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# A CLASS OF PERIODIC JACOBI-PERRON ALGORITHMS IN PURE ALGEBRAIC NUMBER FIELDS OF DEGREE $n \geq 3$

Claude Levesque

The author obtains the periodicity of the Jacobi-Perron algorithm of $(\theta,\ldots,\theta^{n-1})$ where $\theta^n=(D^n-d)/V^n$, with $D$, $d,V \in N^*=\{1,2,\ldots\}$, $d|D$, $D$ and $d$ congruent to $+1 (\mathrm{mod}\ V^{n-1})$ and $D\geq(n-1)d(V+1)/2+1$. The case $V=1$ has been studied by L. Bernstein and the proof for arbitrary $V$ follows exactly the same pattern. Secondary results are then obtained from the main theorem.

## 0. INTRODUCTION

The Jacobi-Perron Algorithm (JPA) is one of the generalizations of the ordinary continued fraction algorithm to higher dimensions [1], and its definition is recalled at the end of this introduction.

Consider pure algebraic number fields of the form $\mathbb{K}=\mathbb{Q}(\omega)$ where

$$\omega^n = M = D^n \pm d > 2 \quad (n \geq 3), \tag{0.1}$$

with $D,d \in \mathbb{N}^* = \{1,2,\ldots\}$, and

$$d|pD \text{ for } n = p^r \ (p \text{ prime}) \text{ or } d|D \text{ for other } n. \tag{0.2}$$

When an $n^{th}$-power $V^n$ divides $M$, let

$$\theta^n = m = M/V^n = \omega^n/V^n . \tag{0.3}$$

L. Bernstein [1] showed that the JPA of $a^{(0)} = (\omega,\ldots,\omega^{n-1})$ is periodic when

$$M = D^n+d, \text{ with } D \geq (n-2)d, \tag{0.4}$$

$$M = D^n-d, \text{ with } D \geq 2(n-1)d. \tag{0.5}$$

Using his methods and ideas described in [2], we obtain

the main result of this paper, viz. the periodicity of the JPA of $a^{(0)} = (\theta,\ldots,\theta^{n-1})$ when

$$(0.6)\quad \begin{cases} M = D^n-d,\ \theta^n=M/V^n, \text{ with } D\geq(n-1)d(V+1)/2+1, \\ D,d \equiv 1 \pmod{V^{n-1}} \text{ and } V\in \mathbb{N}^*. \end{cases}$$

When $V = 1$ in (0.6), note the new bound for D.

Let $t,V \in \mathbb{N}^*$. L. Bernstein [5] obtained the periodicity of the JPA of $a^{(0)} = (\theta,\theta^2)$ when

$$(0.7)\quad M = (tV^3+1)^3 -1,\ \theta^3 = M/V^3 = t^3V^6 + 3t^2V^3 + 3t,$$

after having previously considered in [4] the case $t=V^3+2$ in (0.7). It is worth noting that if $D=tV^n+1$ and $d=1$ in (0.6), we generalize L. Bernstein's case (0.7) to arbitrary dimension. Moreover when $V=nV_0$, $D=n^{n-1}tV_0^n + 1$ and $d=1$ in (0.6), we generalize the case $\theta^3=((9tV_0^3+1)^3-1)/27V_0^3 = 27t^3V_0^6 + 9t^2V_0^3 + tV_0$ previously studied in [6].

Let $a^{(0)}=(a_1^{(0)}, a_2^{(0)},\ldots,a_{n-1}^{(0)})$ be a vector of the real Euclidean vector space $\mathbb{R}^{n-1}$, $n \geq 2$. A sequence $\langle a^{(v)}\rangle$ of vectors of $\mathbb{R}^{n-1}$ is called the JPA of $a^{(0)}$ if for all $v \in \mathbb{N}$,

$$(0.8)\quad a^{(v+1)}=\left(\frac{a_2^{(v)}-b_2^{(v)}}{a_1^{(v)}-b_1^{(v)}},\ldots,\frac{a_{n-1}^{(v)}-b_{n-1}^{(v)}}{a_1^{(v)}-b_1^{(v)}},\ \frac{1}{a_1^{(v)}-b_1^{(v)}}\right),$$

$$(0.9)\quad a_1^{(v)}\neq b_1^{(v)},\ b_i^{(v)}=\left[a_i^{(v)}\right] \quad (i=1,\ldots,n-1),$$

where [ ] is the greatest integer function.

The JPA of $a^{(0)}$ is called periodic, if there exist two integers $\ell,m$ with $\ell\geq 0$, $m\geq 1$ such that

$$(0.10)\quad a^{(v+m)} = a^{(v)} \quad (v=\ell,\ell+1,\ldots).$$

The sequences

$$a^{(0)}, a^{(1)},\ldots,a^{(\ell-1)} \text{ and } a^{(\ell)}, a^{(\ell+1)},\ldots,a^{(\ell+m-1)}$$

are called respectively the preperiod and the period of the periodic JPA, and $\ell$ and m are their respective lengths. When $\ell$ and m are minimal, the preperiod and the period are said to be primitive. If $\ell = 0$, the JPA of $a^{(0)}$ is said to be purely periodic.

Let $\mathbb{K}_n = \mathbb{Q}(a_1^{(0)},\ldots,a_{n-1}^{(0)})$. If the JPA of $a^{(0)}$ becomes

periodic with primitive preperiod of length $\ell$ and primitive period of length m, then H. Hasse and L. Bernstein [5] proved that

$$(0.11) \qquad \prod_{j=\ell}^{\ell+m-1} a_{n-1}^{(j)}$$

is a unit in $\mathbb{K}_n$. Formula (0.11) will be referred to as the Hasse-Bernstein formula.

In this paper, we shall proceed this way: after two preliminary chapters, we shall take care of the cases d=1 and d≠1 in chapters 3 and 4 respectively; in chapter 5, some particular cases will be deduced from the main theorem.

## 1. PRELIMINARIES

The purpose of chapters 3 and 4 is to prove the following main theorem.

THEOREM 1.1 Let $D,d,V,n \in \mathbb{N}^*$ and let $n \geq 3$. Suppose

$$\theta^n = \frac{\omega^n}{V^n} = \frac{D^n-d}{V^n} \in \mathbb{N}^* \text{ , with } D,d \equiv 1 \pmod{V^{n-1}},$$

and

$$d \mid pD \text{ for } n = p^r \ (p \text{ prime}) \text{ or } d \mid D \text{ for other } n.$$

If

$$D \geq (n-1)d(V+1)/2 + 1,$$

the JPA of $a^{(0)} = (\theta,\ldots,\theta^{n-1})$ is periodic. If $\ell$ and m denote the lengths of the primitive preperiod and primitive period respectively, then

$$\ell = (n-1)^2 \text{ and } m = \begin{cases} n & \text{for } d = 1, \\ n^2 & \text{for } d \neq 1. \end{cases}$$

The components of the vectors of the preperiod and the period are the following:

$$b_i^{(0)} = \frac{D^i-1}{V^i} \quad (i=1,\ldots,n-1),$$

$$b_i^{(n)} = \begin{cases} \dfrac{(i+1)D^i - iD^{i-1} - 1}{V^i} & (i=1,\ldots,n-2)\ , \\ \dfrac{nD^{n-1}d^{-1} - (n-1)D^{n-2} - 1}{V^{n-1}} & (i=n-1). \end{cases}$$

For $n \geq 4$ and $2 \leq k \leq n-2$,

$$b_i^{(kn)} = \begin{cases} \dfrac{-1 + \sum_{r=0}^{k} (-1)^r \binom{k}{r} \binom{k+i-r}{k} D^{i-r}}{V^i} & (i=1,\ldots,n-k-1), \\ \dfrac{-1 + \binom{n}{k} D^{n-k}d^{-1} + \sum_{r=1}^{k} (-1)^r \binom{k}{r} \binom{n-r}{k} D^{n-r-k}}{V^{n-k}} & (i=n-k), \\ \dfrac{\left(-1 + \sum_{r=0}^{k-n+i} (-1)^r \binom{n-1-i+r}{r} \binom{n}{i-r} D^{i-r}d^{-1} + (-1)^{k-n+i+1} \sum_{r=0}^{n-1-i} (-1)^r \binom{k}{n-1-i-r} \binom{n-1-r}{k} D^{n-1-r-k}\right)}{V^i} & \\ & (i=n-k+1,\ldots,n-1). \end{cases}$$

Now, for $n \geq 3$,

$$b_i^{((n-1)n)} = \begin{cases} \dfrac{nDd^{-1} - n}{V} & (i=1), \\ \dfrac{-1 + \sum_{r=0}^{i-1} (-1)^r \binom{n-1-i+r}{r} \binom{n}{i-r} D^{i-r}d^{-1} + (-1)^i \binom{n-1}{i}}{V^i} & \\ & (i=2,\ldots,n-1), \end{cases}$$

$$b_i^{(n^2)} = \frac{-1 + \sum_{r=0}^{i} (-1)^r \binom{n-i-1+r}{r} \binom{n}{i-r} D^{i-r}}{V^i} \qquad (i=1,\ldots,n-1).$$

For $n \geq 3$ and $1 \leq k \leq n-2$,

$$b_i^{(n^2+kn)} = \begin{cases} \dfrac{-1 + \sum_{r=0}^{i} (-1)^r \binom{n-i-1+r}{r} \binom{n}{i-r} D^{i-r}}{V^i} & (i=1,\ldots,n-1-k), \\ \dfrac{\left(-1 + \sum_{r=0}^{i+k-n} (-1)^r \binom{n-1-i+r}{r} \binom{n}{i-r} D^{i-r}d^{-1} + \sum_{r=i+k-n+1}^{i} (-1)^r \binom{n-1-i+r}{r} \binom{n}{i-r} D^{i-r}\right)}{V^i} & \\ & (i=n-k,\ldots,n-1). \end{cases}$$

Finally, for $n \geq 3$, $t \in \mathbb{N}^*$ and $n \nmid t$,

$$b_i^{(t)} = \begin{cases} 0 & (i=1,\dots,n-2), \\ V & (i=n-1). \end{cases}$$

We shall prove theorem 1.1 by using L. Bernstein's definitions in [2] and following the pattern of his proof.

