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THE FLUID FLOW THROUGH POROUS MEDIA. REGULARITY OF THE FREE SURFACE

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The fluid flow through an earth dam separating two water reservoirs of different levels gives rise to a free boundary problem. In [1] we have proved the existence of a solution to this problem. In this paper we show that the free boundary is regular.

1. Introduction

In this paper we prove regularity theorems for a free boundary problem related to stationary flow through a homogeneous porous medium.

Let us divide the n-dimensional space $\mathbb{R}^n$ into four parts. The first one is a bounded domain Ω with Lipschitz boundary and denotes the porous medium. Let A be the region of Ω occupied by water. Let K be a closed set, which denotes the impermeable part of the space. The third part consists of water reservoirs $\overline{S_1^o} \setminus K, \dots, \overline{S_r^o} \setminus K$ with levels s_i^o, where S_i^o are open and relatively closed subsets of

$$\{(y,h) \in \mathbb{R}^n \;/\; h < s_i^o \text{ and } (y,h) \notin \overline{\Omega} \cup K\}.$$

We assume that $\overline{S_i^o}$ are disjoint sets. The remainder of $\mathbb{R}^n$, denoted by L, is occupied by air. The pressure of the fluid is given by the function

$$u^o(y,h) = \begin{cases} s_i^o - h & , \text{ if } (y,h) \in \overline{S_i^o} \\ 0 & , \text{ if } (y,h) \in L \end{cases} .$$

We assume

(A1) u^o can be extended to a Lipschitz continuous function on $\mathbb{R}^n$.

(A2) $S := \partial\Omega \setminus K$ is non-empty.

The classes of admissible functions and test functions are given by

$$M = \{v \in H^{1,2}(\Omega) \;/\; v = u^o \text{ on } S\},$$
$$\overset{\circ}{M} = \{v \in H^{1,2}(\Omega) \;/\; v = 0 \text{ on } S\}.$$

The stationary fluid flow through Ω can be described as follows. Find a function u and a set A which satisfy

(P1) $u \in M$ and A is a measurable subset of Ω,

(P2) $u = 0$ almost everywhere in $\Omega \setminus A$,

(P3) for all functions $\zeta \in \overset{\circ}{M}$ we have

$$\int_\Omega \big(\nabla u + I(A)e\big)\nabla\zeta = 0,$$

where $I(A)$ denotes the characteristic function of A and e the vertical unit vector.

The problem (P1) - (P3) was solved in [1] by constructing a minimal supersolution. A pair (u',A') is a supersolution, if

(S1) $u' \in M$ and A' is relative closed in Ω,

(S2) $u' = 0$ almost everywhere in $\Omega \setminus A$,

(S3) for all non-negative functions $\zeta \in \overset{\circ}{M}$ the inequality

$$\int_\Omega \big(\nabla u' + I(A')e\big)\nabla\zeta \geqq 0$$

holds.

For the existence proof in [1] we have used the following assumptions:

(A3) Suppose that Ω^* is an open set of $\mathbb{R}^{n-1}$ and that there are functions $g_-, g_+ : \Omega^* \longrightarrow \mathbb{R}$ such that $g_- < g_+$ and

$$\Omega = \{(y,h) \in \mathbb{R}^n \;/\; y \in \Omega^* \text{ and } g_-(y) < h < g_+(y)\},$$

(A4) The set $\overline{S} \setminus S$ and all the sets $S \cap \Omega_s$, where $s \in \mathbb{R}$ and $\Omega_s = \Omega \cap \overline{\{h = s\}}$, have a finite $(n-2)$-dimensional lower Minkowski content.

Now let (u,A) be a solution of the problem (see [1] Theorem 4.2), that is

(AM) (u,A) is a solution of (S1) - (S3) and (P1) - (P3), and for all supersolutions (u',A') we have

$$J(u,A) \leqq J(u',A') ,$$

where J is the natural functional to the integral in (P3) and (S3) given by

$$J(u,A) = \int_\Omega |\nabla u + I(A)e|^2 .$$

For the proofs in this paper we need further conditions on the boundary of Ω. Let us define

$$S_+ = \{x \in S \;/\; u^o > 0 \text{ in } B(x,\varepsilon) \cap S \text{ for some } \varepsilon > 0\}$$
$$S_o = \{x \in S \;/\; u^o = 0 \text{ in } B(x,\varepsilon) \cap S \text{ for some } \varepsilon > 0\} .$$

We assume

(A5) The set $\{y \in \Omega^* \;/\; (y,g_+(y)) \in S_o\}$ is open and g_+ is a $C^{1,1}$ function on this set.

(A6) If $(y_o,h_o) \in \Omega$ and $(y_o,g_+(y_o)) \in S \setminus S_o$ then there is a ball B^* in Ω^* such that $y_o \in \partial B^*$ and

$$\{(y,g_+(y)) \;/\; y \in B^*\} \subset S_+ .$$

<u>Remark.</u> The pair (u,A) satisfies (AM). By [1] Theorem 4.2 we may assume that

$$A = \Omega \cap \overline{\{u > 0\}} \quad \text{and} \quad \overset{\circ}{A} = \{u > 0\} .$$

Furthermore by [1] Theorem 2.3(4) and [1] Remark 2.4 there is a function $g_A : \Omega^* \longrightarrow \mathbb{R}$ such that $g_- \leqq g_A \leqq g_+$ and

$$A = \{(y,h) \in \Omega \;/\; g_-(y) < h \leqq g_A(y)\} .$$

By [1] Theorem 2.3(6) we have $u(y,h') > 0$ for $(y,h') \in \Omega$ with $g_-(y) < h' \leqq h$ provided that $u(y,h) > 0$.

