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THE NUMBER OF STEPS IN A FINITE JACOBI ALGORITHM

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Let $x = (x_1,\dots,x_n)$ be a rational point and denote by $L(x)$ the length of the finite JACOBI algorithm associated with x. Let $b(x)$ be the minimal common denominator of x. Then $L(x) = \text{const.} \log b(x)$ is true in a certain sense for almost all x.

1. Introduction

The last years saw a renewed interest concerning the length of the continued fraction expansion of a rational number (Heilbronn [5], DIXON [1], [2], TONKOV [11]).
The aim of this paper is to generalize DIXON's theorem [1] to the JACOBI algorithm.

Let $B = \{ x = (x_1,\dots,x_n) \mid 0 \le x_j < 1, 1 \le j \le n \}$

and define $T: B \to B$ as follows:

Let $x = (x_1,\dots,x_n) \in B$ and $x_1 = \dots = x_t = 0, x_{t+1} > 0$

where $0 \le t \le n$, then $Tx = T(0,\dots,0,x_{t+1},\dots,x_n) =$

$$= \left(0,\dots,0,\frac{x_{t+2}}{x_{t+1}} - \left[\frac{x_{t+2}}{x_{t+1}}\right],\dots,\frac{x_n}{x_{t+1}} - \left[\frac{x_n}{x_{t+1}}\right],\frac{1}{x_{t+1}} - \left[\frac{1}{x_{t+1}}\right]\right)$$

We put $x^{(g)} = T^g x$ and define digits $k^{(g)} \in \mathbb{Z}^{n+1}$ as follows:
If $x^{(g)} = (0,\dots,0,x_{t+1}^{(g)},\dots,x_n^{(g)})$, where $x_{t+1}^{(g)} > 0$,

then

$$k_0^{(g+1)} = \dots = k_{t-1}^{(g+1)} = 0$$

$$k_t^{(g+1)} = 1$$

$$k_{t+1}^{(g+1)} = \left[\frac{x_{t+2}^{(g)}}{x_{t+1}^{(g)}}\right],\dots,k_n^{(g+1)} = \left[\frac{1}{x_{t+1}^{(g)}}\right]$$

Then we define $A_i^{(j)} = \delta_{ij}$ for $0 \leq i,j \leq n, A_i^{(n+1)} = 0$ for $1 \leq i \leq n, A_o^{(n+1)} = 1$ and by recursion

$$A_i^{(g+n+1)} = \sum_{j=0}^{n} A_i^{(g+j)} k_j^{(g)}$$

for $0 \leq i \leq n$, $0 \leq g$. An easy induction shows

$$x_i = \frac{A_i^{(g+n+1)} + \sum_{j=1}^{n} A_i^{(g+j)} x_j^{(g)}}{A_o^{(g+n+1)} + \sum_{j=1}^{n} A_o^{(g+j)} x_j^{(g)}}$$

PERRON [6] showed that $x \in B$ is rational iff there exists a $g \geq 0$ such that $x^{(g)} = (0,\ldots,0)$.

We put

$$L(x) = \min \{ g \geq 0 \mid x^{(g)} = (0,\ldots,0) \}$$

Let $x \in B$ be rational then we have a representation

$$x = (\frac{a_1}{b},\ldots,\frac{a_n}{b}),\ a_1,\ldots,a_n,\ b \in Z$$

where $b = b(x) \geq 1$ is minimal (or equivalently g.c.d $(a_1,\ldots,a_n,b) = 1$).

Then (see Schweiger [10])

$$b(x) = A_o^{(L(x) + n + 1)}$$

It is well known (see SCHWEIGER [9]) that there exists a unique probability measure μ on the BOREL sets of B which is equivalent to Lebesgue measure λ and invariant under T. The ergodic dynamical system (B,μ,T) possesses a measure-theoretic entropy $h(T)$ for which the formula

$$h(T) = \int_B (\log \frac{d\lambda T}{d\lambda})\, d\mu$$

holds. The main result of this paper is given as

<u>Theorem 1</u>: Let $\frac{1}{2} < \alpha < 1$, $c > 0$ and define for $z > 0$:

$$A(z) = \{ x \in B \cap \mathbb{Q}^n \mid b(x) \leq z \text{ and } \left| L(x) - \frac{n+1}{h(T)} \log b(x) \right| \geq c (\log b(x))^{\alpha} \}$$

Then there exists a constant $c_0 > 0$

such that

$\text{card } A(z) = O(z^{n+1} \exp(-c_0 (\log z)^{\beta}))$

as $z \to \infty$ where $\beta = \frac{1}{2}(\alpha - \frac{1}{2})$.

