t)=1$ does not preserve the degree) the algebra A does not inherit a grading from the graded (by total degree) algebra $B = (B^*[t], g_t)$. (In case $G = S^1$ one gets an induced Z_2-grading on A since t has degree 2.) To get a converse of (1.12) we therefore assume that A can be equipped with a grading which is compatible with the filtration in the sense that (i) and (ii) above are fulfilled.

<u>(1.16) PROPOSITION: Under the above assumptions there exists an embedding of graded R-algebras $\gamma: \tilde{B} \to A = A \otimes R$ such that γ_ℓ is an isomorphism and $\tilde{B} \otimes_R K_0 = B$ as graded algebras</u>.

<u>Proof</u>: a) case $G = Z_2$: Define $\tilde{B}^n = \{\sum_i a_i t^i,\ a_i \in F_n^{n-i}(A)\} \subseteq A \otimes R$ (compare [7], § 3). Then $\tilde{B} = \bigoplus_n \tilde{B}^n$ is a graded sub-

algebra of $\mathcal{A} = A \otimes R$. Since for each $a \in A$ there exists a power t^s of t such that $a \cdot t^s \in \tilde{\mathcal{B}}^*$ one gets that γ_ℓ is an isomorphism. It remains to show that $\tilde{\mathcal{B}} \otimes_R K_0 \cong B$. But

$\tilde{\mathcal{B}} \otimes_R K_0 \cong \tilde{\mathcal{B}}/t \cdot \tilde{\mathcal{B}} \cong \bigoplus_k \mathcal{F}_k(A)/\mathcal{F}_{k-1}(A) \cong B$. For a fixed degree one has $\tilde{\mathcal{B}}^n/t\,\tilde{\mathcal{B}}^{n-1} = \bigoplus_{i=0}^{n} \mathcal{F}_n^{n-i}(A)/\mathcal{F}_{n-1}^{n-i}(A) \cong B^n$.

b) case $G = S^1$: Define $\tilde{\mathcal{B}}^n = \{\sum_i a_i t^i,\ a_i \in \mathcal{F}_n^{n-2i}\} \overset{\gamma}{\subset} A \otimes R$.

The rest is analogous to the first case.

2. The case $(G,K) = (S^1,\mathbb{Q})$. We now use Sullivan's theory of minimal models (s. [14] or [10]) to handle the non-equivariant realization problem of part 1 in certain cases. Our main result is:

<u>(2.1) Theorem</u>: If $\gamma: \mathcal{B} \to \mathcal{A}$ is an embedding of R-algebras as in (1.8) and $\mathcal{A} = A \otimes R$, where A is a finitely generated K-algebra, then γ can be realized by a relative G-CW-complex (X,F).

This result is an immediate consequence of (1.8) and the following

<u>(2.2) LEMMA: If $\gamma: \mathcal{B} \to \mathcal{A}$ is an elementary embedding of R-algebras and X a h-simple formal G-space which realizes $\mathcal{A}$, i.e. $H^*(X_G) \cong \mathcal{A}$ and there exists a homotopy equivalence of differential graded algebras (d.g.a.'s) $M(X_G) \xrightarrow{h_{X_G}} H^*(X_G)$, where the free R-algebra $M(X_G)$ is the minimal model of X_G, then γ can be realized by attaching a G-cell to X, such that the resulting G-space $Y := X \cup G \times D^n$ is again formal.</u> (Since we have to allow non connected spaces the minimal model of such a space is understood to be the cartesian product of the minimal models of the connected components.)

<u>Proof</u>: By the theorem of Grivel (s.[8] or [10] (20.3)) $M(X) := M(X_G) \otimes_R K_0$ is the minimal model for X and h_{X_G} in-

duces a homotopy equivalence of d.g.a's $h_X: M(X) \to H^*(X)$.

<u>1. case</u>: degree (of the generator) of $A_{/B} = 0$. We then already know from (1.7)a) that $\bar{\pi}: A \underset{R}{\otimes} K_0 \to A_{/B}$ can be realized by a map $\alpha: S^0 \to X$. It remains to show that the resulting G-space $Y := X \underset{\tilde{\alpha}}{\cup} G \times D^1$ (resp. Y_G) is formal. But X_G is obtained from X_G by connecting two different components by a line, which (up to homotopy) means that one takes the wedge of these two components leaving alone the others. We therefore get that Y_G is again formal (s.e.g. [10](15.14)).

<u>2. case</u>: degree of $A_{/B} > 0$. Without restriction we then can assume that X is connected. The morphism $h_X: M(X) \to H^*(X)$ induces a map $\overline{h_X}: \tilde{M}(X)_{/(\tilde{M}(X))^2} \to \tilde{H}^*(X)_{/(\tilde{H}^*(X))^2}$ of the indecomposables.

Hence any algebra map $\bar{\pi}: A \underset{R}{\otimes} K_0 \cong H^*(X) \to A_{/B} \cong \tilde{H}^*(S^{n-1})$ gives a map $\tilde{M}(X)_{/(\tilde{M}(X))^2} \to \tilde{H}^*(X)_{/(\tilde{H}^*(X))^2} \to \tilde{H}^*(S^{n-1})$ which by Sullivan's theory therefore can be realized by a map $\alpha: S^{n-1} \to X$ (s. [14],(10.1)). Put $Y := X \underset{\tilde{\alpha}}{\cup} G \times D^n$ and therefore $Y_G \simeq X_G \underset{i \cdot \alpha}{\cup} D^n$. Then $K = \ker \left(M(X_G) \xrightarrow{h^*_{X_G}} H^*(X_G) \xrightarrow{i^*} H^*(X) \xrightarrow{\alpha^*} \tilde{H}^*(S^{n-1}) \right)$ is a model for Y_G (s.[15] § 2) and h_{X_G} induces a homotopy equivalence of d.g.a.'s $k_{Y_G}: K \to H^*(Y_G)$.

One therefore gets a homotopy equivalence of d.g.a.'s $h_{Y_G}: M(Y_G) \to H^*(Y_G)$, where $M(Y_G)$ is the minimal model of Y_G. By (1.5) Y is h-simple and $q: Y \to X$ realizes γ.

The proof of (2.1) now consists of an iterated application of (2.2) using the decomposition of $\gamma: B \to A$ into elementary embeddings (s.(1.8)) starting with a formal space F which realizes $A = H^*(F)$.

<u>(2.3) Remark</u>: If A in (2.1) is simply connected (and $\dim_K(A) < \infty$) then by (1.12) one can realize A by a formal

space F which is a finite CW-complex.

(2.4) COROLLARY: Any deformation of a finite dimensional connected commutative graded K-algebra B, which fulfils the assumption of (1.16), can be geometrically realized by an h-simple G-space X, with $H^*(X) \cong B$.

(2.5) Remark: The above corollary does not imply that any deformation of $H^*(X)$ for a given space X can be realized by an G-action on X. (s. e.g. [1], Example 3 for a counterexample)

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