2. The behavior of the solution near S_o

From the physical point of view it is evident that for the solution we have

$$\vec{v}\cdot\nu \geqq 0 \quad \text{on} \quad \partial A \cap S_o ,$$

which means that on the boundary between A and the air the fluid flows out of the porous medium. In Theorem 2.8 we prove this, in a weak sense, on that part of S_o which is contained in the graph of g_+ .

2.1 NOTATION. We suppose that $x_o = (y_o,h_o) \in \Omega$ and $x_1 = (y_o,g_+(y_o)) \in S_o$. Let Z^* be a ball with centre y_o and define

$$Z = \{(y,h) \in \mathbb{R}^n \,/\, y \in Z^* \text{ and } h_o < h < h_1\} ,$$

where h_1 is a constant greater than $\sup g_+$. If Z^* is small enough we have $\overline{Z} \cap \partial\Omega \subset S_o$ and $\{(y,h_o) \,/\, y \in \overline{Z^*}\} \subset \Omega$. Let $u = 0$ on $\overline{Z} \setminus \Omega$ and

$$u^{\downarrow}(y,h) = \int_h^{h_1} u(y,s)\,ds \qquad \text{for} \quad (y,h) \in \overline{Z} .$$

2.2 LEMMA. <u>There is a bounded measurable function</u> $\theta : Z^* \longrightarrow \mathbb{R}$ <u>such that</u> $\theta(y) = 0$ <u>whenever</u> $g_A(y) < g_+(y)$ <u>and such that for</u> $\zeta \in C_o^\infty(Z)$ <u>we have</u>

$$\int_Z \big(\nabla u + I(A)e\big)\nabla\zeta = \int_{Z^*} \theta(y)\,\zeta(y,g_+(y))\,dy .$$

PROOF. Let $u_\Omega \in M$ be the solution of the variational equality

$$\int_\Omega \big(\nabla u_\Omega + e\big)\nabla\eta = 0 \qquad \text{for} \quad \eta \in \mathring{M} .$$

We have $u_\Omega \in C^1(\overline{Z \cap \Omega})$ by (A5) and $0 \leqq u \leqq u_\Omega$ by [1] Lemma 1.5(1) and [1] Theorem 1.7. Using [1] Lemma 3.8 we conclude for non-negative functions $\zeta \in C_o^\infty(Z)$

$$\int_Z (\nabla u + I(A)e)\nabla\zeta \leqq \int_Z I(A)\,\partial_h\zeta = \int_{Z^*} \zeta(y,g_A(y))\,dy \ .$$

Hence by (P3)

$$\int_Z (\nabla u + I(A)e)\nabla\zeta \leqq \int_{Z^* \cap \{g_A = g_+\}} \zeta(y,g_+(y))\,dy \ .$$

Now we apply [1] Lemma 3.8 to the function $u - u_\Omega$ and get

$$\int_Z (\nabla u + I(A)e)\nabla\zeta \geqq \int_Z \nabla u_\Omega \nabla\zeta$$

$$= \int_{Z^*} \nabla u_\Omega(y,g_+(y))\cdot(-\nabla g_+(y) + e)\,\zeta(y,g_+(y))\,dy \ .$$

The assertion follows from the Radon - Nikodym theorem.

A consequence of this lemma is the following statement (see [1] Theorem 3.2).

2.3 LEMMA. *The function* $u^{\downarrow}$ *defined in* 2.1 *belongs to* $H^{2,p}(Z)$ *for* $1 \leqq p < \infty$ *and* $\Delta u^{\downarrow} = (1-\theta)I(A)$.

Our aim is to show that θ is non - positive. To do this, we solve a variational problem in Z .

2.4 DEFINITION. *We define*

$$M^{\downarrow} = \{v \in H^{1,2}(Z) \ / \ 0 \leqq v \leqq u^{\downarrow} \text{ and } v = u^{\downarrow} \text{ on } \partial Z^* \times \mathbb{R}\}$$

and let $w \in M^{\downarrow}$ *be the solution of the variational inequality*

$$\int_Z (\nabla w \nabla(v-w) + v - w) - \int_{Z^*} (u(v-w))(y,h_o)\,dy \geqq 0$$

for $v \in M^{\downarrow}$. *Then we define* $u' = -\partial_h w$.

2.5 THEOREM. *The functions* w *and* $u^{\downarrow}$ *have the properties*

(1) $w \in H^{2,p}(Z)$ *for* $1 \leqq p < \infty$.

(2) $u' = u$ *on* ∂Z .

(3) $\Delta w = I(\{w > 0\}) - \min(\theta,0)\,I(\{w = u^{\downarrow} > 0\})$.

(4) <u>If</u> D <u>is an open subset of</u> $Z \cap \Omega$ <u>and</u> $w = u^{\downarrow}$ <u>in</u> D <u>then</u> $\theta \leqq 0$ <u>almost everywhere in</u> D.

(5) <u>The Lebesgue measure of</u> $Z \cap \partial\{w > 0\}$ <u>is zero</u>.