<u>Remarks</u>: (1) Let $M(z) = \{ x \in B \cap \mathbb{Q}^n \mid b(x) \leq z \}$

then

$$\lim_{z\to\infty} \frac{\text{card } A(z)}{\text{card } M(z)} = 0$$

Since the relation

$$\text{card } M(z) = \sum_{1\leq b\leq z} b^n - \sum_{2\leq k\leq z} \text{card } M\left(\left[\frac{z}{k}\right]\right)$$

implies

$$\lim_{z\to\infty} \frac{\text{card } M(z)}{z^{n+1}} = \frac{1}{(n+1)\zeta(n+1)}$$

this follows at once from theorem 1.

(2) The "direct" generalization of DIXON's result is to consider $(n+1)$ - tuples $(a_1,\ldots,a_n,d) \in \mathbb{Z}^{n+1}$ and to consider

$B(z) = \{ y = (a_1,\ldots,a_n,d) \mid 0 \leq a_i < d < z \text{ for } 1\leq i\leq n \text{ and}$

$$\left| L(y) - \frac{n+1}{h(T)} \log d \right| \geq c (\log d)^{\alpha} \}$$

where $L(y) = L((\frac{a_1}{d},\ldots,\frac{a_n}{d}))$. From theorem 1 we will deduce

<u>Corollary</u>: There exists a constant $c_o > 0$, such that

$$\text{card } B(z) = O\left(z^{n+1}\exp(-c_o(\log z)^{\beta})\right)$$

$$\text{as } z \to \infty \quad \text{where} \quad \beta = \frac{1}{2}\left(\alpha - \frac{1}{2}\right).$$

2. Some auxiliary results

We define for all $x \in B$ with $x_1 > 0$

$$f(x) := \frac{d\lambda T}{d\lambda}(x) = \frac{1}{x_1^{n+1}}$$

as usual we put

$$B_g(x) = \{y \in B \mid k^{(j)}(y) = k^{(j)}(x),\ 1 \leq j \leq g\}$$

<u>Lemma 1</u>: There is a constant θ with the following property:

Let $x \in B$ such that $x_1^{(j)} > 0$ for $0 \leq j \leq g - 1$.

Then

$$\left| \sum_{j=0}^{g-1} \log f(T^j x) + \log \mu(B_g(x)) \right| \leq \theta$$

<u>Proof</u>: It is known (see SCHWEIGER [9]) that there exist constants $c_1, c_2 > 0$ such that

$$c_1 (A_o^{(g+n+1)}(x))^{-n-1} \leq \mu(B_g(x)) \leq c_2 (A_o^{(g+n+1)})^{-n-1}$$

as long as $x_1^{(g-1)} > 0$. By induction we have

$$\prod_{j=0}^{g-1} \frac{1}{x_1^{(j)}} = \sum_{j=1}^{n} A_o^{(g+j)} x_j^{(g)} + A_o^{(g+n+1)}$$

Therefore

$$\left| \sum_{j=0}^{g-1} \log x_1^{(j)} + \log A_o^{(g+n+1)} \right| \leq \log(1+n)$$

which proves lemma 1.

For $p \geq 1$ we denote the norm in $L_p(B,\mu)$ by $\| \ \|_p$. It is known that

$$h(T) = - \lim_{g\to\infty} \frac{\log \mu(B_g(x))}{g}$$

holds almost every where and in $L_1(B,\mu)$.

We need an estimate of the convergence:

<u>Lemma 2</u>: There exist constants $c_3, c_4 > 0$ such that for $k,g \in N$

and

$$k < c_4 \frac{\sqrt[4]{g}}{\log g}$$

the inequality

$$\| \log \mu(B_g(\cdot) + gh(T) \|_{2k} \leq \sqrt{c_3}k^2\sqrt{g}$$

is valid.