DEFINITIONS. Let $f_{s,k} = f_{s,k}(V\theta,D) = f_{s,k}(\omega,D)$ with

$$(1.1)\quad f_{s,k}(\omega,D) = \begin{cases} \sum_{j=0}^{k-s} \binom{s+j}{j} \omega^{k-s-j} D^j & (0 \leq s \leq k \leq n-1), \\ \omega^{k+1} & (s=-1;\ k=0,\dots,n-1), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $d_0 f_{s,k} = f_{s,k}$ and for $0 \leq k \leq n-1$ let

$$(1.2)\quad d_1 f_{s,k} = \begin{cases} f_{s,k} - f_{s,k-1} & \text{if } k < n-1, \\ f_{s,n-1} d^{-1} - f_{s,n-2} & \text{if } k = n-1. \end{cases}$$

Suppose $d_m f_{s,k} = d_{m-1} f_{s,k} - d_{m-1} f_{s,k-1}$, so that for $1 \leq m \leq s+1 \leq k+1 \leq n$

$$(1.3)\quad d_m f_{s,k} = \begin{cases} \sum_{r=0}^{m} (-1)^r \binom{m}{r} f_{s,k-r} & \text{if } k < n-1, \\ f_{s,n-1} d^{-1} + \sum_{r=1}^{m} (-1)^r \binom{m}{r} f_{s,n-1-r} & \text{if } k = n-1. \end{cases}$$

For $0 \leq m \leq k-1 \leq n-2$, let

$$(1.4)\quad F_{m,k} = F_{m,k}(\omega,D) = \sum_{j=0}^{k-m-1} (-1)^j \binom{m+j}{m} f_{m+j,n-1} d^{-1} + (-1)^{k-m} \sum_{j=1}^{m+1} \binom{k-j}{k-m-1} d_j f_{k,n-1},$$

$$(1.5)\quad E_{m,k} = E_{m,k}(\omega,D) = \sum_{j=0}^{k-m-1} (-1)^j \binom{m+j}{m} f_{m+1+j,n-1} d^{-1} + (-1)^{k-m} \sum_{j=1}^{m+1} \binom{k-j}{k-m-1} d_j f_{k+1,n-1}.$$

A useful characterization for $F_{m,k}$ was obtained by L. Bernstein [2] and will be used also as a definition: for $0 \le m < k \le n-2$, for $m = n-2 \le k$ and for $m = n-1 \le k$,

$$(1.6)\quad F_{m,k} = \sum_{j=0}^{k-m} (-1)^j \binom{m+j}{m} f_{m+j,n-1} d^{-1} + (-1)^{k-m+1} \sum_{j=0}^{m} (-1)^j \binom{k+1}{m-j} f_{k,n-2-j} .$$

Finally, for $0 \le m \le n-1$ and $0 \le t \le n-1-m$, let

$$(1.7)\quad H_{m,t} = \sum_{j=0}^{n-1-m} (-1)^j \binom{m+j}{j} f_{m+j,n-1} d^{\alpha(j)},$$

$$(1.8)\quad G_{m,t} = \sum_{j=0}^{n-1-m} (-1)^j \binom{m+j}{j} f_{m+1+j,n-1} d^{\alpha(j)},$$

where

$$\alpha(j) = \alpha_t(j) = \begin{cases} -1 & \text{for } 0 \le j \le t, \\ 0 & \text{elsewhere.} \end{cases}$$

Note that $df_{s,k}$ is the product of d with $f_{s,k}$. Instead of $E_{m,k}$, L. Bernstein [2] used the notation $F_{m,k}$ with a bold-faced m; he also used $d^m f_{s,k}$ instead of $d_m f_{s,k}$.

Let us collect together some important equalities and inequalities we shall use later without always referring explicitly to them.

LEMMA 1.1. The following formulas hold:

$$(1.9)\quad \frac{1}{D-\omega} = \frac{f_{0,n-1}}{d},$$

$$(1.10)\quad f_{s,k}(D,D) = \binom{k+1}{k-s} D^{k-s} \qquad (0 \le k \le n-1 \ ; \ -1 \le s \le k),$$

$$(1.11)\quad f_{s,k} - f_{s,k}(D,D) = (\omega - D) f_{s+1,k} \qquad (1 \le k \le n-1 ; -1 \le s \le k-1),$$

$$(1.12)\quad \begin{cases} F_{m,n-2} = F_{m,n-1+s} \\ E_{m,n-2} = E_{m,n-1+s} \end{cases} \qquad (0 \le m \le n-2 ; \ s \in \mathbb{N}),$$

$$(1.13)\quad E_{m-1,k} - E_{m,k} = F_{m,k+1} \qquad (1 \le m \le k-1 \le n-2),$$

$$(1.14)\quad F_{m,k} - F_{m,k}(D,D) = (\omega-D)E_{m,k} \quad (0\le m\le k-1\le n-2)\ ,$$

$$(1.15)\quad H_{m,t} - H_{m,t}(D,D) = (\omega-D)G_{m,t} \quad (0\le m\le n-1; 0\le t\le n-1-m),$$

$$(1.16)\quad G_{m-1,t} - G_{m,t-1} = H_{m,t} \quad (1\le m\le n-1;\ 1\le t\le n-1-m).$$

Finally for $1\le s\le k\le m\le n-1$, the formulas

$$(1.17)\quad d_s f_{k-1,m} = \sum_{j=0}^{s} \binom{s}{j} (D-1)^{s-j} f_{k-1-j,m-s}\ ,$$

$$(1.18)\quad d_k f_{k-1,m}(D,D) = D^{m-k+1} + (D-1)\sum_{j=0}^{k-1} \binom{k}{j}\binom{m-k+1}{k-j} (D-1)^{k-1-j} D^{m-2k+1+j}$$

hold, but m can be equal to n-1 only when d=1.

PROOF. These results are obtained in [2] or as in [2]. Q.E.D.

LEMMA 1.2. For D,d, $V \in \mathbb{N}^*$ and $k = 1,\dots,n-1$,

$$(1.19)\quad D^k - 1 < V^k \theta^k = \omega^k < D^k,$$

$$(1.20)\quad Vf_{s,k} < f_{s-1,k} d^{-1} \quad (s=2,\dots,k),$$

$$(1.21)\quad Vf_{s,k-1} < f_{s,k} d^{-1} \quad (s=0,\dots,k-1),$$

$$(1.22)\quad \begin{cases} (V+1)f_{1,n-1} < f_{0,n-1} d^{-1}, \\ (V+1)f_{0,n-2} < f_{0,n-1} d^{-1}, \end{cases}$$

$$(1.23)\quad d_m f_{s,k-1} < d_m f_{s,k} \quad (1\le m\le s+1\le k\le n-1).$$

PROOF. See [2] for (1.19). Now (1.20) is equivalent to

$$Vd\left(\sum_{j=0}^{k-s} \binom{s+j}{j} \omega^{k-s-j} D^j\right) < \omega^{k-s+1} + \sum_{j=0}^{k-s} \binom{s+j}{j+1} \omega^{k-s-j} D^{j+1},$$

but $Vd\binom{s+j}{j} < \binom{s+j}{j+1} D$ because of the hypothesis

$$(1.24)\quad D \ge 1 + (n-1)d(V+1)/2.$$

Relation (1.21) is similarly obtained. For the most important inequalities (1.22), note that by (1.19) we have

$$(V+1)df_{1,n-1} < (V+1)d\sum_{j=1}^{n-1} jD^{n-1-j}D^{j-1} = (V+1)dn(n-1)D^{n-2}/2,$$

$$(V+1)df_{0,n-2} < (V+1)d\sum_{j=0}^{n-2} D^{n-2} = (V+1)d(n-1)D^{n-2},$$

and

$$f_{0,n-1} > \sum_{j=0}^{n-1} \omega^{n-1-j}\omega^{j} = n\omega^{n-1} > n(D^{n-1}-1);$$

consequently (1.22) holds if

$$D \geq (n-1)d(V+1)/2 + 1/D^{n-2}; \tag{1.25}$$

however (1.25) occurs because of (1.24). Finally, if $k < n-1$, then (1.23) follows from (1.17) and (1.21); if $k = n-1$, then

$$d_m f_{s,n-1} \geq \frac{1}{d}\left(\sum_{j=0}^{m}\binom{m}{j}(D-1)^{m-j}f_{s-j,n-1-m}\right)$$

$$> V\left(\sum_{j=0}^{m}\binom{m}{j}(D-1)^{m-j}f_{s-j,n-2-m}\right) \geq d_m f_{s,n-2}.$$ Q.E.D.

LEMMA 1.3. For $0 \leq m < k \leq n-1$,

$$0 \leq f_{m+2,n-1} < E_{m,k} \leq f_{m+1,n-1}d^{-1}, \tag{1.26}$$

$$0 \leq f_{m+1,n-1} < F_{m,k} \leq f_{m,n-1}d^{-1}. \tag{1.27}$$

PROOF. (i) We want to prove

$$\sum_{j=0}^{k-m-1}(-1)^j\binom{m+j}{m}f_{m+j+1,n-1}d^{-1} - f_{m+2,n-1}$$

$$+(-1)^{k-m}\sum_{j=1}^{m+1}\binom{k-j}{k-m-1}d_j f_{k+1,n-1} > 0.$$

When $k-m$ is even, it is sufficient to show

$$f_{m+1,n-1}d^{-1} > (m+2)f_{m+2,n-1},$$

$$\binom{m+2s}{m}f_{m+1+2s,n-1} > \binom{m+2s+1}{m}f_{m+2+2s,n-1} \left(s=1,\ldots,\frac{k-m-2}{2}\right),$$

by proceeding as in part (a) of [2,p.190]. When $k = m+1$, it is sufficient to show

$$f_{m+1,n-1}d^{-1} > f_{m+2,n-1} + \sum_{j=1}^{m+1} d_j f_{m+2,n-1}d^{-1},$$

as in part (b) of [2,p.190], and when $k-m$ is odd and $\geq 3$, it is enough to prove

$$\binom{k-1}{m} f_{k,n-1}d^{-1} > \sum_{j=1}^{m+1} \binom{k-j}{n-m-1} d_j f_{k+1,n-1}.$$

Similarly, one shows $f_{m+1,n-1}d^{-1} - E_{m,k} \geq 0$.

(ii) Finally, (1.27) is obtained as (1.26), though, for the inequality

$$f_{m,n-1}d^{-1} > (m+2)f_{m+1,n-1},$$

one has to proceed as for (1.22). Q.E.D.

We shall also need the following lemmas involving interesting combinatorial identities.