<u>PROOF</u>. Assertion (1) can be reduced to [2] Corollaire I.1 with the reflection method used in [1] Theorem 3.4(1). The proof of (2) is similar to [1] Theorem 3.5(2). Since w and $u^{\downarrow}$ belong to $H^{2,1}(Z)$ we have $\Delta w = 0$ a.e. in $\{w = 0\}$ and $\Delta w = \Delta u^{\downarrow}$ a.e. in $\{w = u^{\downarrow}\}$. By definition of $M^{\downarrow}$ we have

$$\{w > 0\} \subset \{u^{\downarrow} > 0\} = \{u > 0\} \subset A$$

which implies $\Delta w = \Delta u^{\downarrow} = 1 - \theta$ a.e. in $\{w = u^{\downarrow} > 0\}$ by 2.3. The variational inequality for w implies $\Delta w \geqq 1$ a.e. in $\{w > 0\}$ and $\Delta w = 1$ in $\{u^{\downarrow} > w > 0\}$. We conclude $\theta \leqq 0$ a.e. in $\{w = u^{\downarrow} > 0\}$ and (3) is proved. Now let D be an open set as in (4). We have $\theta \leqq 0$ a.e. in $D \cap \{w > 0\}$. Furthermore $D \setminus \overline{\{w > 0\}}$ is contained in $\Omega \setminus A$, hence $\theta = 0$ in $D \setminus \overline{\{w > 0\}}$ by definition of θ in 2.2. Using (5) we conclude $\theta \leqq 0$ a.e. in D. Assertion (5) is proved in 6.1.

We translate the properties of the function w to u'.

<u>2.6 THEOREM</u>.

(1) $0 \leqq u' \leqq u$ <u>in</u> Z.

(2) <u>There is a function</u> $g : Z^{*} \longrightarrow \mathbb{R}$ <u>such that</u>

$$\{w = u^{\downarrow}\} \cap Z = \{(y,h) \in Z \ / \ h \geqq g(y)\} .$$

(3) <u>For</u> $\zeta \in C_0^{\infty}(Z \cap \Omega)$ <u>we have</u>

$$\int_Z \left(\nabla u' + I(\overline{\{w > 0\}})e\right)\nabla\zeta = \int_{Z^*} \max(-\theta(y),0)\,\zeta(y,g(y))\,dy .$$

<u>PROOF</u>. 2.5(3) and 2.5(5) yields

$$\int_Z \left(\nabla u' + I(\overline{\{w > 0\}})e\right)\nabla\zeta = \int_{Z \cap \{w = u^{\downarrow} > 0\}} \min(\theta,0)\,\partial_h\zeta$$

for $\zeta \in \overset{\circ}{H}{}^{1,2}(Z)$. We take $\zeta = \max(u' - u,0)$ as test function and obtain using (P3)

$$\int_Z |\nabla\zeta|^2 = \int_Z \big(\nabla u' + I(A)e\big)\nabla\zeta$$

$$= \int_{Z\cap\{w=u^{\downarrow}>0\}} \min(\theta,0)\,\partial_h\zeta + \int_Z I(A\setminus\overline{\{w>0\}})\,\partial_h\zeta\,.$$

The first term is zero since $u' = u$ a.e. in $\{w = u^{\downarrow}\}$, and the second term is zero since $u' - u = -u \leqq 0$ in $Z\setminus\overline{\{w>0\}}$. That proves $u' \leqq u$. We conclude $u^{\downarrow} \geqq w$ and $\partial_h u^{\downarrow} \leqq \partial_h w$, hence (2) and the second inequality in (1) is proved. Now take $\zeta = \min(u',0)$ as test function and calculate

$$\int_Z |\nabla\zeta|^2 = \int_Z \nabla u'\nabla\zeta = \int_{Z\cap\{w=u^{\downarrow}>0\}} \min(\theta,0)\,\partial_h\zeta - \int_Z \partial_h\zeta$$

$$= \int_{Z\cap\{w=u^{\downarrow}\}} \min(\theta,0)\,\partial_h\zeta = -\int_{Z^*} \min(\theta(y),0)\,\zeta(y,g(y))\,dy \leqq 0\,.$$

That proves $u' \geqq 0$. To prove (3) we look at the set

$$N = Z\cap\Omega\cap\{w = u^{\downarrow} = 0\}\,.$$

In $N\cap A$ we have $0 = \Delta u^{\downarrow} = 1-\theta$ a.e., and in $N\setminus A$ we have $\theta = 0$ by 2.2, hence $\theta \geqq 0$ a.e. in N. For $\zeta \in C_0^{\infty}(Z\cap\Omega)$ we conclude

$$\int_{Z\cap\{w=u^{\downarrow}>0\}} \min(\theta,0)\,\partial_h\zeta = \int_{Z\cap\{w=u^{\downarrow}\}} \min(\theta,0)\,\partial_h\zeta$$

$$= -\int_{Z^*} \min(\theta(y),0)\,\zeta(y,g(y))\,dy\,.$$

Now we are able to prove the main theorem of this section. This result is used in 3.4.

<u>2.7 THEOREM.</u>

(1) $w(y,h) = \int_h^{g_+(y)} u(y,s)\,ds$ <u>for</u> $(y,h)\in Z$,

(2) $\Delta w \geqq 1$ <u>almost everywhere in</u> $Z\cap\{u>0\}$,

(3) $\theta \leqq 0$ <u>almost everywhere in</u> Z^*.

An equivalent formulation of 2.7 is

2.8 THEOREM. For non-negative functions $\zeta \in C_0^\infty(Z)$ we have $\int_Z (\nabla u + I(A)e)\nabla\zeta \leqq 0$.

PROOF OF 2.7. Define $u' = u$ in $\Omega \setminus Z$ and

$$A' = (A \setminus Z) \cup (\Omega \cap \overline{Z \cap \{w > 0\}}) .$$

Using 2.6 we see that (u',A') is a supersolution in the sense of (S1) - (S3) (the proof is similar to [1] Theorem 3.7). Since the pair (u,A) minimizes the functional J in the class of supersolutions (see assumption (AM) in 1.) we conclude

$$0 \geqq J(u,A) - J(u',A') = \int_\Omega |\nabla(u - u') + I(A \setminus A')e|^2 + 2R ,$$

where

$$R = \int_\Omega (\nabla u' + I(A')e)\big(\nabla(u - u') + I(A \setminus A')\big) \geqq 0$$

by (S3), since $u - u' \in M$ is non-negative. We have proved that $\nabla u' = \nabla u$ a.e. in $Z \cap \{w > 0\}$. This implies $u' = u$ and $w = u^+$ in Z. Then the theorem follows by 2.5(4) and 2.5(3).

