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ON CERTAIN TOPOLOGIES ON A VECTOR SPACE (*)

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Let E and F be two vector spaces over the field K of the real or complex numbers such that $\langle E,F\rangle$ is a dual pair. If $\mathcal{F}$ is a locally convex topology on E, compatible with the dual pair $\langle E,F\rangle$, then $\mathcal{F}^f$ is the finest topology on F, such that it coincides with the weak topology $\sigma(F,E)$ in the $\mathcal{F}$-equicontinuous sets of F. In ([1],p.270) G. Köthe asks if every space $F[\mathcal{F}^f]$ is a topological vector space. Y. Kômura, [2], gives an example of a space $F[\mathcal{F}^f]$ which is not a topological vector space. In this paper we give a method for obtaining spacesof type $F[\mathcal{F}^f]$ which are not regular.

The vector spaces we use here are defined over the field K of the real or complex numbers. If X is a topological space, A is a subset of X and $\mathcal{F}$ is the topology of X, we write $A[\mathcal{F}]$ for A provided with the induced topology by $\mathcal{F}$. If $\langle E,F\rangle$ is a dual pair, we represent by $\sigma(E,F)$ the weak topology on E. If G is a locally convex space G' is the topological dual of G. Given a topological space X we say that $X[\mathcal{V}]$ is the associated k-space to X if a set is closed in X, for the topology $\mathcal{V}$, if, and only if, it intersects every compact set of X in a compact set. If X is a completely regular space we denote by C(X) the vector space over K, of all continuous functions on X, valued in K, provided with the compact-open topology. The topologies used here are separated

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THEOREM 1. Let X be a completely regular space. If $\mathcal{F}$ is the topology of C(X), there is a set A in the topological dual of C(X), such that $A[\mathcal{F}^f]$ is homeomorphic to the associated k-space to X.

Proof: Given $x\in X$, let u_x be the linear form on C(X), defined by

$$u_x(f)=f(x), \text{ for every } f\in C(X).$$

Let E be the topological dual of C(X). We define the mapping ρ from X into E such that

$$\rho(x)=u_x, \text{ for every } x\in X.$$

The mapping ρ is injective since C(X) separates points of X. If $\{x_i : i\in I,\geq\}$ is a net in X which converges to x_o, the net $\{f(x_i) : i\in I,\geq\}$ converges to $f(x_o)$, for every $f\in C(X)$ and, therefore, the net $\{u_{x_i} : i\in I,\geq\}$ converges to u_{x_o} in $E[\sigma(E,C(X))]$. Conversely, if the net $\{u_{x_i} : x\in I,\geq\}$ converges to u_{x_o} in $E[\sigma(E,C(X)]$ and $\{x_i : i\in I,\geq\}$ is not convergent to x_o in X, then there is in X a neighbourhood U of x_o, such that, given any index $i\in I$, there is an index $j\in I$, $j\geq i$, $x_j\notin U$. Since X is completely regular, there is an element $g\in C(X)$, such that $g(x_o)=1$, and $g(x)=0$, for every x of X which is not in U. Obviously, the net $\{u_{x_i}(g) : i\in I,\geq\}$ is not convergent to $u_{x_o}(g)$, which is a contradiction. Thus, if $A = \rho(X)$ we have that ρ is a homeomorphism from X onto $A[\sigma(E,C(X))]$.

Let us see now that A is closed in $E[\sigma(E,C(X))]$. Indeed, let u be a point of the $\sigma(E,C(X))$-closure of A in E and let $\mathcal{U}$ be a filter in A which converges to u in $E[\sigma(E,C(X))]$. Since $\mathcal{U}$ is a Cauchy filter in $A[\sigma(E,C(X))]$ it follows that $\rho^{-1}(\mathcal{U})$ is a Cauchy filter in X for the coarsest uniformity $\mathcal{R}$, such that every $f\in C(X)$ is uniformly continuous for $\mathcal{R}$. Thus, $\rho^{-1}(\mathcal{U})$ converges to an element z of the completion $\hat{X}$ of the uniform space $[X;\mathcal{R}]$. If $h\in C(X)$ there is a function $\bar{h}$ in $C(\hat{X})$, its restriction to X coincides with h and, therefore, the filter

$$\{h(W) : W\in\rho^{-1}(\mathcal{U})\}$$

converges to $\bar{h}(z)$. Since $\mathcal{U}$ converges to u in $E[\sigma(E,C(X))]$, we have that $u(h)=\bar{h}(z)$. Let us suppose that $u\notin A$. If M is any compact set of X, then $z\notin M$ and, therefore, there is a function $\bar{k}\in C(\hat{X})$ such that $\bar{k}(z)=1$, $k(x)=0$, for every $x\in M$. If k is the restriction of $\bar{k}$ to X we have that $u(k)=\bar{k}(z)=1$, whence M is not the support of u and, therefore, u is not continuous in C(X), [4] and [5], which is a contradiction.