LEMMA 1.4. Let $k,m,n \in \mathbb{N}$. (i) For $n \geq k$,

$$(1.28) \qquad \sum_{r=k}^{n} \binom{m+r}{m}\binom{r}{k} = \sum_{r=k}^{n} \binom{m+r}{r-k}\binom{m+k}{m} = \binom{m+k}{m}\binom{m+1+n}{n-k}.$$

(ii) For $0 \leq k \leq m \leq n-1$,

$$(1.29) \qquad \sum_{r=0}^{k} \binom{n-m-1+r}{r}\binom{m-r}{k-r} = \binom{n}{k}.$$

(iii) For $0 \leq m \leq k \leq n-1$,

$$(1.30) \qquad \sum_{r=0}^{m} (-1)^r \binom{k}{m-r}\binom{n-1-r}{k} + \sum_{r=0}^{m} (-1)^r \binom{k+1}{m-r}\binom{n-1-r}{k+1} = \binom{k}{m}\binom{n}{k+1}.$$

(iv) For $0 \leq m \leq k \leq n-1$,

$$(1.31) \qquad \sum_{r=0}^{n-k-1} (-1)^r \binom{r+k}{m}\binom{n}{1+r+k} = \sum_{r=0}^{m} (-1)^r \binom{k}{m-r}\binom{n-1-r}{k}.$$

PROOF. (i) Formula (1.28) follows easily by induction on $n$.

(ii) If $k = m$ or $k = 0$ or $m = n-1$, then (1.29) is true. The proof will be by induction on $n$; (1.29) is true for $n = 1$, as is easily seen. Suppose the formula holds for $0<k<m<n-1$. Now for $0<k<m<n$,

$$\sum_{r=0}^{k} \binom{n-m+r}{r} \binom{m-r}{k-r} = \sum_{r=0}^{k} \binom{n-m+r}{r} \left( \binom{m-r-1}{k-r-1} + \binom{m-r-1}{k-r} \right)$$

$$= \sum_{r=0}^{k-1} \binom{n-m+r}{r} \binom{m-r-1}{k-r-1} + \sum_{r=0}^{k} \binom{n-m+r}{r} \binom{m-r-1}{k-r}.$$

Apply the induction hypothesis to obtain

$$\sum_{r=0}^{k} \binom{n-m+r}{r} \binom{m-r}{k-r} = \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

(iii) Formula (1.30) is easily obtained by induction on $n$.

(iv) Formula (1.31) is true if $k = n-1$ or $m = 0$.

The proof will be by induction on $n$. Suppose the formula holds for $0<m\leq k<n-1$. Now for $0<m\leq k<n$,

$$\sum_{r=0}^{n-k} (-1)^r \binom{r+k}{m} \binom{n+1}{l+r+k}$$

$$= \sum_{r=0}^{n-k-1} (-1)^r \binom{r+k}{m} \binom{n}{l+r+k} + \sum_{r=0}^{n-k} (-1)^r \binom{r+k}{m} \binom{n}{n+k}$$

$$= \sum_{r=0}^{n-k-1} (-1)^r \binom{r+k}{m} \binom{n}{l+r+k} + \sum_{r=0}^{n-k} (-1)^r \binom{k+r-1}{m} \binom{n}{r+k}$$

$$+ \sum_{r=0}^{n-k} (-1)^r \binom{k+r-1}{m-1} \binom{n}{r+k}$$

$$= \binom{k-1}{m} \binom{n}{k} + \binom{k-1}{m-1} \binom{n}{k} - \sum_{r=0}^{n-k-1} (-1)^r \binom{k+r}{m-1} \binom{n}{l+r+k}$$

$$= \binom{n}{k} \binom{k}{m} - \sum_{r=0}^{m-1} (-1)^r \binom{k}{m-r-1} \binom{n-1-r}{k}$$

$$= \sum_{r=0}^{m} (-1)^r \binom{k}{m-r} \binom{n-r}{k} . \qquad \text{Q.E.D.}$$

LEMMA 1.5. Let $k,s,r,i,j,n,D \in \mathbb{N}$. (i) For $0 \le s \le r$,

$$(1.32) \qquad \sum_{j=s}^{r} \binom{k}{j-s} \binom{r}{j} = \binom{k+r}{k+s} .$$

(ii) For $0 \le s \le k$,

$$(1.33) \qquad \sum_{j=0}^{r} \binom{k}{j+s} \binom{r}{j} = \binom{k+r}{k-s} .$$

(iii) For $0 \le s \le j-1$,

$$(1.34) \qquad \sum_{r=s+1}^{j} (-1)^r \binom{j}{r} D^{j-r} = (-1)^{s+1} \sum_{r=s+1}^{j} \binom{j}{r} \binom{r-1}{s} (D-1)^{j-r} .$$

(iv) For $2 \le k \le n$ and $n-k+1 \le i \le n-1$,

$$(1.35) \qquad \sum_{j=0}^{n-1-i} \binom{k-1+i-n+j}{j} \binom{i+j}{n-k} = \sum_{j=0}^{n-1-i} (-1)^j \binom{k-1}{n-1-i-j} \binom{n-1-j}{k-1} .$$

(v) For $2 \le k \le n$ and $n-k+2 \le i \le n$,

$$(1.36) \qquad \sum_{j=0}^{n-k+1} (-1)^j \binom{k+j-1}{n-i} \binom{n}{k+j-1} = \binom{n}{i} \binom{i-1}{n+1-k} .$$

(vi) For $2 \le k \le n-1$ and $n-k+1 \le i \le n-1$,

$$(1.37) \qquad \sum_{j=0}^{n-k} \binom{k-1-n+i+j}{j} \binom{k-1+j}{n-1-i} = \sum_{j=0}^{n-k} (-1)^j \binom{j+k-1}{n-1-i} \binom{n}{k+j} .$$

(vii) For $2 \le k \le n-1$ and $n-k+1 \le i \le n-1$,

$$(1.38) \qquad \sum_{j=0}^{n-1-i} \binom{k-1+i-n+j}{j} \binom{i+j}{n-k} = \sum_{j=0}^{n-k} \binom{k-1-n+i+j}{j} \binom{k-1+j}{n-1-i} .$$

PROOF. (i) Formula (1.32) is proved by induction on $k$.

(ii) Similarly for formula (1.33).

(iii) For (1.34), one expands $D^{j-r} = ((D-1)+1)^{j-r}$ .

(iv) Formula (1.35) is true for $k = 2$, for $k = n$ and for $n = 2$. The proof will be by induction on n and (1.30) will be used at a certain step. Suppose the formula holds for $2<k<n$ . Now for $2<k<n+1$ ,

$$\sum_{j=0}^{n-i} \binom{k-2+i-n+j}{j} \binom{i+j}{n+1-k}$$

$$= \sum_{j=0}^{n-1-i} \binom{k-2+i-n+j}{j} \binom{i+j}{n-k+1} + \binom{k-2}{n-i} \binom{n}{n+1-k}$$

$$= \sum_{j=0}^{n-1-i} (-1)^j \binom{k-2}{n-1-i-j} \binom{n-1-j}{k-2} + \binom{k-2}{n-i} \binom{n}{k-1}$$

$$= \binom{k-2}{n-1-i} \binom{n}{k-1} - \sum_{j=0}^{n-1-i} (-1)^j \binom{k-1}{n-1-i-j} \binom{n-1-j}{k-1} + \binom{k-2}{n-i} \binom{n}{k-1}$$

$$= \sum_{j=1}^{n-i} (-1)^j \binom{k-1}{n-i-j} \binom{n-j}{k-1} + \binom{k-1}{n-i} \binom{n}{k-1}$$

$$= \sum_{j=0}^{n-i} (-1)^j \binom{k-1}{n-i-j} \binom{n-j}{k-1} .$$

(v) For (1.36), note that

$$\binom{k+j-1}{n-i} \binom{n}{k+j-1} = \binom{n}{i} \binom{i}{n-k+1-j}.$$

(vi) Formula (1.37) is obtained by induction on n with the help of (1.36). Let us just point out that

$$\sum_{j=0}^{n+1-k} \binom{k-2-n+i+j}{j} \binom{k-1+j}{n-i}$$

$$= \sum_{j=0}^{n-k} \binom{k-1-n+(i-1)+j}{j} \binom{k-1+j}{n-1-(i-1)} + \binom{i-1}{n+1-k} \binom{n}{i} .$$

(vii) Formula (1.38) follows from (1.35), (1.37) and (1.31). Q.E.D.

## 2. INEQUALITIES

To calculate the greatest integers contained in the components of the vectors a's, one need prove the following lemmas.

LEMMA 2.1. The following inequalities hold:

$$(2.1)\qquad 0 < \frac{1+(\omega-D)d_k f_{k,s}}{1+(\omega-D)d_k f_{k,r}} < 1 \quad (0\le k\le r\le s-1\le n-2),$$

$$(2.2)\qquad 0 < \frac{V^{r-s+1}(D-\omega)d_1 f_{0,s}}{1+(\omega-d)f_{0,r}} < 1 \quad (0\le s\le r-1\le n-3),$$

$$(2.3)\qquad 0 < \frac{V^{r-s+1}(D-\omega)d_{k+1} f_{k,s}}{1+(\omega-D)d_k f_{k,r}} < 1 \quad (1\le k\le s\le r-1\le n-2),$$

$$(2.4)\qquad 0 < \frac{1+(\omega-D)E_{m,k-1}}{1+(\omega-D)d_k f_{k,r}} < 1 \quad (0\le m\le k-2\le r-2\le n-3),$$

$$(2.5)\qquad 0 < \frac{1+(\omega-D)E_{m,k-1}}{1+(\omega-D)E_{r,k-1}} < 1 \quad (0\le m\le r-1\le k-3\le n-4),$$

$$(2.6)\qquad 0 < \frac{V^{n-r+k-1-m}(D-\omega)d_{k+1} f_{k,r}}{1+(\omega-D)E_{m,k-1}} < 1 \quad (0\le m\le k-2\le r-2\le n-3),$$

$$(2.7)\qquad 0 < \frac{V^{r-m}(D-\omega)F_{r,k}}{1+(\omega-D)E_{m,k-1}} < 1 \quad (0\le m\le r-2\le k-3\le n-4).$$

PROOF. To prove (2.1), see [2]. To come up with the left inequality in (2.2) and (2.3), one has to prove $d_{k+1}f_{k,s} > 0$ and $d_k f_{k,r} < f_{0,n-1}d^{-1}$, but $d_{k+1}f_{k,s} = d_k f_{k,s} - d_k f_{k,s-1} > 0$ by (1.23), and $d_k f_{k,r} < f_{k,r} < f_{0,n-1}d^{-1}$ by lemma 1.2 .