3. Continuity properties of the free surface

Let $x_0 = (y_0,h_0)$ be an element of $\Omega \cap \partial A$ and $x_1 = (y_0,g_+(y_0))$. We distinguish the cases $x_1 \in S_0$ and $x_1 \in S \setminus S_0$, and we prove the following two theorems.

3.1 THEOREM. If $x_0 \in \Omega \cap \partial A$ then x_1 cannot be an element of $S \setminus S_0$.

3.2 THEOREM. If $x_0 \in \Omega \cap \partial A$ and $x_1 \in S_0$ then the solution u is zero in a neighborhood of x_1, that is, $x_1 \notin \overline{A}$.

To prove the second theorem we start with some notation and some lemmas.

3.3 NOTATION. We consider the case $x_1 \in S_o$. Let $\rho > 0$ and $h_o < \mu < g_+(y_o) -$, and define

$$Z = \{(y,h) \in \mathbb{R}^n \ / \ |y - y_o| < \rho \ \text{and} \ \mu < h < g_+(y)\} .$$

If ρ is small enough then we have $\overline{Z} \cap \partial\Omega \subset S_o$ and $\{(y,h_o) \ / \ |y - y_o| \leqq \rho\} \subset \Omega$. Let w be the function from 2.7.

3.4 LEMMA. *If $x_1 \in S_o$ then there is a constant $C < \infty$ such that $\partial_h u \leqq C u$ almost everywhere in a neighborhood of x_1. Hence this inequality holds in Z if $g_+(y_o) - \mu$ and ρ are small enough.*

PROOF. Let δ be a small positive number and define

$$\partial_h^\delta u(y,h) = \frac{1}{\delta}\big(u(y,h) - u(y,h-\delta)\big) \quad \text{for} \quad (y,h) \in \overline{Z} .$$

Then $\partial_h^\delta u \in C^o(\overline{Z}) \cap H^{1,2}(Z)$ and $\partial_h^\delta u$ is a harmonic function in $Z \cap \{u > 0\}$. Now let v^δ be the solution of the Dirichlet problem

$$\Delta v^\delta = 0 \quad \text{in} \ Z , \quad v^\delta = \max(\partial_h^\delta u, 0) \quad \text{on} \ \partial Z .$$

Since $v^\delta \geqq 0$ and $\partial_h^\delta u \leqq 0$ on $Z \cap \partial\{u > 0\}$ the maximum principle yields $v^\delta \geqq \partial_h^\delta u$ on $Z \cap \{u > 0\}$. Since $g_+ \in C^{1,1}$ by (A5) we have a representation

$$v^\delta(x) = \int_{\partial Z} P(x,\xi) \max(\partial_h^\delta u(\xi), 0) \, dH^{n-1}(\xi)$$

where P is the Poisson kernel with $0 \leqq P(x,\xi) \leqq c(n,g_+,\rho)$ for $\xi \in \Omega \cap \partial Z$ and $\operatorname{dist}(x, \Omega \cap \partial Z) \geqq \frac{\rho}{2}$. Choose ρ and μ such that $u \in H^{1,2}(\partial B(y_o,\rho) \times \mathbb{R})$ and such that

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{B(y_o,\rho) \times [\mu - \delta, \mu]} |\partial_h u| = \int_{B(y_o,\rho)} |\partial_h u(y,\mu)| \, dy$$

exists. Then we get an estimate

$$\partial_h^\delta u \leqq v^\delta \leqq c(n,g_+,\rho) \, \|\partial_h u\|_{L_1(\Omega \cap \partial Z)}$$

in a neighborhood of x_1, where the right side does not depend on δ. If we choose $g_+(y_o) - \mu$ and ρ small enough, we conclude $\partial_h^\delta u \leqq c(n,g_+,\rho,u)$ in $Z \cap \{u > 0\}$.

Now we apply 6.3 to the domain $D = Z \cap \{u > 0\}$ and the sets (choose μ' with $\mu < \mu' < g_+(y_o)$)

$$K = \{(y,h) \in \overline{D} \ / \ |y - y_o| \leqq \frac{\rho}{2}, \ \mu' \leqq h \leqq g_+(y)\},$$
$$L = \{(y,h) \in \partial D \ / \ |y - y_o| = \rho \ \text{or} \ h = \mu\},$$

and the functions $v = \partial_h^\delta u$ and w from 2.7. We have $\Delta w \geqq 1$ in D and $\{w > 0\} \cap \overline{D} = \{u > 0\} \cap \overline{D}$ by 2.7 and $w = 0$ on $\partial D \setminus L$. u and v are harmonic in D, and $u \geqq 0$ in $\overline{D}$, and $v \leqq 0$ on $\partial D \setminus L$, and the supremum of v on L does not depend on δ as shown above. All this implies that there are constants $\varepsilon > 0$ and $C < \infty$ such that

$$\varepsilon \, \partial_h^\delta u - C u + w \leqq 0 \quad \text{in } K,$$

where C does not depend on δ. Letting $\delta \to 0$ we get

$$\partial_h u \leqq \frac{C}{\varepsilon} u \quad \text{a.e. in } K,$$

and the lemma is proved.