Then A is closed in $[\sigma(E,C(X))]$.

Given a closed set H of $E[F^f]$ let P be any compact subset of $A[\sigma(E,C(X))]$. Then P is homeomorphic to a compact set of X and, therefore, P is equicontinuous in C(X) and, hence $P\cap H$ is $\sigma(E,C(X))$-compact. Thus, $H\cap A$ is closed in the associated k-space $A[S]$ to $A[\sigma(E,C(X)]$ On the other hand, let B be a closed set in $A[S]$ and let T be a closed set of $E[\sigma(E,C(X))]$ which is equicontinuous in C(X). Since A is closed in $E[\sigma(E,C(X)]$ we have that $T\cap A$ is $\sigma(E,C(X))$-closed and equiconti - nuous in C(X), whence $T\cap A$ is compact in $A[\sigma(E,C(X))]$. Then $B\cap T = B\cap(T\cap A)$ is $\sigma(E,C(X))$-compact and, therefore, B is closed in $E[F^f]$. Then it results that F^f coincides with S in A, hence we obtain that $A[F^f]$ is homeomorphic to the associated k-space to X. q.e.d.

Let X be a completely regular space such that the associated k-space $X[V]$ is not regular. If F is the topology of C(X), there is in the topological dual E of C(X) a subset A such that $A[F^f]$ is homeomorphic to $X[V]$ according to Theorem 1. Then $E[F^f]$ is not regular since it has a non-regular subspace $A[F^f]$. Thus, it is sufficient to find an example of a completely regular space X such that its associated k-space is not regular to obtain a topology F^f which is not regular.

Example. Let ω be the first infinite ordinal. Let ω_1 be the first non-countable ordinal. We denote by S(T) the set of all ordinals non-larger than $\omega(\omega_1)$, with the order topology. Let X be the topological product space SxT and let F be the topology of X.

Let (ω,α) a point of X such that α is a limit point of T, different from ω_1. There are countable infinitely many points of T preceding α and, therefore, α has in T a countable fundamental system of neighbourhoods. Thus, we can choose a sequence $\{\beta_n\}_{n=1}^{\infty}$, strictly increasing, which converges to α. If we put $\alpha_n=\beta_n+1$, the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is strictly increasing, it converges to α, and for every n, α_n is an isolated point of T. For every positive integer p we write

$$A_{p,\alpha} = \{(p,\delta)\in X:\alpha_p \leq \delta \leq \alpha\}$$

Since p and α_p are isolated points of S and T, respectively, we have that $A_{p,\alpha}$ is an open set in X.

Let

$$U_{n,\alpha} = \{(\omega,\alpha)\}\cup(\bigcup_{p=n}^{\infty} A_{p,\alpha})$$

We shall see that $U_{n,\alpha}$ is closed in X. In fact, if $(\lambda,\mu)\in X$, $(\lambda,\mu)\notin U_{n,\alpha}$ we shall consider the following cases: a) $\mu>\alpha$. Then the set $\{(\eta,\delta)\in X: \eta\in S, \delta>\alpha\}$ is a neighbourhood of (λ,μ) which does not intersect $U_{n,\alpha}$. b) $\lambda<n$. Then the set $V=\{(\eta,\delta)\in X: \eta<n,\ \delta\in T\}$ is a neighbourhood of (λ,μ) such that $V\cap U_{n,\alpha}=\emptyset$. c) $\omega>\lambda\geq n$, $\mu\leq\alpha$. Then $\mu<\alpha_\lambda$ since if $\alpha_\lambda\leq\mu\leq\alpha$ we have that $(\lambda,\mu)\in A_{\lambda,\alpha}\subset U_{n,\alpha}$. The set $V=\{(\eta,\delta)\in X: \eta<\alpha_\lambda\ \delta=\lambda\}$ is a neighbourhood of (λ,μ) with $V\cap U_{n,\alpha}=\emptyset$. d) $\lambda=\omega, \mu<\alpha$. We can find a positive integer n_o such that $\mu<\alpha_{n_o}$. The set $\{(\eta,\delta)\in X:\ \eta>n_o,\ \delta<\alpha_{n_o}\}$ is a neighbourhood of (λ,μ), which does not intersect $U_{n,\alpha}$

We define a topology $\mathcal{U}$ on X in the following way: if (λ,β) is a point of X which is different from (ω,α), with α as a limit point of T, different of ω_1 we take as fundamental system of neighbourhoods of (λ,β) the $\mathcal{F}$-neighbourhoods of (λ,β). If α is a limit point of T, different from ω_1 we take as neighbourhood basis of (ω,α) the sets $\{U_{n,\alpha}\}_{n=1}^{\infty}$. It is easy to show that $\mathcal{U}$ is finer than $\mathcal{F}$.