The right side of (2.2) is equivalent to

$$V^{r-s+1}d_1 f_{0,s} + f_{0,r} < f_{0,n-1}d^{-1} ;$$

however, by lemma 1.2,

$$V^{r-s+1}d_1 f_{0,s} + f_{0,r} \le V^{r-s+1} f_{0,s} + f_{0,r} < Vf_{0,r} + f_{0,r}$$
$$\le (V+1)f_{0,n-2} < f_{0,n-1}d^{-1} .$$

Similarly, for the right side of (2.3),

$$V^{r-s+1}d_{k+1}f_{k,s} + d_k f_{k,r} < V^{r-s+1}f_{k,s} + f_{k,r} < Vf_{k,r} + f_{k,r}$$
$$\leq (V+1)f_{1,n-1} < f_{0,n-1}d^{-1} .$$

To come up with the left inequality of (2.4), we have to show $E_{m,k-1} < f_{0,n-1}d^{-1}$, but this follows from (1.26) and lemma 1.2; for the right hand inequality, we have to show $d_k f_{k,r} < E_{m,k-1}$, but by lemma 1.3, $d_k f_{k,r} \leq f_{k,r} \leq f_{m+2,n-1} < E_{m,k-1}$.

For the right inequality of (2.5), we have to show $E_{m,k-1} - E_{r,k-1} > 0$, but by (1.13) and (1.27),

$$-E_{r,k-1} + E_{m,k-1} = \sum_{j=m}^{r-1} (-E_{j+1,k-1} + E_{j,k-1}) = \sum_{j=m+1}^{r} F_{j,k} > 0 .$$

For the right inequality of (2.6) one shows

$$V^{n-r+k-1-m}d_{k+1}f_{k,r} + E_{m,k-1} \leq V^{n-r+k-1}f_{k,r} + f_{1,n-1}$$
$$< V^{n-r}f_{1,r} + f_{1,n-1} \leq Vf_{1,n-1} + f_{1,n-1} < f_{0,n-1}d^{-1} .$$

Similarly for the right inequality of (2.7), one shows

$$V^{r-m}F_{r,k} + E_{m,k-1} \leq V^{r-m}f_{r,n-1}d^{-1} + f_{m+1,n-1}d^{-1}$$
$$< Vf_{m+1,n-1}d^{-1} + f_{m+1,n-1}d^{-1} \leq (V+1)f_{1,n-1}d^{-1}$$
$$< f_{0,n-1}d^{-1} . \quad \text{Q.E.D.}$$

LEMMA 2.2. The following inequalities hold:

$$(2.8) \qquad V < \frac{V}{1+\omega-D} < V+1 ,$$

$$(2.9) \qquad V < \frac{V(1+(\omega-D)d_k f_{k,s})}{1+(\omega-D)d_k f_{k,s+1}} < V+1 \quad (1 \leq k \leq s \leq n-2) \text{ or } (k=0,\ 0 \leq s \leq n-3) ,$$

$$(2.10) \qquad \frac{d_k f_{k-1,r}(D,D)-1}{V^{r-k+1}} < \frac{d_k f_{k-1,r}}{V^{r-k+1}} < \frac{d_k f_{k-1,r}(D,D)}{V^{r-k+1}} \quad (1 \leq k \leq r \leq n-1) ,$$

$$(2.11)\qquad \frac{F_{m,k}(D,D)-1}{V^{n-1-m}} < \frac{F_{m,k}}{V^{n-1-m}} < \frac{F_{m,k}(D,D)}{V^{n-1-m}} \quad (0\le m\le k-1\le n-3),$$

$$(2.12)\qquad V < \frac{V(1+(\omega-D)d_k f_{k,n-1})}{1+(\omega-D)E_{k-2,k-1}} < V+1 \quad (2\le k\le n-2)\ ,$$

$$(2.13)\qquad V < \frac{V(1+(\omega-D)E_{m,k-1})}{1+(\omega-D)E_{m-1,k-1}} < V+1 \quad (1\le m\le k-2\le n-3)\ .$$

PROOF. One comes up with (2.8) by showing $\omega<D$ and $(V+1)(D-\omega) < 1$ or $V+1 < f_{0,n-1}d^{-1}$. Now (2.9) is equivalent to the inequalities

$$d_k f_{k,s} < d_k f_{k,s+1} \text{ and } (V+1)d_k f_{k,s+1} - Vd_k f_{k,s} < f_{0,n-1}d^{-1}.$$

As for (2.10) and (2.11), they are equivalent to the inequalities

$$0 < d_k f_{k,r} < f_{0,n-1}d^{-1} \text{ and } 0 < E_{m,k} < f_{0,n-1}d^{-1}$$

which were obtained in [2]. One comes up with (2.12) by proving

$$d_k f_{k,n-1} < f_{k,n-1} < E_{k-2,k-1}$$

and

$$(V+1)E_{k-2,k-1} - Vd_k f_{k,n-1} < (V+1)E_{k-2,k-1} \le$$
$$(V+1)f_{k-1,n-1}d^{-1} \le (V+1)f_{1,n-1}d^{-1} < f_{0,n-1}d^{-1}.$$

Finally for (2.13) one proves $E_{m-1,k-1} - E_{m,k-1} > 0$ and

$$(V+1)E_{m-1,k-1} - VE_{m,k-1} < (V+1)E_{m-1,k-1} \le$$
$$(V+1)f_{1,n-1}d^{-1} < f_{0,n-1}d^{-1}\ .\quad \text{Q.E.D.}$$

LEMMA 2.3. For $d \ne 1$, the following inequalities hold:

$$(2.14)\qquad 0 < \frac{1+d(\omega-D)E_{m,n-1}}{1+d(\omega-D)E_{r,n-1}} < 1 \quad (0\le m\le r-1\le n-3)\ ,$$

$$(2.15)\qquad 0 < \frac{V^{r-m}d(D-\omega)F_{r,n-1}}{1+d(\omega-D)E_{m,n-1}} < 1 \quad (0\le m\le r-2\le n-3)\ ,$$

$$(2.16)\qquad 0 < \frac{1+(\omega-D)G_{m,t}}{1+d(\omega-D)E_{r,n-1}} < 1 \quad (0\le m\le r-1\le n-3;\ 0\le t\le \underline{\min}(r-1,n-1-m)),$$

$$(2.17)\qquad 0 < \frac{1+(\omega-D)G_{m,t}}{1+(\omega-D)G_{r,s}} < 1 \quad (0\le m\le r-1\le n-5;\ 0\le s\le t-1\le \underline{\min}(n-2-m,n-5)),$$

$$(2.18)\qquad 0 < \frac{V^{m-r}d(D-\omega)F_{m,n-1}}{1+(\omega-D)G_{r,s}} < 1 \quad (0\le r\le m-2\le n-3;\ 0\le s\le \underline{\min}(m-2,n-1-r)),$$

$$(2.19)\qquad 0 < \frac{V^{m-r}(D-\omega)H_{m,t}}{1+(\omega-D)G_{r,s}} < 1 \quad (0\le r\le m-2\le n-5;\ 0\le t\le s-1\le \underline{\min}(n-5,n-2-r)).$$

PROOF. The proof is easily achieved with the help of these few remarks. For (2.14) and (2.15), note that $1+d(\omega-D)E_{m,n-1} > 0$, because $dE_{m,n-1} \le f_{m+1,n-1} \le f_{1,n-1} < f_{0,n-1}d^{-1}$. For (2.16), note that $1+(\omega-D)G_{m,t} \ge 1+(\omega-D)E_{m,n-1} > 0$; moreover, by (1.13) and lemma 1.3,

$$G_{m,t} - dE_{m+1,n-1} \ge E_{m,n-1} - E_{m+1,n-1} - (d-1)E_{m+1,n-1}$$
$$> F_{m+1,n-1} - dE_{m+1,n-1} > f_{m+2,n-1} - df_{m+2,n-1}d^{-1} = 0 .$$

For (2.17), it is enough to prove

$$G_{m,t} > dE_{m+1,n-1} \ge dE_{r,n-1} \ge G_{r,s} .$$

For (2.18), note

$$V^{m-r}dF_{m,n-1} + G_{r,s} \le V^{m-r}df_{m,n-1}d^{-1} + dE_{r,n-1}$$
$$< Vf_{1,n-1} + f_{r+1,n-1} \le (V+1)f_{1,n-1} < f_{0,n-1}d^{-1} .$$

Finally, concerning (2.19), one shows $H_{m,t} \ge F_{m,n-1} > 0$, and $V^{m-r}H_{m,t} + G_{r,s} \le V^{m-r}dF_{m,n-1} + dE_{r,n-1} \le (V+1)f_{1,n-1} < f_{0,n-1}d^{-1}$. Q.E.D.

LEMMA 2.4. For $d \ne 1$, the following inequalities hold:

$$(2.20)\qquad V < \frac{V}{1+(\omega-D)d^{-1}} < V+1 ,$$

$$(2.21)\qquad \frac{dF_{m,n-1}(D,D)-1}{V^{n-1-m}} < \frac{dF_{m,n-1}}{V^{n-1-m}} < \frac{dF_{m,n-1}(D,D)}{V^{n-1-m}} \qquad (0\le m\le n-2)\ ,$$

$$(2.22)\qquad V < \frac{V(1+d(\omega-D)E_{m,n-1})}{1+d(\omega-D)E_{m-1,n-1}} < V+1 \quad (1\le m\le n-2)\ ,$$

$$(2.23)\qquad \frac{H_{k,m}(D,D)-1}{V^{n-1-k}} < \frac{H_{k,m}}{V^{n-1-k}} < \frac{H_{k,m}(D,D)}{V^{n-1-k}} \qquad (0\le k\le n-3;\ 0\le m\le \underline{\min}(n-3,n-1-k))\ ,$$

$$(2.24)\qquad V < \frac{V(1+d(\omega-D)E_{k,n-1})}{1+(\omega-D)G_{k-1,0}} < V+1 \quad (1\le k\le n-3)\ ,$$

$$(2.25)\qquad V < \frac{V(1+(\omega-D)G_{m,t-1})}{1+(\omega-D)G_{m-1,t}} < V+1 \quad (1\le m\le n-4;\ 1\le t\le \underline{\min}(n-4,n-m))\ .$$

PROOF. The inequalities in (2.21) and (2.23) are equivalent to

$$0<dE_{m,n-1} < f_{0,n-1}d^{-1} \text{ and } 0 <G_{k,m} < f_{0,n-1}d^{-1}$$

respectively. Concerning (2.24), it is sufficient to prove

$$G_{k-1,0} > dE_{k,n-1} \text{ and } (V+1)G_{k-1,0} < f_{0,n-1}d^{-1}\ .$$

The rest is obtained as usual. Q.E.D.