3.5 LEMMA. *If* $x_1 \in S_o$ *then there is a constant* $C < \infty$ *such that* $u(y,h) \leqq C w(y,h_o)$ *for* $(y,h) \in Z$.

PROOF. If $h_o < \mu' < \mu$ then 3.4 implies

$$u(y,h) \leqq u(y,\mu') + C \int_{\mu'}^{h} u(y,s)\, ds \leqq u(y,\mu') + C w(y,\mu').$$

Integration over μ' yields $u(y,h) \leqq \left((\mu - h_o)^{-1} + C\right) w(y,h_o)$.

3.6 LEMMA. *If* $x_o \in \Omega \cap \partial A$ *and* $x_1 \in S_o$ *then for* $0 < \alpha < 1$ *there is a constant* $C < \infty$ *such that*

$$u(y,h) \leqq C\, |y - y_o|^{1+\alpha} \quad \textit{for} \quad (y,h) \in Z.$$

PROOF. We have $w \in C^{1,\alpha}(\overline{Z})$ by 2.5(1). Since $w \geqq 0$ and $w(x_o) = 0$ we conclude $\nabla w(x_o) = 0$. Hence for some $C < \infty$

$$w(y,h_o) \leqq C\, |y - y_o|^{1+\alpha} \quad \text{for} \quad |y - y_o| \leqq \rho.$$

Then the lemma follows from 3.5.

3.7 PROOF OF THEOREM 3.2. Suppose $\sigma \in \mathbb{R}$, and suppose $\psi \in C^{1,1}(\mathbb{R}^{n-1})$ with $\inf \psi \geqq \sigma + 1$. Define

$$D = \{(y',h') \,/\, |y'| < 1 \text{ and } \sigma < h' < \psi(y')\},$$

and let v be harmonic in D, continuous in $\overline{D}$, and $v = 0$ on $\partial D \cap \operatorname{graph}(\psi)$. Then we have the well known estimate (see Kellogg's Theorem)

$$|\nabla v(y',\psi(y'))| \leqq c(n,\sup|\nabla^2\psi|) \sup_{\partial D} |v| \quad \text{for} \quad |y'| \leqq \tfrac{1}{2}.$$

We apply this to prove 3.2. Choose a non - negative function $\varphi \in C_0^\infty(B(0,\frac{1}{2}))$ with $\varphi(0) > 0$ and define

$$g_\rho(y) = g_+(y) - \rho\,\varphi(\tfrac{1}{\rho}(y - y_0))$$
$$Z_\rho = \{(y,h) \in \mathbb{R}^n \,/\, |y - y_0| < \rho \text{ and } \mu < h < g_\rho(y)\}.$$

Let v_ρ be the solution of the Dirichlet problem

$$\Delta v_\rho = 0 \quad \text{in} \quad Z_\rho,$$
$$v_\rho = u \text{ on } \partial Z_\rho \setminus \operatorname{graph}(g_\rho), \quad v_\rho = 0 \text{ on } \partial Z_\rho \cap \operatorname{graph}(g_\rho).$$

Using 3.6 we conclude for $x = (y,g_\rho(y))$ with $|y| \leqq \frac{\rho}{2}$

$$|\nabla v_\rho(x)| \leqq \frac{1}{\rho} c(n,g_+,\varphi) \sup_{\partial Z} |u| \leqq C\rho^\alpha$$

if ρ is small enough. Since $v_\rho \geqq 0$ this implies

$$(\nabla v_\rho(x) + e)\cdot\nu_{Z_\rho}(x) = -|\nabla v_\rho(x)| + \left(1 + |\nabla g_\rho(y)|^2\right)^{-1/2}$$
$$\geqq c(n,g_+,\varphi) - C\rho^\alpha > 0$$

if ρ is small enough. Hence

$$\int_Z \left(\nabla v_\rho + I(\overline{Z_\rho})e\right)\nabla\zeta \geqq 0$$

for all non - negative functions $\zeta \in C_0^\infty(Z)$, if we set $v_\rho = 0$ on $Z \setminus Z_\rho$. We apply the theory of supersolutions to the domain Z (see [1] Theorem 2.8) and get

$$\int_Z \left(\nabla \min(v_\rho,u) + I(\overline{Z_\rho} \cap A)e\right)\cdot\nabla\zeta \geqq 0$$

for non - negative $\zeta \in C_0^\infty(Z)$. Using the notations

$$u' = \min(v_\rho,u) \text{ in } Z, \quad u' = u \text{ in } \Omega \setminus Z,$$
$$A' = (A \setminus Z) \cup (\overline{Z_\rho} \cap A \cap \Omega),$$

we can prove that the pair (u',A') is a supersolution (see the proof of [1] Theorem 3.7). As in 2.8 we conclude $u' = u$. This implies $u = 0$ in $Z \setminus Z_\rho$, which proves the theorem.

The prove of Theorem 3.1 is quite different.

3.8 LEMMA. *Suppose* $x_1 \in S \setminus S_o$ *and let* B^* *be as in* (A6). *Then* $u(y,h) > 0$ *for* $(y,h) \in \Omega$ *with* $y \in B^*$.