Let us see now that $X[\mathcal{U}]$ is completely regular. Let U be a neighbourhood of $(\lambda,\mu)\in X$ for the topology $\mathcal{U}$. Let us suppose that (λ,μ) is different of (ω,α), with α a limit point of T, different from ω_1. Then U is a neighbourhood of (λ,μ) for the topology $\mathcal{F}$ and, since X is a compact space, there is a real-valued continuous function f on X such that $f(\lambda,\mu)=1$, $f(\eta,\delta)=0$, $(\eta,\delta)\in X$, $(\eta,\delta)\notin U$. Obviously the function f is continuous in $X[\mathcal{U}]$. If α is a limit point of T, different from ω_1, let V be an $\mathcal{U}$-neighbourhood of (ω,α). Then there is a positive integer n_o such that $U_{n_o,\alpha}\subset V$. Let us see that $U_{n_o,\alpha}$ is open in $X[\mathcal{U}]$. If $(\gamma,\delta)\in U_{n_o,\alpha}$ then $\gamma=\omega$ and $\mu=\alpha$ or there is a positive integer $p\geq n_o$, such that $(\gamma,\delta)\in A_{p,\alpha}$. If $\gamma=\omega$ and $\mu=\alpha$, $U_{n_o,\alpha}$ is an $\mathcal{U}$-neighbourhood of (γ,δ). In the second case, since $A_{p,\alpha}$ is open in X, we have that $U_{n_o,\alpha}$ is an $\mathcal{U}$-neighbourhood of (γ,δ). In the other hand $U_{n_o,\alpha}$ is closed in $X[\mathcal{U}]$ since $U_{n_o,\alpha}$ is closed in $X[\mathcal{F}]$. It follow that the function g which thakes the value 1 in $U_{n_o,\alpha}$ and $g(x)=0$, for every $x\in X\sim U_{n_o,\alpha}$ is continuous in $X[\mathcal{U}]$. Then we conclude that $X[\mathcal{U}]$ is completely regular.

If $\mathcal{V}$ is the topology on X such that $X[\mathcal{V}]$ is the associated k-space to $X[\mathcal{U}]$, let us see that $X[\mathcal{V}]$ is not a regular space. First we shall see that $Y=\{(\omega,\delta)\in X: \delta<\omega_1\}$ is closed in $X[\mathcal{V}]$. We take a compact set M in $X[\mathcal{U}]$ and we shall see that $M\cap Y$ is compact. Let us suppose that

$M \cap Y$ is not compact, then there is a sequence $\{(\omega,y_n)\}_{n=1}^{\infty}$ in $Y \cap M$ such that

$$y_1 < y_2 < \dots < y_n < \dots$$

for the order induced by T. Let α be the first element of T larger than y_p, p=1,2,... Then $\alpha \neq \omega_1$ and $\{(\omega,y_p)\}_{p=1}^{\infty}$ converges to (ω,α) in $X[\mathcal{F}]$. Since $M \cup \{(\omega,\alpha)\}$ is compact in $X[\mathcal{U}]$, then $M \cup \{(\omega,\alpha)\}$ is compact in $X[\mathcal{F}]$ and, therefore, $\{(\omega,y_p)\}_{p=1}^{\infty}$ converges to (ω,α) in $X[\mathcal{U}]$. On the other hand, $(\omega,y_p) \notin U_{n,\alpha}$ n,p=1,2,.., which is a contradiction. Then Y is closed in $X[\mathcal{V}]$. Let W be the complementary set of Y in X. Then W is a neighbourhood of (ω, ω_1) for the topology $\mathcal{V}$. We shall see that for the topology $\mathcal{V}$ every neighbourhood of (ω, ω_1) contained in W is not closed. Let us suppose that H is a $\mathcal{V}$-neighbourhood $\mathcal{V}$-closedof (ω,ω_1) contained in W. Let $\mathring{H}$ be the interior of H in $X[\mathcal{V}]$. It is immediate that the set $\{(\eta,\delta) \in X: \eta \in S\}$ is compact in $X[\mathcal{U}]$ and, therefore,it is compact in $X[\mathcal{V}]$, for every $\delta \in T$. Then the topologies $\mathcal{F}$ and $\mathcal{V}$ coincide in $Z=\{(\eta,\omega_1): \eta \in S\}$ and, therefore, there is a positive integer n_1 such that the set $\{(n,\omega_1): n_1 < n\omega\}$ is contained in $\mathring{H} \cap Z$. Let $X_n = \{(\eta,\delta) \in X: \eta < n, \delta \in T$. Then X_n is compact in $X[\mathcal{U}]$. If $n > n_1$, we have that $\mathring{H} \cap X_n$ is a neighbourhood of (n,ω_1), for the topology $\mathcal{F}$ and, therefore, there is an element $\alpha_n \in T$, $\alpha_n \neq \omega_1$ such that the set $\{(n,\delta): \alpha_n \leq \delta \leq \omega_1\}$ is contained in $\mathring{H}$. Let α be the least of all elements in T, larger than α_n, n=1,2,... Then $\alpha \neq \omega_1$ and the sequence $\{(n,\alpha)\}_{n=1}^{\infty}$ is contained in H and it converges to (ω,α) in $X[\mathcal{F}]$ and also in $X[\mathcal{V}]$, since $\mathcal{F}$ and $\mathcal{V}$ coincide in the compact set $\{(\eta,\alpha): \eta \in S\}$, hence $(\omega,\alpha) \in H$, which is a contradiction. q.e.d.