<u>Proof</u>: Note that $x_1^{(j)} > 0$ for all $j \geq 0$ is true almost every where. By lemma 1 it is sufficient to prove

$$\| \sum_{j=0}^{g-1} \log f(T^j x) - gh(T) \|_{2k} \leq \sqrt{c_3}k^2\sqrt{g}$$

Put $y(x) = \log f(x) - h(T)$ then we have to show

$$\int_B \left| \sum_{j=0}^{g-1} y(T^j x) \right|^{2k} d\mu(x) \leq (c_3 g k^4)^k$$

Let

$$\sigma(g) = \sup_{k^{(1)},\ldots,k^{(g)}} \operatorname{diam} B(k^{(1)},\ldots,k^{(g)}$$

An easy consequence of the KUZMIN theorem for JACOBI's algorithm (see SCHWEIGER [9]) gives:

Let

f_g be a simple function, constant on the sets

$B(k^{(1)},\dots,k^{(g)})$ and $m \in L_2(B,\mu)$, then

$$\int_B f_g(x)\, m(T^h x)\, d\mu(x) = \int_B f_g \int_B m + ||f_g||_1\, ||m||_1\, O(\sigma(h-g))$$

Now let ϕ_s be the conditional expectation of y with respect to the partition by the sets $B(k^{(1)},\dots,k^{(s)})$.

Then (see PHILIPP [7], Hilfssatz 2)

$$\int_B \phi_s^{p_1}(T^{i_1}x)\dots\phi_s^{p_r}(T^{i_r}x)\, d\mu(x) =$$

$$= \int_B \phi_s^{p_1}(T^{i_1}x)\dots\phi_s^{p_j}(T^{i_j}x)\, d\mu(x).$$

$$\int_B \phi_s^{p_{j+1}}(T^{i_{j+1}}x)\dots\phi_s^{p_r}(T^{i_r}x)\, d\mu(x) +$$

$$+ \quad ||\phi_s^{p_1}\dots\phi_s^{p_j}||_1\, ||\phi_s^{p_{j+1}}\dots\phi_s^{p_r}||_1\, O(\sigma(i_{j+1}-i_j-s))$$

where $0 \le i_1<\dots<i_j<\dots<i_r, p_i \ge 1$, and $i_{j+1} - i_j \ge s$.

Now fix $\delta > 0$ and $s \in \mathbb{N}$. Let $\gamma_s \in L_2(B,\mu)$ be defined as

$$\gamma_s(x) = \phi_s(x) \qquad \text{if } x_1 \ge \delta$$

$$\gamma_s(x) = 0 \qquad \text{if } 0 \le x_1 < \delta$$

For small δ we have

$$||y-\phi_s||_2 \le ||y-\gamma_s||_2$$

For $x_1 \geq \delta$ the mean value theorem implies

$$|y(x) - \gamma_s(x)| \leq \frac{\sigma(s)}{\delta}$$

Since $\frac{d\mu}{d\lambda} \leq c_5$ we calculate

$$\int_{x_1<\delta} | y - \gamma_s |^2 d\mu = \int_{x_1<\delta} y^2 d\mu \leq$$

$$\leq c_5 \int_{x_1<\delta} y^2 d\lambda \leq c_6\ \delta((n+1) \log \delta + h(T))^2$$

Therefore

$$||y-\gamma_s||_2 \leq (\delta^{-2}\sigma(s)^2 + c_6\delta((n+1) \log \delta + h(T))^2)^{\frac{1}{2}}$$

$$\leq \delta^{-1}\ \sigma(s) + \sqrt{c_6}\ \sqrt{\delta}\ |\ (n+1) \log \delta + h(T)\ |$$

$$\leq \delta^{-1}\ \sigma(s) + c_7\delta^{\frac{1}{4}}$$

We choose $\delta = \sigma(s)^{\frac{4}{5}}$ and obtain

$$||y-\phi_s||_2 \leq c_8\sigma(s)^{\frac{1}{5}}$$

Now we use the result of FISCHER [3] that $\sigma(s) = O(q^s)$, $0 < q < 1$ and conclude

$$||y - \phi_s||_2 \leq \eta^{-5}$$

where $\eta > 1$. Following a method given by PHILIPP ([7],p.84) we see

$$| \int_B y^{p_1}(T^{i_1}x)\dots y^{p_r}(T^{i_r}x)\ d\mu\ (x) -$$

$$- \int_B y^{p_1}(T^{i_1}x)\dots y^{p_j}(T^{i_j}x) d\mu(x)\ \cdot$$

$$\cdot \int_B y^{p_{j+1}}(T^{i_{j+1}}x)\dots y^{p_r}(T^{i_r}x)\,d\mu(x) \;|$$

$$\leq c_9^{\,p_1+\dots+p_r} \exp(-c_{10}(i_{j+1}-i_j))$$

Now we are in position to apply a result of PHILIPP ([8], Hilfssatz 1.3) to obtain our result. (In his notation we have $\psi(s) = \text{const. } 5(\log s)^{-1}$ and $L(x) = \text{const.}$).