PROOF. Suppose there is a $y \in B^*$ with $(y,h) \in \Omega \setminus A$. Then u is zero in a neighborhood of $(y, g_+(y))$. This is a contradiction to the properties of B^* in (A6), hence

$$(B^* \times \mathbb{R}) \cap \Omega \subset \mathring{A} = \{u > 0\} .$$

3.9 NOTATION. Suppose $x_1 \in S \setminus S_o$. By 3.8 we can choose B^* in (A6) such that $\{(y,h_o) \,/\, y \in \overline{B^*}\} \subset \Omega$ and $u(x_2) > 0$ for some $x_2 = (y_2,h_o)$ with $y_2 \in B^*$ and $y_2 \neq y_o$. Choose a ball B_2 with centre x_2 such that $x_o \in \Omega \setminus \overline{B_2}$, and choose $\eta \in C_o^\infty(B_2)$ with $0 \leqq \eta \leqq 1$ and $\eta(x_2) > 0$. Let D be a domain in $(B^* \times \mathbb{R}) \cap \Omega$ with smooth boundary such that $\overline{D} \subset \Omega$ and $\{(y,h) \,/\, y \in B^* \text{ and } |h - h_o| \leqq \varepsilon_o\} \subset D$ for some $\varepsilon_o > 0$.

3.10 LEMMA. *There is a ball* B_o *with centre* x_o *such that* $\overline{B_o}$ *and* $\overline{B_2}$ *are disjoint, and a domain* D' *with smooth boundary such that*

$$D \subset D' \subset D \cup B_o , \quad \text{and} \quad x_o \in D' , \quad \text{and} \quad D' \setminus B_o = D \setminus B_o ,$$

so that the solution v' *of the Dirichlet problem*

$$\Delta v' = 0 \quad \text{in} \quad D' , \quad v' = \eta u \quad \text{on} \quad \partial D' ,$$

satisfies the inequality

$$(\nabla v' + e) \cdot \nabla v' \geqq 0 \quad \text{in} \quad D' \cap B_o .$$

PROOF. Let v be the solution of the Dirichlet problem

$$\Delta v = 0 \quad \text{in} \quad D , \quad v = \eta u \quad \text{on} \quad \partial D .$$

Since $v \geqq 0$ and $v(x_2) > 0$ we have by the Hopf maximum principle at the point x_o

$$(\nabla v + e)\cdot\nabla v = - |\nabla v|\, \nabla v\cdot\nu_D > 0 \,.$$

Hence we can choose $\delta_o > 0$ and B_o such that

$$(\nabla v + e)\cdot\nabla v \geqq \delta_o \quad \text{in} \quad \overline{D} \cap \overline{B}_o \,.$$

If we swell out the domain D near x_o to a suitable domain D', we get the result (use the $C^{1,\alpha}$ a priori estimates for the Dirichlet problem).

3.11 PROOF OF THEOREM 3.1. We use the notation in 3.10. Define

$$u' = \max(v',u) \quad \text{in} \quad D', \quad u' = u \quad \text{in} \quad \Omega \setminus D' .$$

Since $v' \leqq u$ on $\partial D'$ we have $u' \in M$. Using (P3) and taking into account that $v' + h$ is harmonic in D' and that $D' \setminus A$ is contained in $D' \cap B_o$ by 3.8 we conclude

$$\int_{D'\cap A} |\nabla(u'-u)|^2 = \int_{D'\cap A} \nabla v'\cdot\nabla(u'-u) - \int_\Omega \nabla u\cdot\nabla(u'-u)$$

$$= \int_{D'\cap A} \nabla v'\cdot\nabla(u'-u) + \int_A e\cdot\nabla(u'-u) = \int_{D'\cap A} (\nabla v'+e)\cdot\nabla(u'-u)$$

$$= \int_{D'} \nabla(v'+h)\cdot\nabla(u'-u) - \int_{D'\setminus A} (\nabla v'+e)\cdot\nabla(u'-u)$$

$$= - \int_{D'\setminus A} (\nabla v'+e)\cdot\nabla v' \leqq 0$$

by 3.10. Hence $\nabla(u'-u) = 0$ in D. Since $u' = u = 0$ on $\partial D \setminus B_o \setminus B_2$ this implies $v' \leqq u$ in D, and we have proved that $u(x_o) \geqq v'(x_o) > 0$, that is, $x_o \notin \partial A$.

4. The case $n = 2$

In the case $n = 2$ Theorem 3.2 can be proved in another way. We use the notation in 3.3 and suppose $x_o \in \Omega \cap \partial A$. Define

$$C_\mu = \{y \,/\, y_o - \rho \leqq y \leqq y_o + \rho \text{ and } u(y,\mu) = 0\} .$$

We have to show that $y_o \in \mathring{C}_\mu$ for some μ with $\mu < g_+(y_o)$. Let I be a bounded connected component of $\mathbb{R} \setminus C_\mu$ and $h_o < \mu < \mu' < g_+(y_o)$. Then we have $I \subset [y_o-\rho, y_o+\rho]$ and by 2.7

$$\Delta w \geqq 1 \quad \text{in} \quad D = Z \cap \{u > 0\} ,$$
$$w = 0 \quad \text{on} \quad \{(y,h) \in \partial D \,/\, h > \mu\} .$$

We apply 6.2 to the sets $K = \bar{D} \cap \{h \geqq \mu'\}$ and $L = \bar{D} \cap \{h = \mu\}$ and get

$$D \subset \{h < \mu'\} , \text{ that is, } y_o \in \mathring{C}_{\mu'}$$
or
$$w(y,\mu) > c(n)(\mu'-\mu)^2 \quad \text{for some } y \in I .$$

But since $w(y_o,\mu) = 0$ we have $w(y,\mu) \leqq C\rho$ for $|y-y_o| \leqq \rho$, where C does not depend on ρ and μ. Hence if δ is a small positive number and $\mu' - \mu > \delta$, and if ρ is small enough we conclude that $y_o \in \mathring{C}_{\mu'}$ if C_μ has a bounded connected component. Since $y_o \in C_h$ for all h with $h_o < h < g_+(y_o)$ we distinguish two cases.