In the next two sections, we shall develop the JPA of

$$a^{(0)} = (\theta,\theta^2,\ldots,\theta^{n-1}) = (\omega/V,\omega^2/V^2,\ldots,\omega^{n-1}/V^{n-1})$$

and suppose that the components of the vectors $b^{(k)}$ are the following: For $n\ge 3$,

$$(2.26)\qquad b_i^{(0)} = \frac{D^i-1}{V^i} \quad (i=1,\ldots,n-1);$$

$$(2.27)\qquad b_i^{(n)} = \frac{d_1 f_{0,i}(D,D)-1}{V^i} \quad (i=1,\ldots,n-1);$$

for $2\le k\le n-1$ ,

$$(2.28)\qquad b_i^{(kn)} = \begin{cases} \dfrac{d_k f_{k-1,k+i-1}(D,D)-1}{V^i} & (i=1,\dots,n-k)\ , \\ \dfrac{F_{n-1-i,k-1}(D,D)-1}{V^i} & (i=n-k+1,\dots,n-1); \end{cases}$$

$$(2.29)\qquad b_i^{(n^2)} = \frac{dF_{n-i-1,n-1}(D,D)-1}{V^i} \quad (i=1,\dots,n-1);$$

for $1\le k\le n-2$,

$$(2.30)\qquad b_i^{(n^2+kn)} = \begin{cases} \dfrac{dF_{n-1-i,n-1}(D,D)-1}{V^i} & (i=1,\dots,n-1-k), \\ \dfrac{H_{n-1-i,i+k-n}(D,D)-1}{V^i} & (i=n-k,\dots,n-1); \end{cases}$$

finally for $n\nmid t$ , $t \in \mathbb{N}^*$ ,

$$(2.31)\qquad b_i^{(t)} = \begin{cases} 0 & (i=1,\dots,n-2), \\ V & (i=n-1)\ . \end{cases}$$

## 3. THE CASE d=1

Suppose $d=1$ in this chapter. First,

$$a_i^{(1)} = \begin{cases} \dfrac{1+(\omega-D)f_{0,i}}{V^i(1+\omega-D)} & (i=1,\dots,n-2)\ , \\ \dfrac{V}{1+\omega-D} & (i=n-1)\ . \end{cases}$$

If $n\ge 4$ and if $2\le j\le n-2$, we come up with

$$a_i^{(j)} = \begin{cases} \dfrac{1+(\omega-D)f_{0,j+i-1}}{V^i(1+(\omega-D)f_{0,j-1})} & (i=1,\dots,n-1-j)\ , \\ \dfrac{V^{n-i}(D-\omega)d_1 f_{0,i+j-n}}{1+(\omega-D)f_{0,j-1}} & (i=n-j,\dots,n-2)\ , \\ \dfrac{V(1+(\omega-D)f_{0,j-2})}{1+(\omega-D)f_{0,j-1}} & (i=n-1)\ . \end{cases}$$

Now, if $n\ge 3$, we get

$$a_i^{(n-1)} = \begin{cases} \dfrac{V^{n-i}(D-\omega)d_1 f_{0,i-1}}{1+(\omega-D)f_{0,n-2}} & (i=1,\ldots,n-2) , \\ \dfrac{V(1+(\omega-D)f_{0,n-3})}{1+(\omega-D)f_{0,n-2}} & (i=n-1) , \end{cases}$$

$$a_i^{(n)} = \frac{d_1 f_{0,i}}{V^i} \quad (i=1,\ldots,n-1) ,$$

and

$$a_i^{(n+1)} = \begin{cases} \dfrac{1+(\omega-D)d_1 f_{1,i+1}}{V^i(1+\omega-D)} & (i=1,\ldots,n-2) , \\ \dfrac{V}{1+\omega-D} & (i=n-1) . \end{cases}$$

If $n\geq 4$ and $2\leq j\leq n-2$ , we have

$$a_i^{(n+j)} = \begin{cases} \dfrac{1+(\omega-D)d_1 f_{1,j+i}}{V^i(1+(\omega-D)d_1 f_{1,j})} & (i=1,\ldots,n-1-j) , \\ \dfrac{V^{n-i}(D-\omega)d_2 f_{1,i+j+1-n}}{1+(\omega-D)d_1 f_{1,j}} & (i=n-j,\ldots,n-2) , \\ \dfrac{V(1+(\omega-D)d_1 f_{1,j-1})}{1+(\omega-D)d_1 f_{1,j}} & (i=n-1) . \end{cases}$$

Finally, for $n\geq 3$ ,

$$a_i^{(2n-1)} = \begin{cases} \dfrac{V^{n-i}(D-\omega)d_2 f_{1,i}}{1+(\omega-D)d_1 f_{1,n-1}} & (i=1,\ldots,n-2) , \\ \dfrac{V(1+(\omega-D)d_1 f_{1,n-2})}{1+(\omega-D)d_1 f_{1,n-1}} & (i=n-1) . \end{cases}$$

Suppose now

(3.1) $n\geq 4$ and $2\leq k\leq n-2$ .

We first obtain

$$a_i^{(kn)} = \begin{cases} \dfrac{d_k f_{k-1,k+i-1}}{V^i} & (i=1,\ldots,n-k) , \\ \dfrac{F_{n-1-i,k-1}}{V^i} & (i=n-k+1,\ldots,n-1) , \end{cases}$$

and

$$a_i^{(kn+1)} = \begin{cases} \dfrac{1+(\omega-D)d_k f_{k,k+i}}{V^i(1+\omega-D)} & (i=1,\ldots,n-1-k) , \\ \dfrac{1+(\omega-D)E_{n-2-i,k-1}}{V^i(1+\omega-D)} & (i=n-k,\ldots,n-2) , \\ \dfrac{V}{1+\omega-D} & (i=n-1) . \end{cases}$$

If $n\geq 5$ and $2\leq k\leq n-3$ , then for $2\leq j\leq n-k-1$ , we come up with

$$a_i^{(kn+j)} = \begin{cases} \dfrac{1+(\omega-D)d_k f_{k,k+j+i-1}}{V^i(1+(\omega-D)d_k f_{k,k+j-1})} & (i=1,\ldots,n-k-j) , \\ \dfrac{1+(\omega-D)E_{n-j-i-1,k-1}}{V^i(1+(\omega-D)d_k f_{k,k+j-1})} & (i=n-k-j+1,\ldots n-j-1) , \\ \dfrac{V^{n-i}(D-\omega)d_{k+1} f_{k,k+j+i-n}}{1+(\omega-D)d_k f_{k,k+j-1}} & (i=n-j,\ldots n-2) , \\ \dfrac{V(1+(\omega-D)d_k f_{k,k+j-2})}{1+(\omega-D)d_k f_{k,k+j-1}} & (i=n-1) . \end{cases}$$

Now, under condition (3.1) , we have

$$a_i^{(kn+n-k)} = \begin{cases} \dfrac{1+(\omega-D)E_{k-1-i,k-1}}{V^i(1+(\omega-D)d_k f_{k,n-1})} & (i=1,\ldots,k-1) , \\ \dfrac{V^{n-i}(D-\omega)d_{k+1} f_{k,i}}{1+(\omega-D)d_k f_{k,n-1}} & (i=k,\ldots,n-2) , \\ \dfrac{V(1+(\omega-D)d_k f_{k,n-2})}{1+(\omega-D)d_k f_{k,n-1}} & (i=n-1) . \end{cases}$$

If $n\geq 5$ and $3\leq k\leq n-2$ , we come up with

$$a_i^{(kn+n-k+1)} = \begin{cases} \dfrac{1+(\omega-D)E_{k-2-i,k-1}}{V^i(1+(\omega-D)E_{k-2,k-1})} & (i=1,\ldots,k-2)\ , \\ \dfrac{V^{n-i}(D-\omega)d_{k+1}f_{k,i+1}}{1+(\omega-D)E_{k-2,k-1}} & (i=k-1,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)d_kf_{k,n-1})}{1+(\omega-D)E_{k-2,k-1}} & (i=n-1)\ . \end{cases}$$

Moreover, if $n\geq 6$ , $4\leq k\leq n-2$ and $n-k+2\leq j\leq n-2$ , we have

$$a_i^{(kn+j)} = \begin{cases} \dfrac{1+(\omega-D)E_{n-j-i-1,k-1}}{V^i(1+(\omega-D)E_{n-j-1,k-1})} & (i=1,\ldots,n-j-1)\ , \\ \dfrac{V^{n-i}(D-\omega)d_{k+1}f_{k,k+j+i-n}}{1+(\omega-D)E_{n-j-1,k-1}} & (i=n-j,\ldots,2n-k-j-1), \\ \dfrac{V^{n-i}(D-\omega)F_{2n-j-i-1,k}}{1+(\omega-D)E_{n-j-1,k-1}} & (i=2n-k-j,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)E_{n-j,k-1})}{1+(\omega-D)E_{n-j-1,k-1}} & (i=n-1)\ . \end{cases}$$

Finally, when $n\geq 5$ , $3\leq k\leq n-2$ , we end up with

$$a_i^{(kn+n-1)} = \begin{cases} \dfrac{V^{n-i}(D-\omega)d_{k+1}f_{k,k+i-1}}{1+(\omega-D)E_{0,k-1}} & (i=1,\ldots,n-k)\ , \\ \dfrac{V^{n-i}(D-\omega)F_{n-i,k}}{1+(\omega-D)E_{0,k-1}} & (i=n-k+1,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)E_{1,k-1})}{1+(\omega-D)E_{0,k-1}} & (i=n-1)\ , \end{cases}$$

and when $n\geq 4$ , $k=2$ , we have

$$a_i^{(3n-1)} = \begin{cases} \dfrac{V^{n-i}(D-\omega)d_3f_{2,i+1}}{1+(\omega-D)E_{0,1}} & (i=1,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)d_2f_{2,n-1})}{1+(\omega-D)E_{0,1}} & (i=n-1)\ . \end{cases}$$

Therefore, when $n\geq 3$ , we conclude

$$a_i^{((n-1)n)} = \begin{cases} \dfrac{d_{n-1}f_{n-2,n-1}}{V} = \dfrac{F_{n-2,n-2}}{V} & (i=1)\ , \\ \dfrac{F_{n-1-i,n-2}}{V^i} & (i=2,\ldots,n-1)\ , \end{cases}$$

$$a_i^{((n-1)n+1)} = \begin{cases} \dfrac{1+(\omega-D)E_{n-2-i,n-2}}{V^i(1+(\omega-D)d^{-1})} & (i=1,\ldots,n-2)\ , \\ \dfrac{V}{1+(\omega-D)d^{-1}} & (i=n-1)\ . \end{cases}$$

As $d=1$, then $E_{n-2-i,n-2} = E_{n-2-i,n-3}$ and $d_{n-2}f_{n-2,n-1} = E_{n-3,n-2}$, whereupon

$$a^{(n^2-n+1)} = a^{(n^2-2n+1)}\ . \tag{3.2}$$

A complete proof by induction that the a's are correct is long though easy to give, and involves always the same pattern. It will be left to the reader.