For $g \in \mathbb{N}$ and real $w,c > 0$, we define $K(g,c,w) =$

$$= \{B_g(x) \mid 1 \leq w\mu(B_g(x)), |\log \mu(B_g(x)) + gh(T)| \geq c(\log w)^{\alpha}\}$$

$$\kappa(g,c,w) = \text{card } K(g,c,w)$$

<u>Lemma 3</u>: For any $c > 0$ there are constants c_{11}, $w_0 > 0$ such that for $w > w_0$ the inequality

$$|\log w - h(T)\, g| \geq c\,(\log w)^{\alpha}$$

implies

$$\kappa(g,c,w) < w \exp(-c_{11}(\log w)^{\beta})$$

<u>Remark</u>: Lemma 3 essentially replaces Lemma 6 in [1] the proof of which contains a gap:

The function $f(\xi) = (\log \xi - n\lambda + 4\theta)^{2k}\xi^{-2}$ discussed in [1], p. 421 is decreasing in the domain

$$]\,0, \exp(n\lambda - 4\theta)\,[\;\cup\;]\,\exp(n\lambda - 4\theta + k), \infty\,[\,.$$

Hence, $|\log x - n\lambda| > k + 4$ and $Q_n \leq x$ do not imply $f(Q_n) \geq f(x)$.

<u>Proof</u>: There are constants $c_{12}, c_{13} > 0$

such that

$$c_{12} \leq (k_n^{(g+1)})^{n+1} \mu(B_1(T^g x)) \leq c_{13}$$

Therefore, for $\frac{n}{n+1} < \sigma < 1$ we

obtain

$$\sum_{k^{(g+1)}} \mu(B_1(T^g x))^\sigma = c_{14} < \infty \text{ (say)}$$

Define

$$\lambda(g,w) = \text{card } \{ B_g(x) \mid 1 \leq w\mu(B_g(x)) \}$$

It is well known that there is a constant $c_{15} > 0$ such that

$$\mu(B_{g+1}(x)) \leq c_{15} \mu(B_g(x) \mu(B_1(T^g x))$$

This shows

$$\lambda(g+1,w) \leq \sum_{k^{(g+1)}} \lambda(g, c_{15} w \mu(B_1(T^g x)))$$

and, putting $c_{16} = c_{15}^\sigma \cdot c_{14}$,

we obtain

$$\lambda(g,w) \leq w^\sigma c_{16}^g$$

Now let

$$c_{17} = \frac{1-\sigma}{2 \log c_{16}}$$

then for $g < c_{17} \log w$

$$\kappa(g,c,w) \leq \lambda(g,w) < w^{\frac{1+\sigma}{2}}$$

This shows

$$\kappa(g,c,w) < w \exp(-c_{11}(\log w)^{\beta})$$

for sufficiently large w.

Since $\mu(B_g(x)) \leq \delta^{-g}$ ($\delta > 1$)

the other case leads to

$$c_{17} \log w \leq g \leq c_{18} \log w$$

We put

$$k = \left[c_{19} \ (\log w)^{\beta} \right]$$

with a small constant $c_{19} > 0$ to be chosen later. We introduce the function

$$q(t) = (\log t + gh(T))^{2k} t$$

If $t_1 < t_2$ and $|\log t_i + gh(T)| \geq 2k$ both for $i = 1,2$

then $g(t_1) < q\ (t_2)$.

We take $t_1 = w^{-1}$ and $t_2 = \mu(B_g(x))$, where $B_g(x) \in K(g,c,w)$

and

$$| \log w - h(T)g | \geq c(\log w)^{\alpha}$$

Since $\beta < \frac{1}{4}$ we may apply Lemma 2 in our case. Then we calculate

$$\kappa(g,c,w) = \sum_{B_g(x) \in K(g,c,w)} 1 < \sum g(\mu(B_g))(g(w^{-1}))^{-1} =$$

$$= w(\log w - gh(T))^{-2k} \Sigma (\log \mu(B_g) + gh(T))^{2k} \mu(B_g) =$$

$$= w(\log w - gh(T))^{-2k} || \log B_g(\cdot) + gh(T) ||_{2k}^{2k} \leq$$

$$\leq w(c(\log w)^{\alpha})^{-2k} (c_3 k^4 g)^k \leq$$

$$\leq w \exp(c_{19} (\log w)^{\beta} (\log c_3 + 4 \log c_{19} - 2 \log c + \log c_{18}))$$

Here we used $1 + 4\beta - 2\alpha = 0$ and $g \leq c_{18} \log w$. If we choose c_{19} sufficiently small, we obtain the lemma.