First case: $y_o \in \mathring{C}_h$ for some h with $h_o < h < g_+(y_o)$. Then Theorem 3.2 is proved.

Second case: For all $h_o < \mu < g_+(y_o) - \delta$ we have

$$C_\mu = [y_o, y_1] \quad \text{for some } y_1 \in [y_o, y_o+\rho] ,$$
or
$$C_\mu = [y_1, y_o] \quad \text{for some } y_1 \in [y_o-\rho, y_o] .$$

Since $C_h \subset C_{h'}$ for $h \leqq h'$ (see Remark in 1.) we conclude if ρ is chosen small enough

$$C_\mu = \{y_o\} \quad \text{for all } \mu \text{ with } h_o < \mu < g_+(y_o) - \delta ,$$
or
$$C_\mu = [y_o, y_o+\rho] \quad \text{for all } \mu \text{ with } h_o < \mu < g_+(y_o) - \delta ,$$
or
$$C_\mu = [y_o-\rho, y_o] \quad \text{for all } \mu \text{ with } h_o < \mu < g_+(y_o) - \delta .$$

This implies $\partial_h I(A) = 0$ in

$$Q = \{(y,h) \,/\, |y - y_o| < \rho \text{ and } h_o < h < g_+(y_o) - \delta\}$$

which yields $\Delta u = 0$ in Q by (P3). Since $u \geqq 0$ and $u(y_o,h) = 0$ for $h \geqq h_o$. we conclude $u = 0$ in Q by the maximum principle, and Theorem 3.2 is proved.

5. Regularity of the free surface

In 3.1 and 3.2 we have proved that the free surface has the following property.

5.1 REMARK. *If* $x_o \in \Omega \cap \partial A$ *then the solution* u *is zero in a neighborhood of* $(x_o, g_+(y_o))$.

From this we conclude that the free surface is Lipschitz continuous (see the following theorem). For the proof we need the maximum principle in 6.3.

5.2 THEOREM. *The set* $U = \{y \in \Omega^* \,/\, (y, g_A(y)) \in \Omega \cap \partial A\}$ *is an open set and* g_A *is a Lipschitz continuous function on* U . *Moreover* $\Omega \cap \partial A = \{(y, g_A(y)) \,/\, y \in U\}$.

PROOF. Suppose $y_o \in U$ and $h_o = g_A(y_o)$. Then $(y_o . h_o)$ is a point on the free surface $\Omega \cap \partial A$, and by 5.1 we can choose a cylinder

$$Z = \{(y,h) \in \mathbb{R}^n \,/\, |y - y_o| < \rho \text{ and } h_o < h < h_1\}$$

as in 2.1 such that $\overline{Z} \subset \Omega$ and $\{(y, h_1) \,/\, |y - y_o| \leqq \rho\} \subset \Omega \setminus A$. The function

$$w(y,h) = \int_h^{h_1} u(y,s)\, ds \quad \text{for} \quad (y,h) \in \overline{Z}$$

is of class $H^{2,p}(Z)$ for $1 \leqq p < \infty$ by 2.5(1) and we have

$$\Delta w = I(A) \quad \text{in } Z$$

(see 2.3). Let $\delta > 0$ and $b \in \mathbb{R}^{n-1}$. We apply 6.3 to the domain $D = Z \cap \{w > 0\}$, and the sets

$$K = \{(y,h) \in \overline{D} \,/\, |y - y_o| \leqq \rho - \delta \text{ and } h \geqq h_o + \delta\} ,$$
$$L = \partial D \cap \partial Z ,$$

and the function

$$v = b \cdot \nabla_y w .$$

Since $w \in C^1(Z)$ with $w \geqq 0$, we have $w = 0$ and $v = 0$ on $\partial D \setminus L = Z \cap \partial\{w > 0\}$. Since $\overline{Z} \cap \{w > 0\} = \overline{Z} \cap \{u > 0\} = \overline{Z} \cap \mathring{A}$, we have $\Delta w = 1$ and $\Delta v = 0$ in D. Moreover

$$v \leqq |b| \sup_{\overline{Z}} |\nabla w| .$$

All this implies that there are constants, $\varepsilon > 0$ and $C < \infty$ depending on δ, such that for all b with $|b| < \varepsilon$

$$b \cdot \nabla_y w - C u + w \leqq 0 \quad \text{in } K .$$

Hence $(b + e) \cdot \nabla w \leqq 0$ in K for all b with $|b| < C^{-1}\varepsilon$. We conclude that there is a $\varepsilon(K) > 0$ such that for (y,h) in $K \cap \partial A$

$$\{(y',h') / h_o < h' < h \text{ and } |y'-y| < \varepsilon(K)(h-h')\} \subset Z \cap \mathring{A} .$$

This completes the demonstration.

<u>5.3 THEOREM. Let</u> U <u>be as in</u> 5.2. <u>Then</u> $g_A : U \longrightarrow \mathbb{R}$ <u>is an analytic function. The free surface is analytic.</u>

<u>PROOF.</u> See 5.2, [4] Theorem 2, and [5] Theorem 1.