Because of lemmas 2.1 and 2.2, it is sufficient to show that the b's previously given are indeed integers. By hypothesis, $\dfrac{D^i-1}{V^i}$ is an integer. One uses formula (1.18) to conclude that

$$\frac{d_k f_{k-1,k+i-1}(D,D)-1}{V^i} \quad (k=1,\ldots,n-1;i=1,\ldots,n-k)$$

is an integer. Now from (1.6) and (1.28), we obtain

$$\begin{aligned} F_{n-1-i,n-1}(D,D) = F_{n-1-i,n-2}(D,D) &= \sum_{j=0}^{i} (-1)^j \binom{n-1-i+j}{j}\binom{n}{i-j} D^{i-j} \\ &= \sum_{j=0}^{i} (-1)^j \left( \sum_{r=j}^{i} \binom{n-1-i+r}{r}\binom{r}{j} \right) D^{i-j} \\ &= \sum_{r=0}^{i} \sum_{j=0}^{r} (-1)^j \binom{n-1-i+r}{r}\binom{r}{j} D^{i-j}\ , \end{aligned}$$

whereupon

$$F_{n-1-i,n-1}(D,D) = D^i + \sum_{r=1}^{i} \binom{n-1-i+r}{r} D^{i-r}(D-1)^r \tag{3.3}$$

$$(i=1,\ldots,n-1).$$

Consequently,

$$(3.4)\qquad \frac{F_{n-1-i,n-1}(D,D)-1}{V^i} \quad (i=1,\ldots,n-1)$$

is an integer. Finally, for $2\le k\le n-2$ and $n-k+1\le i\le n-1$, we have

$$\frac{F_{n-1-i,k-1}(D,D)-1}{V^i} = \frac{F_{n-1-i,n-1}(D,D)-1}{V^i} + \frac{A}{V^i}$$

with

$$A = -\sum_{j=k-n+1+i}^{i} (-1)^j \binom{n-1-i+j}{j}\binom{n}{i-j} D^{i-j}$$

$$+ (-1)^{k-n+1+i} \sum_{j=0}^{n-1-i} (-1)^j \binom{k}{n-1-i-j}\binom{n-1-j}{k} D^{n-1-j-k} ;$$

however (3.4) is an integer; consequently, we show that $(F_{n-1-i,k-1}(D,D)-1)/V^i$ is an integer, if we show

$$\sum_{j=k-n+1+i}^{i} (-1)^j \binom{n-1-i+j}{j}\binom{n}{i-j} =$$

$$\sum_{j=0}^{n-1-i} (-1)^{k-n+1+i+j} \binom{k}{n-1-i-j}\binom{n-1-j}{k},$$

but this follows from formula (1.31).

Finally, the Hasse-Bernstein formula provides us with the unit

$$(3.5)\qquad \varepsilon = \prod_{j=n^2-2n+1}^{n^2-n} a_{n-1}^{(j)} = \frac{F_{0,n-2}}{1+(\omega-D)E_{0,n-3}} = f_{0,n-1} = \frac{1}{D-V\theta}.$$

## 4. THE CASE $d \neq 1$

In this chapter, $d\neq 1$ . The values of $a^{(j)}, 0\le j\le n^2-n+1$, are those given in chapter 3. Now, for $n\ge 4$ and $2\le j\le n-2$, we have

$$a_i^{((n-1)n+j)} = \begin{cases} \dfrac{1+(\omega-D)E_{n-j-i-1,n-2}}{V^i(1+(\omega-D)E_{n-j-1,n-2})} & (i=1,\ldots,n-j-1)\ , \\ \dfrac{V^{n-i}(D-\omega)F_{2n-i-j-1,n-1}}{1+(\omega-D)E_{n-j-1,n-2}} & (i=n-j,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)E_{n-j,n-2})}{1+(\omega-D)E_{n-j-1,n-2}} & (i=n-1)\ . \end{cases}$$

Then, for $n\geq 3$ we come up with

$$a_i^{(n^2-1)} = \begin{cases} \dfrac{V^{n-i}(D-\omega)F_{n-i,n-1}}{1+(\omega-D)E_{0,n-2}} & (i=1,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)E_{1,n-2})}{1+(\omega-D)E_{0,n-2}} & (i=n-1)\ , \end{cases}$$

$$a_i^{(n^2)} = \frac{dF_{n-i-1,n-1}}{V^i} \quad (i=1,\ldots,n-1)\ ,$$

$$a_i^{(n^2+1)} = \begin{cases} \dfrac{1+d(\omega-D)E_{n-2-i,n-1}}{V^i(1+\omega-D)} & (i=1,\ldots,n-2)\ , \\ \dfrac{V}{1+\omega-D} & (i=n-1)\ . \end{cases}$$

For $n\geq 4$ and $2\leq j\leq n-2$, we have

$$a_i^{(n^2+j)} = \begin{cases} \dfrac{1+d(\omega-D)E_{n-j-i-1,n-1}}{V^i(1+d(\omega-D)E_{n-j-1,n-1})} & (i=1,\ldots,n-j-1)\ , \\ \dfrac{V^{n-i}d(D-\omega)F_{2n-i-j-1,n-1}}{1+d(\omega-D)E_{n-j-1,n-1}} & (i=n-j,\ldots,n-2)\ , \\ \dfrac{V(1+d(\omega-D)E_{n-j,n-1})}{1+d(\omega-D)E_{n-j-1,n-1}} & (i=n-1)\ . \end{cases}$$

Then for $n\geq 3$ comes

$$a_i^{(n^2+n-1)} = \begin{cases} \dfrac{V^{n-i}d(D-\omega)F_{n-i,n-1}}{1+d(\omega-D)E_{0,n-1}} & (i=1,\ldots,n-2)\ , \\ \dfrac{V(1+d(\omega-D)E_{1,n-1})}{1+d(\omega-D)E_{0,n-1}} & (i=n-1)\ . \end{cases}$$

Let $n\geq 4$ and let $1\leq k\leq n-3$ ; we obtain

$$a_i^{(n^2+kn)} = \begin{cases} \dfrac{dF_{n-1-i,n-1}}{V^i} & (i=1,\ldots,n-1-k)\ , \\ \dfrac{H_{n-1-i,i+k-n}}{V^i} & (i=n-k,\ldots,n-1)\ , \end{cases}$$

$$a_i^{(n^2+kn+1)} = \begin{cases} \dfrac{1+d(\omega-D)E_{n-2-i,n-1}}{V^i(1+\omega-D)} & (i=1,\ldots,n-2-k)\ , \\ \dfrac{1+(\omega-D)G_{n-2-i,i+k-n+1}}{V^i(1+\omega-D)} & (i=n-1-k,\ldots n-2)\ , \\ \dfrac{V}{1+\omega-D} & (i=n-1). \end{cases}$$

For $n\geq 5$ , $1\leq k\leq n-4$ and $2\leq j\leq n-k-2$ , we come up with

$$a_i^{(n^2+kn+j)} = \begin{cases} \dfrac{1+d(\omega-D)E_{n-j-i-1,n-1}}{V^i(1+d(\omega-D)E_{n-j-1,n-1})} & (i=1,\ldots,n-j-k-1)\ , \\ \dfrac{1+(\omega-D)G_{n-1-i-i,i+k+j-n}}{V^i(1+d(\omega-D)E_{n-j-1,n-1})} & (i=n-j-k,\ldots, n-j-1)\ , \\ \dfrac{V^{n-i}d(D-\omega)F_{2n-i-j-1,n-1}}{1+d(\omega-D)E_{n-j-1,n-1}} & (i=n-j,\ldots,n-2)\ , \\ \dfrac{V(1+d(\omega-D)E_{n-j,n-1})}{1+d(\omega-D)E_{n-j-1,n-1}} & (i=n-1)\ . \end{cases}$$

Now for $n\geq 4$ and $1\leq k\leq n-3$ comes

$$a_i^{(n^2+kn+n-k-1)} = \begin{cases} \dfrac{1+(\omega-D)G_{k-i,i-1}}{V^i(1+d(\omega-D)E_{k,n-1})} & (i=1,\ldots,k)\ , \\ \dfrac{V^{n-i}d(D-\omega)F_{n+k-i,n-1}}{1+d(\omega-D)E_{k,n-1}} & (i=k+1,\ldots,n-2), \\ \dfrac{V(1+d(\omega-D)E_{k+1,n-1})}{1+d(\omega-D)E_{k,n-1}} & (i=n-1)\ . \end{cases}$$

If $n\geq 5$ and $2\leq k\leq n-3$, we then have

$$a_i^{(n^2+kn+n-k)} = \begin{cases} \dfrac{1+(\omega-D)G_{k-i-1,i}}{V^i(1+(\omega-D)G_{k-1,0})} & (i=1,\ldots,k-1)\ , \\ \dfrac{V^{n-i}d(D-\omega)F_{n+k-i-1,n-1}}{1+(\omega-D)G_{k-1,0}} & (i=k,\ldots,n-2)\ , \\ \dfrac{V(1+d(\omega-D)E_{k,n-1})}{1+(\omega-D)G_{k-1,0}} & (i=n-1)\ . \end{cases}$$

If $n\geq 6$, $3\leq k\leq n-3$ and $1\leq j\leq k-2$, then we get

$$a_i^{(n^2+kn+n-k+j)} = \begin{cases} \dfrac{1+(\omega-D)G_{k-i-j-1,i+j}}{V^i(1+(\omega-D)G_{k-j-1,j})} & (i=1,\ldots,k-j-1)\ , \\ \dfrac{V^{n-i}d(D-\omega)F_{n+k-j-i-1,n-1}}{1+(\omega-D)G_{k-j-1,j}} & (i=k-j,\ldots,n-2-j)\ , \\ \dfrac{V^{n-i}(D-\omega)H_{n+k-j-i-1,j+i-n+1}}{1+(\omega-D)G_{k-j-1,j}} & (i=n-1-j,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)G_{k-j,j-1}}{1+(\omega-D)G_{k-j-1,j}} & (i=n-1)\ . \end{cases}$$