3. A general result

The following theorem shows that the results of DIXON and this paper essentially are statements on information theory which may be extended to general classes of algorithms (see e.g. FISCHER [4]).

Theorem 2: For real $w, c, > 0$ define

$$\hat{A}(c,w) = \{B_g = B_g(x) \mid \mu(B_g)\, w \geq 1,$$
$$, \; |g + (h(T))^{-1} \log \mu(B_g)| \geq c |\log \mu(B_g)|^{\alpha}\}$$

Then there exist constants $c_{20}, w_0 > 0$, depending on c and a, such that for $w \geq w_0$

$$\text{card } \hat{A}(c,w) < \exp(-c_{20} (\log w)^{\beta})$$

Proof:

Suppose, $B_g \in \hat{A}(c,w)$. Then, for w sufficiently large,

we have either

(a) $\mu(B_g)\, w \geq \exp((\log w)^{\frac{1}{2}})$

or

(b) $\exp((\log w)^{\frac{1}{2}}) > w\,\mu(B_g) \geq 1,$

$$|gh(T) - \log w| \geq \frac{ch(T)}{2}(\log w)^{\alpha},$$

$$|gh(T) + \log \mu(B_g)| \geq \frac{ch(T)}{2}(\log w)^{\alpha}.$$

The number of cylinders B_g satisfying

(a) is bounded by $w \exp(-(\log w)^{\frac{1}{2}})$.

If B_g is subject to conditions (b) then $B_g \varepsilon K(g, \frac{ch(T)}{2}, w)$.

Since $g \leq c_{18} \log w$ (see e.g. SCHWEIGER [10]), we obtain by Lemma 3 the upper bound

$$\sum_{g \leq c_{18}\log w} \kappa(g, \frac{ch(T)}{2}, w) < w \exp(-c_{21}(\log w)^{\beta})$$

This completes the proof.

From Theorem 2 we deduce

<u>Lemma 4</u>: For $z \geq 1$, $c > 0$ put

$A^{\neq}(c,z) = \{ x \varepsilon B \cap \mathbb{Q}^n | b(x) \leq z,\ x_1^{L(x)-1} > 0.$

$|L(x)h(T) - (n+1)\log b(x)| \geq ch(T)(\log b(x))^{\alpha} \}$

Then there exist c_{22}, $z_0 > 0$ such that for $z \geq z_0$

card $A^{\neq}(c,z) \leq z^{n+1} \exp(-c_{22}(\log z)^{\beta})$

<u>Proof</u>: We may assume $z_1 \leq b(x)$.

Then $x \in A^{\neq}(c,z)$ implies that the corresponding cylinder $B_g(x) \in \hat{A}(c^{\neq},w)$, where $g = L(x)$, $c^{\neq} =$

$= c(n+1)^{-1}$ and $w = z^{n+1}$ and $w = z^{n+1} c_1^{-1}$

(we use the estimate $c_1 \leq b(x)^{n+1} \mu(B_g(x)) \leq c_2$).

4. Proof of Theorem 1

For $x \in B \cap \mathbb{Q}^n$ we define: $t = t(x) = \min \{s | x_1^{(s)} = 0\}$

$$x' = \left(\frac{A_1^{(t+n+1)}}{A_0^{(t+n+1)}}, \ldots, \frac{A_n^{(t+n+1)}}{A_0^{(t+n+1)}}\right)$$

$$x'' = T^t x$$

The mapping $x \mapsto (x',x'')$ is injective and

$$x' \in \{x \in P \cap \mathbb{Q}^n | x_1^{(L(x)-1} > 0\}$$

The recursion relations for the denominators imply

$b(x')\, b(x'') \leq b(x) \leq (n+1)\, b(x')\, b(x'')$

If $b(x) \leq z$, $b(x'') = k$, then $b(x') \leq \frac{z}{k}$.

We want to apply Lemma 4 with c replaced by $\frac{c}{2}$.