6. Appendix

<u>6.1 THEOREM. Let</u> $D \subset \mathbb{R}^n$ <u>be open, and suppose that</u> $w \in H^{2,p}(D)$ <u>with</u> $p \geqq 2$ <u>satisfy</u>

$$\Delta w \geqq 1 \quad \text{in } D \cap \{w > 0\} \quad \text{and} \quad w \geqq 0 \quad \text{in } D .$$

<u>Then the Lebesgue measure of</u> $D \cap \partial\{w > 0\}$ <u>is zero.</u>

<u>PROOF.</u> Suppose D has smooth boundary and define for $\delta \geqq 0$

$$K = \{v \in H^{1,2}(D) \;/\; v \geqq 0 \text{ and } v = w \text{ on } \partial D\},$$

$$N_\delta = \{x \in D \;/\; \operatorname{dist}(x, D \cap \{w = 0\}) \leqq \delta\},$$

$$f_\delta = \Delta w \text{ in } D \setminus N_\delta , \quad f_\delta = 1 \text{ in } N_\delta .$$

Let $w_\delta \in K$ be the solution of the variational inequality

$$\int_D \big(\nabla w_\delta \nabla(v - w_\delta) + f_\delta(v - w_\delta)\big) \geqq 0 \quad \text{for all } v \in K .$$

We have $w_\delta \in H^{2,p}(D)$ by [3] Théorème II.1 with $w_o = w$ and $\Delta w_\delta = f_\delta I(\{w_\delta > 0\})$. Choose $0 \leqq \delta_2 \leqq \delta_1$. Then

$$\int_D |\nabla \max(w_{\delta_2}-w_{\delta_1},0)|^2 \leqq \int_D (f_{\delta_1}-f_{\delta_2})\max(w_{\delta_2}-w_{\delta_1},0) \leqq 0 .$$

Hence

(1) $w_{\delta_2} \leqq w_{\delta_1}$ for $0 \leqq \delta_2 \leqq \delta_1$.

We have

$$\int_D |\nabla(w_\delta - w)|^2 \leqq \int_D (f_o - f_\delta)(w_\delta - w) .$$

Hence

(2) $w_\delta \longrightarrow w$ in $H^{1,2}(D)$ for $\delta \to 0$.

Since f_δ is regular in N_δ we conclude by [4] Theorem 2 and (1) that $D \cap \partial\{w_\delta > 0\} = \overset{\bullet}{N}_\delta \cap \partial\{w_\delta > 0\}$ has Lebesgue measure zero. Then (1) and (2) yields for $\delta \to 0$

$$f_o I(\{w > 0\}) = \Delta w \leftarrow \Delta w_\delta = f_\delta \; I(D \cap \overline{\{w_\delta > 0\}})$$

$$\to f_o \; I(\bigcap_{\delta>0} (D \cap \overline{\{w_\delta > 0\}}))$$

in the sense of distributions, which proves the theorem.

6.2 THEOREM. Let $D \subset \mathbb{R}^n$ be an open and bounded set, $L \subset \partial D$ closed, and $K \subset \overline{D}$ closed such that $K \cap L$ is empty. Suppose

$$w \in C^o(\overline{D}) , \quad \Delta w \geqq 1 \text{ in } D , \quad w \leqq 0 \text{ on } \partial D \setminus L .$$

Then there is a constant $c(n) < \infty$ such that

$$w \leqq c(n)\,\mathrm{dist}(K,L)^2 \text{ on } L \text{ implies } w \leqq 0 \text{ in } K .$$

PROOF. Choose $d \in C^2(\mathbb{R}^n)$ such that $d \geqq 0$, $d = 0$ in K , and $d \geqq 1$ on L . Then choose $\delta > 0$ such that $\delta \Delta d \leqq 1$ in D. For suitable d and δ we have $\delta \geqq c(n)\,\mathrm{dist}(K,L)^2$. If $w \leqq \delta$ on L then $\Delta(w - \delta d) \geqq 0$ in D and $w - \delta d \leqq 0$ on ∂D . The maximum principle yields $w \leqq \delta d$ in D , hence $w \leqq 0$ in K .

6.3 THEOREM. Suppose D , K , L as in 6.2 and let u , v , w be continuous functions in $\overline{D}$ with $\{w > 0\} \subset \{u > 0\}$ and

$$\Delta u \leqq 0 \text{ in } D , \quad u \geqq 0 \text{ on } \partial D ,$$

$$\Delta v \geqq 0 \text{ in } D , \quad v \leqq 0 \text{ on } \partial D \setminus L , \quad v \leqq 1 \text{ on } L ,$$

$$\Delta w \geqq 1 \text{ in } D , \quad w \leqq 0 \text{ on } \partial D \setminus L .$$

Then there are constants $\varepsilon = \varepsilon(n,\mathrm{dist}(K,L)) > 0$ and

$C = C(n, \mathrm{dist}(K,L), \sup w\ ,\ \inf\{u(x) / w(x) \geqq \varepsilon\}) < \infty$ <u>such that</u>

$$\varepsilon v - Cu + w \leqq 0 \quad \underline{\text{in}} \quad K.$$

<u>PROOF.</u> For $0 < \varepsilon, C < \infty$ the function $w' = \varepsilon v - Cu + w$ is continuous in $\overline{D}$ and

$$\Delta w' \geqq 1 \quad \text{in } D\,, \quad w' \leqq 0 \quad \text{on } \partial D \setminus L\,.$$

Define $\varepsilon = \frac{1}{2} c(n)\, \mathrm{dist}(K,L)^2$, where $c(n)$ is the constant in 6.2. If $x \in L$ and $w(x) \leqq \varepsilon$ then

$$w'(x) \leqq \varepsilon v(x) + w(x) \leqq 2\varepsilon\,.$$

If $w(x) > \varepsilon$ then

$$w'(x) \leqq \varepsilon - C \inf_{\{w \geqq \varepsilon\}} u + \sup_{\overline{D}} w \leqq 2\varepsilon$$

for a suitable C since $\inf\{u(x) / x \in \overline{D} \text{ and } w(x) \geqq \varepsilon\} > 0$. By 6.2 the theorem follows.

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