Now, if $n\geq 5$, $2\leq k\leq n-3$, we have

$$a_i^{(n^2+kn+n-1)} = \begin{cases} \dfrac{V^{n-i}d(D-\omega)F_{n-i,n-1}}{1+(\omega-D)G_{0,k-1}} & (i=1,\ldots,n-k-1)\ , \\ \dfrac{V^{n-i}(D-\omega)H_{n-i,k+i-n}}{1+(\omega-D)G_{0,k-1}} & (i=n-k,\ldots,n-2)\ , \\ \dfrac{V(1+(\omega-D)G_{1,k-2})}{1+(\omega-D)G_{0,k-1}} & (i=n-1)\ , \end{cases}$$

but if $n\geq 4$, $k=1$, we have

$$a_i^{(n^2+2n-1)} = \begin{cases} \dfrac{V^{n-i}d(D-\omega)F_{n-i,n-1}}{1+(\omega-D)G_{0,0}} & (i=1,\ldots,n-2)\ , \\ \dfrac{V(1+d(\omega-D)E_{1,n-1})}{1+(\omega-D)G_{0,0}} & (i=n-1)\ , \end{cases}$$

Finally, for $n\geq 3$ , we end up with

$$a_i^{(2n^2-2n)} = \begin{cases} \dfrac{dF_{n-2,n-1}}{V} & (i=1), \\ \dfrac{H_{n-i-1,i-2}}{V^i} & (i=2,\ldots,n-1)\ , \end{cases}$$

and

$$a_i^{(2n^2-2n+1)} = \begin{cases} \dfrac{1+(\omega-D)G_{n-2-i,i-1}}{V^i(1+\omega-D)} & (i=1,\ldots,n-2)\ , \\ \dfrac{V}{1+\omega-D} & (i=n-1)\ , \end{cases}$$

whereupon, as is easily seen,

$$(4.1) \qquad a^{(2n^2-2n+1)} = a^{(n^2-2n+1)}\ .$$

When $d\neq 1$ , we conclude from the last section that

$$\frac{d_k f_{k-1,k+i-1}(D,D)-1}{V^i} \quad (k=1,\ldots,n-2;\ i=1,\ldots,n-k-1)$$

is an integer. Now, for $1\leq k\leq n-1$ ,

$$d_k f_{k-1,n-1} = \sum_{r=0}^{k}(-1)^r\binom{k}{r} f_{k-1,n-1-r} + f_{k-1,n-1}d^{-1} - f_{k-1,n-1};$$

consequently,

$$\frac{d_k f_{k-1,n-1}(D,D)-1}{V^{n-k}} \quad (k=1,\ldots,n-1)$$

is an integer if and only if

$$\frac{d^{-1}f_{k-1,n-1}(D,D) - f_{k-1,n-1}(D,D)}{V^{n-k}} = \binom{n}{k} D^{n-k}d^{-1}\binom{\frac{1-d}{V^{n-k}}}{}$$

is an integer; from now on, one takes care of the case $n = p^r$ with the help of L. Bernstein's remark [2,p.200].

Similarly, $(F_{n-1-i,k-1}(D,D)-1)/V^i$ is equal to

$$\left(\frac{1-d}{V^i}\right)\left(\sum_{j=0}^{k-n+i}(-1)^j\binom{n-1-i+j}{j}\binom{n}{i-j}D^{i-j}d^{-1}\right)+\frac{B-1}{V^i}$$

where

$$B = \sum_{j=0}^{k-n+i}(-1)^j\binom{n-1-i+j}{j}\binom{n}{i-j}D^{i-j}$$

$$+ (-1)^{k-n+1+i}\sum_{j=0}^{n-1-i}(-1)^j\binom{k}{n-1-i-j}\binom{n-1-j}{k}D^{n-1-j-k}.$$

We have shown that $(B-1)/V^i$ is an integer, so that

$$\frac{F_{n-1-i,k-1}(D,D)-1}{V^i} \quad (k=2,\ldots,n-1;\ i=n-k+1,\ldots,n-1)$$

is an integer. Obviously (2.29) is an integer.

Finally, we have the same conclusion for

$$\frac{H_{n-1-i,i+k-n}(D,D)-1}{V^i} \quad (k=1,\ldots,n-2;\ i=n-k,\ldots,n-1)$$

which is equal to

$$\frac{F_{n-1-i,n-1}(D,D)-1}{V^i}+\left(\frac{d-1}{V^i}\right)\sum_{j=i+k-n+1}^{i}(-1)^j\binom{n-1-i+j}{j}\binom{n}{i-j}D^{i-j}d^{-1}.$$

The Hasse-Bernstein formula supplies the unit

$$\varepsilon = \prod_{j=n^2-2n+1}^{2n^2-2n+1} a_{n-1}^{(j)} = \left(\frac{F_{0,n-2}}{1+(\omega-D)E_{0,n-3}}\right)\left(\frac{dF_{0,n-1}}{1+(\omega-D)E_{0,n-2}}\right)$$

$$\left(\frac{H_{0,0}}{1+d(\omega-D)E_{0,n-1}}\right)\left(\prod_{j=1}^{n-3}\frac{H_{0,j}}{1+(\omega-D)G_{0,j-1}}\right)$$

$$= \left(f_{0,n-1}d^{-1}\right)\left(f_{0,n-1}\right)\left(f_{0,n-1}d^{-1}\right)\prod_{j=1}^{n-3}f_{0,n-1}d^{-1}$$

$$= \left(f_{0,n-1}d^{-1}\right)^{n_d},$$

$$\varepsilon = \frac{d}{(V\theta - D)^n}, \tag{4.2}$$

a unit already obtained by L. Bernstein and H. Hasse [1] .

The b's written down in theorem 1.1 are now easily obtained.

## 5. REMARKS AND EXAMPLES

We want to give another form for $V^i a_i^{(n^2)} = dF_{n-1-i,n-1} = dF_{n-1-i,n-2}$ $(i=1,\ldots,n-1)$:

$$\begin{aligned} dF_{n-1-i,n-1} &= \sum_{r=0}^{i} \sum_{j=0}^{i-r} (-1)^r \binom{n-1-i+r}{r} \binom{n-1-i+r+j}{j} D^j \omega^{i-r-j} \\ &= \sum_{r=0}^{i} \sum_{j=0}^{i-r} (-1)^r \binom{n-1-i+r+j}{n-1-i} \binom{r+j}{j} D^j \omega^{i-r-j} \\ &= \sum_{r=0}^{i} \sum_{j=r}^{i} (-1)^r \binom{n-1-i+j}{j} \binom{j}{r} D^{j-r} \omega^{i-j} \\ &= \sum_{j=0}^{i} \sum_{r=0}^{j} (-1)^r \binom{n-1-i+j}{j} \binom{j}{r} D^{j-r} \omega^{i-j} \\ &= \sum_{j=0}^{i} \binom{n-1-i+j}{j} \omega^{i-j} (D-1)^j . \end{aligned}$$

Under the conditions of theorem 1.1, we have therefore proved that the JPA of $a^{(0)}$, with

$$a_i^{(0)} = \sum_{j=0}^{i} \binom{n-1-i+j}{j} \theta^{i-j} \left(\frac{D-1}{V}\right)^j \quad (i=1,\ldots,n-1),$$

is purely periodic of length $n$ if $d = 1$ or of length $n^2$ if $d \neq 1$. When $V = 1$, this result was obtained by L. Bernstein [3] .

Suppose now

$$\theta^n = \frac{(tV^n+1)^n - 1}{V^n}, \quad \text{with } t, V \in \mathbb{N}^* . \tag{5.1}$$

Here $D = tV^n+1$. For $1 \leq s \leq k \leq m+1 \leq n$ , we have by (1.17)

$$d_s f_{k-1,m}(D,D) = \sum_{j=0}^{s} \binom{s}{j} \binom{m-s+1}{k-j} D^{m-s-k+1+j}(D-1)^{s-j}$$

$$= \sum_{j=0}^{s} \sum_{r=0}^{m-s-k+1+j} \binom{s}{j}\binom{m-s+1}{k-j} \binom{m-s-k+1+j}{r} (tV^n)^{s-j+r}$$

$$= \sum_{j=0}^{s} \sum_{r=s-j}^{m-k+1} \binom{s}{j} \binom{m-s+1}{k-j}\binom{m-s-k+1+j}{m-k+1-r} (tV^n)^r$$

$$= \sum_{j=k-m+s-1}^{s} \sum_{r=s-j}^{m-k+1} \binom{s}{j} \binom{m-s+1}{k+r-s} \binom{k+r-s}{k-j} (tV^n)^r$$

$$= \sum_{r=0}^{m-k+1} \binom{m+1-s}{k+r-s} \left( \sum_{j=s-r}^{s} \binom{s}{j} \binom{k+r-s}{k-j} \right) (tV^n)^r .$$

Let $s_1 = s-r$ and $k_1 = k+r-s$. Then, with the help of (1.32) or (1.33) according to whether $s \geq r$ or not, we deduce

$$\sum_{j=s-r}^{r} \binom{k+r-s}{k-j}\binom{s}{j} = \sum_{j=s_1}^{s} \binom{k_1}{j-s_1}\binom{s}{j} = \binom{k_1+s}{k_1+s_1} = \binom{k+r}{k}.$$

Therefore

$$(5.2) \qquad d_s f_{k-1,m}(D,D) = \sum_{r=0}^{m-k+1} \binom{m+1-s}{k+r-s} \binom{k+r}{k} (tV^n)^r .$$

For $D = tV^n+1$, we have by (3.3) and (1.29)

$$F_{n-1-i,n-2}(D,D) = \sum_{r=0}^{i} \sum_{j=r}^{i} \binom{n-1-i+r}{r}\binom{i-r}{j-r} (tV^n)^j$$

$$= \sum_{j=0}^{i} \left( \sum_{r=0}^{i} \binom{n-1-i+r}{r}\binom{i-r}{j-r} \right) (tV^n)^j ,$$

$$(5.3) \qquad F_{n-1-i,n-2}(D,D) = \sum_{j=0}^{i} \binom{n}{j} (tV^n)^j .$$

Now $F_{n-1-i,k-1}(D,D) = T_1+T_2$ with

$$T_1 = \sum_{j=0}^{k-n+i-1} (-1)^j \binom{n-1-i+j}{j}\binom{n}{i-j} D^{i-j} ,$$

$$T_2 = (-1)^{k+i-n} \sum_{j=1}^{n-i} \binom{k-1-j}{k-1-n+i} d_j f_{k-1,n-1} .$$

Using (5.2), one comes up with

$$(5.4)\quad T_2 = (-1)^{k+i-n} \sum_{r=0}^{n-k} \binom{k+r}{r} B(n,i,k,r)\,(tV^n)^r\ ,$$

where $B(n,i,k,r)$ is equal to

$$(5.5)\quad \sum_{j=1}^{n-i} \binom{k-1-j}{k-1-n+i}\binom{n-j}{k+r-j} = \sum_{j=0}^{n-1-i} \binom{k-1+i-n+j}{j}\binom{i+j}{n-k-r}\ .$$