Then we choose $0 < c_{23} < \frac{c_{22}}{n}$

(where c_{22} satisfies the assertion of lemma 4 with c replaced by $\frac{c}{2}$). An upper bound for the number of points x, satisfying $b(x) \leq z$ and $b(x'') \geq \exp(c_{23}(\log z)^{\beta})$, is given by:

$$\sum_{k \geq \exp(c_{23}(\log z)^{\beta})} \left(\frac{z}{k}\right)^{n+1} k^{n-1} < c_{24} z^{n+1} \exp(-c_{23}(\log 2)^{\beta})$$

Since card $M(z^{\frac{1}{2}}) = O(z^{\frac{n+1}{2}})$, we now need only to consider those points x for which

$$b(x'') \leq \exp\,(c_{23}(\log z)^{\beta})$$

$$b(x) \geq \sqrt{z}$$

We remark that $L(x) = L(x') + L(x'')$.

Now we calculate

$$c(\log b(x))^{\alpha} \leq \left| L(x) - \frac{n+1}{h(T)} \log b(x) \right| \leq$$

$$\leq \left| L(x') - \frac{n+1}{h(T)} \log b(x') \right| + L(x'')$$

$$+ \frac{n+1}{h(T)} \left| \log \frac{b(x)}{b(x')} \right|$$

$$\leq \left| L(x') - \frac{n+1}{h(T)} \log b(x') \right| + c_{18} \log b(x'')$$

$$+ c_{25} \log b(x'')$$

$$\leq \left| L(x') - \frac{n+1}{h(T)} \log b(x') \right| + c_{26} (\log b(x))^{\beta}$$

This implies

$$| L(x') - \frac{n+1}{h(T)} \log b(x') | \geq \frac{c}{2} (\log b(x'))^{\alpha}$$

Therefore $x' \in A^{\neq} (\frac{c}{2} , z)$.

By lemma 4 the number of these points x' is bounded by

$$z^{n+1} \exp(-c_{22}(\log z)^{\beta})$$

Since $b(x'') \leq \exp (c_{23}(\log z)^{\beta})$, for each of these x' there are at most $\exp (nc_{23}(\log z)^{\beta})$ possible values for x".

Hence

$$\begin{aligned} \text{card } A(z) &= O(z^{n+1} \exp((-c_{22} + nc_{23}) (\log z)^{\beta}) \\ &\quad + z^{n+1} \exp(-c_{23}(\log z)^{\beta}) + z^{\frac{n+1}{2}}) = \\ &= O(z^{n+1} \exp(-c_{o}(\log z)^{\beta}) \end{aligned}$$

This shows theorem 1.

<u>Proof of the corollary</u>: We may restrict ourselves to the case

$\sqrt{z} \leq d \leq z$. Let $f = g.c.d(a_1,\ldots,a_n,d)$ and $b = \frac{d}{f}$.

If

$$f \geq \exp (\frac{ch(T)}{2(n+1)} (\log d)^{\alpha})$$

then

$$b \leq z \exp (- \frac{ch(T)}{2^{1+\alpha}(n+1)} (\log z)^{\alpha})$$

The number of (n+1) - tuples y satisfying this condition is not greater than

$$\sum_b b^n \frac{z}{b} = O(z^{n+1} \exp(-c_o (\log z)^\alpha))$$

Now let

$$f \leq \exp\left(\frac{ch(T)}{2(n+1)} (\log d)^\alpha \right)$$

Then $x = \left(\frac{a_1}{d}, \ldots, \frac{a_n}{d} \right) \varepsilon A \left(\frac{z}{f} \right)$

(when we consider the sets A (z) with the constant c replaced by $\frac{c}{2}$):

$$\left| L(x) - \frac{n+1}{h(T)} \log b \right| \geq$$

$$\geq \left| L(x) - \frac{n+1}{h(T)} \log d \right| - \frac{n+1}{h(T)} \log f$$

$$\geq \frac{c}{2} (\log d)^\alpha$$

Note, that L (x) = L (y).

Therefore the number of (n+1) - tuples y in the second case is bounded by

$$\sum_{k \leq z} \text{card } A\left(\frac{z}{k}\right) \leq \sum_{k=1}^{\left[\frac{z}{z_o}\right]} \text{card } A\left(\frac{z}{k}\right) + \sum_{k=\left[\frac{z}{z_o}\right]}^{\infty} \left(\frac{z}{k}\right)^{n+1}$$

$$= O(z^{n+1} \exp(-c_o (\log z)^\beta))$$

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