By (1.28),

$$T_1 = \sum_{j=0}^{k-n+i-1} \sum_{r=j}^{i} (-1)^j \binom{n-1-i+r}{r}\binom{r}{j} D^{i-j} = T_3 - T_4$$

with

$$T_3 = \sum_{j=0}^{i} \sum_{r=j}^{i} (-1)^j \binom{n-1-i+r}{r}\binom{r}{j} D^{i-j}\ ,$$

$$T_4 = \sum_{j=k-n+i}^{i} \sum_{r=j}^{i} (-1)^j \binom{n-1-i+r}{r}\binom{r}{j} D^{i-j}\ .$$

From (5.3), we have

$$(5.6)\qquad T_3 = F_{n-1-i,n-2}(D,D) = \sum_{r=0}^{i} \binom{n}{r} (tV^n)^r.$$

Finally,

$$T_4 = \sum_{r=k-n+i}^{i} \binom{n-1-i+r}{r} D^{i-r} \sum_{j=k-n+i}^{r} (-1)^j \binom{r}{j} D^{r-j}$$

so that by (1.34), $(-1)^{k+i-n} T_4$ is equal to

$$\sum_{r=k-n+i}^{i} \sum_{j=k-n+i}^{r} \binom{n-1-i+r}{r}\binom{r}{j}\binom{j-1}{k-1-n+i} D^{i-r}(D-1)^{r-j}$$

$$= \sum_{j=k-n+i}^{i} \sum_{r=j}^{i} \binom{n-1-i+r}{r}\binom{r}{j}\binom{j-1}{k-1-n+i} D^{i-r}(D-1)^{r-j}$$

$$= \sum_{j=k-n+i}^{i} \sum_{r=0}^{i-j} \binom{n-1-i+r+j}{r+j}\binom{r+j}{j}\binom{j-1}{k-1-n+i} D^{i-r-j}(D-1)^{r}$$

$$= \sum_{j=k-n+i}^{i} \sum_{r=0}^{i-j} \sum_{s=0}^{i-r-j} C_1 (tV^n)^{r+s}$$

$$= \sum_{j=k-n+i}^{i} \sum_{r=0}^{i-j} \sum_{s=r}^{i-j} C_2 (tV^n)^{s}$$

$$= \sum_{s=0}^{n-k} \sum_{j=k-n+i}^{i-s} \sum_{r=0}^{s} C_2 (tV^n)^s$$

with

$$C_1 = \binom{n-1-i+r+j}{r+j} \binom{r+j}{j} \binom{j-1}{k-1-n+i} \binom{i-r-j}{s},$$

$$C_2 = \binom{n-1-i+r+j}{r+j} \binom{r+j}{j} \binom{j-1}{k-1-n+i} \binom{i-r-j}{s-r}$$

$$= \binom{n-1-i+r+j}{r} \binom{n-1-i+j}{j} \binom{j-1}{k-1-n+i} \binom{i-r-j}{s-r}.$$

Using (1.29) and changing $s$ for $r$, we end up with

$$(5.7) \qquad T_4 = (-1)^{k-n+i} \sum_{r=0}^{n-k} \binom{n}{r} A(n,i,k,r) (tV^n)^r$$

where $A(n,i,k,r)$ is equal to

$$(5.8) \sum_{j=i+k-n}^{i-r} \binom{n-1-i+j}{j} \binom{j-1}{k-1-n+i} = \sum_{j=0}^{n-k-r} \binom{k-1+i-n+j}{j} \binom{k-1+j}{n-1-i}.$$

Now for $r=0$, we deduce from (1.38) the equality

$$\binom{n}{r} A(n,i,k,r) = \binom{k+r}{r} B(n,i,k,r) ;$$

it is direct to verify the last equality also holds for $r=n-k$.

Collecting together all those facts, we have proved the following result.

THEOREM 5.1. *Let* $n \geq 3$ *and*

$$\theta^n = \frac{(tV^n+1)^n-1}{V^n} \text{ , with } t,V \in \mathbb{N}^* .$$

*For* $V=1$, *let* $t \geq n-1$. *Then the JPA of* $(\theta,\ldots,\theta^{n-1})$ *is periodic. If* $\ell$ *and* $m$ *denote the lengths of the primitive preperiod and primitive period respectively, then*

$$\ell = (n-1)^2 \text{ and } m=n .$$

*The components of the vectors of the preperiod and the period are the following:*

$$b_i^{(0)} = \sum_{r=1}^{i} \binom{i}{r} t^r V^{nr-i} \quad (i=1,\dots,n-1);$$

$$b_i^{(n)} = \sum_{r=1}^{i} \binom{i}{r}\binom{r+1}{1} t^r V^{nr-i} \quad (i=1,\dots,n-1);$$

<u>for</u> $2 \le k \le n-1$ ,

$$b_i^{(kn)} = \begin{cases} \displaystyle\sum_{r=1}^{i} \binom{i}{r}\binom{r+k}{k} t^r V^{nr-i} \quad (i=1,\dots,n-k), \\ \displaystyle\sum_{r=1}^{i} \binom{n}{r} t^r V^{nr-i} \\ \quad + (-1)^{k-n+i} \sum_{r=1}^{n-k} \left( \binom{k+r}{k} B - \binom{n}{r} A \right) t^r V^{nr-i} \\ \qquad\qquad (i=n-k+1,\dots,n-1), \end{cases}$$

<u>with</u>

$$B = B(n,i,k,r) = \sum_{j=0}^{n-1-i} \binom{k-1+i-n+j}{j} \binom{i+j}{n-k-r} ,$$

$$A = A(n,i,k,r) = \sum_{j=0}^{n-k-r} \binom{k-1+i-n+j}{j} \binom{k-1+j}{n-1-i} ;$$

<u>finally</u>, <u>for</u> $n \ge 3$, $s \in \mathbb{N}^*$ <u>and</u> $n \nmid s$,

$$b_i^{(s)} = \begin{cases} 0 \quad (i=1,\dots,n-2), \\ V \quad (i=n-1), \end{cases}$$

It is also easy to deduce the following result.

<u>THEOREM 5.2.</u> <u>Let</u> $n \ge 3$ <u>and</u>

$$\theta^n = \frac{(n^{n-1} t\, V^n + 1)^n - 1}{(nV)^n} \text{ , } \underline{\text{with}} \quad t,\ V \in \mathbb{N}^* .$$

<u>Then the JPA of</u> $(\theta,\dots,\theta^{n-1})$ <u>is periodic. If</u> $\ell$ <u>and</u> $m$ <u>denote the lengths of the primitive preperiod and primitive period respectively, then</u>

$$\ell = (n-1)^2 \quad \underline{\text{and}} \quad m=n .$$

<u>The components of the vectors of the preperiod and the period are the following:</u>

$$b_i^{(0)} = \sum_{r=1}^{i} \binom{i}{r} n^{(n-1)r-i}\, t^r\, V^{nr-i} \quad (i=1,\dots,n-1);$$

$$b_i^{(n)} = \sum_{r=1}^{i} \binom{i}{r}\binom{r+1}{1} n^{(n-1)r-i} t^r\, V^{nr-i} \quad (i=1,\dots,n-1);$$

<u>for</u> $2 \le k \le n-1$,

$$b_i^{(kn)} = \begin{cases} \displaystyle\sum_{r=1}^{i} \binom{i}{r}\binom{r+k}{k} n^{(n-1)r-i}\, t^r\, V^{nr-i} & (i=1,\dots,n-k), \\ \displaystyle\sum_{r=1}^{i} \binom{n}{r} n^{(n-1)r-i}\, t^r\, V^{nr-i} & \\ \quad + (-1)^{k-n+i} \displaystyle\sum_{r=1}^{n-k} \left( \binom{k+r}{k} B - \binom{n}{r} A \right) n^{(n-1)r-i}\, t^r V^{nr-i} & \\ \qquad\qquad (i=n-k+1,\dots,n-1), & \end{cases}$$

<u>where</u>

$$B = B(n,i,k,r) = \sum_{j=0}^{n-i-1} \binom{k-1+i-n+j}{j} \binom{i+j}{n-k-r},$$

$$A = A(n,i,k,r) = \sum_{j=0}^{n-k-r} \binom{k-1+i-n+j}{j} \binom{k-1+j}{n-1-i};$$

<u>finally</u>, <u>for</u> $n \ge 3$, $s \in \mathbb{N}^*$ <u>and</u> $n \nmid s$,

$$b_i^{(s)} = \begin{cases} 0 & (i=1,\dots,n-2), \\ nV & (i=n-1). \end{cases}$$

It is left to the reader to state a theorem similar to the last one, when

$$\theta^n = \frac{(r^{n-1} t\, V^n+1)^n-1}{(rV)^n}, \quad \text{with } r,t,V \in \mathbb{N}^* \text{ and } r \mid n.$$

Finally, note that the bound $D \ge (n-1)d(V+1)/2+1$, which was deduced from (1.25), is probably almost as best as possible. This at least the case when $n=3$ and $V=1$. In fact, proceeding as in [5], we have showed that the JPA of $(\omega,\omega^2)$, where $\omega^3 = 8V^3-V$, is periodic with $\ell=4$ and $m=9$; here $D \ge 2d$. However, when $\omega^3 = 125\, V^3-3V$, we have not obtained periodicity after having achieved thirteen steps of the JPA of $(\omega,\omega^2)$; therefore, when $n=3$ and $V=1$,

the bound $D \geq 5d/3$ cannot be used.

To conclude this work, we want to point out that theorem 1.1 can be stated in a more general form; instead of assuming $D,d \equiv 1 \pmod{V^{n-1}}$ and $D \geq (n-1)d(V+1)/2 + 1$, one just has to suppose that (2.26) to (2.30) are integers.

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