Note. In [3], from the space X used above, an example of a completely regular space, such that its associated k-space L is not completely regular is constructed. Using an analogous method to the example given here it is possible to show that L is not regular.

THEOREM 2. There is a Banach space G and a locally convex topology $\mathcal{W}$ on G, compatible with the dual pair $\langle G,G' \rangle$, such that $G'[\mathcal{W}^{\mathcal{F}}]$ is a non-regular topological space.

Proof: The notations we use here are the same as in the example given above. We take the vector space E on K of all continuous functions in X, with values in K, provided with the topology of uniform convergence on every compact set of $X[\mathcal{U}]$. For every $x \in X$ we consider

the linear form u_x on E defined by $u_x(h)=h(x)$, for $h\in E$. If ρ is the mapping from X into $E'[\sigma(E',E)]$ such that $\rho(x)=u_x$, $x\in X$, reasoning in the same way as in Theorem 1, we have that X is homeomorphic to $\rho(X)=A$ for the induced topology by $\sigma(E',E)$. Since X is compact it follows that A is a closed subset of $E'[\sigma(E',E)]$ If $\mathcal{E}$ is the topology of E let us see that $A[\mathcal{E}^f]$ is homeomorphic to the associated k-space $X[\mathcal{V}]$ to $X[\mathcal{U}]$. Given a closed subset F of $E'[\mathcal{E}^f]$ let D be any compact subset of $X[\mathcal{U}]$ Then $\rho(D)$ is equicontinuous in E and, therefore, $\rho(D)\cap F$ is compact in $E'[\sigma(E',E)]$, hence $D\cap\rho^{-1}(F)$ is compact in $X[\mathcal{U}]$, whence it follows that ρ is continuous from $X[\mathcal{V}]$ into $A[\mathcal{E}^f]$. Let Q be a closed subset of $X[\mathcal{V}]$. Let N be a closed subset of $E'[\sigma(E',E))]$, equicontinuous in E. Since A is a closed in $E'[\sigma(E',E)]$ we have that $N\cap A$ is $\sigma(E',E)$-closed and equicontinuous in E and, therefore, $\rho^{-1}(N)$ is compact in $X[\mathcal{U}]$. Thus, $Q\cap\rho^{-1}(N)$ is compact and then $N\cap\rho(Q)$ is compact in $E'[\sigma(E',E)]$ and, therefore, $\rho(Q)$ is closed in $A[\mathcal{E}^f]$ and so ρ^{-1} is continuous from $A[\mathcal{E}^f]$ into $X[\mathcal{V}]$. Since $X[\mathcal{V}]$ is non-regular we have that $E'[\mathcal{E}^f]$ is not regular.

Let $G=C(X)$. Since X is compact it results that C(X) is a Banach space and E' is a dense subspace of $G'[\sigma(G',G)]$. Let J be a co-basis of E' in G'. Let $\mathcal{M}$ be the family of all sets of $E'[\sigma(E',E)]$ such that if $M\in\mathcal{M}$, M is closed absolutely convex and equicontinuous in E. Let L be the linear hull of $\mathcal{J}$. In L let $\mathcal{L}$ be the family of all sets such that if $B\in\mathcal{L}$ then B is the absolutely convex hull of a finite subset. Let

$$\mathcal{H} = \{M+B : M\in\mathcal{M},\ B\in\mathcal{L}\}$$

Let $\mathcal{W}$ be the topology on G of the uniform convergence on every element of $\mathcal{H}$. Since $\mathcal{H}$ covers G' and every element of $\mathcal{H}$ is equicontinuous in G, we have that $\mathcal{W}$ is compatible with the dual pair $\langle G,G'\rangle$. On the other hand, if $M\in\mathcal{M}$ and $B\in\mathcal{L}$ we have that $(M+B)\cap E'=M$ and, therefore, E' is closed in $G'[\mathcal{W}^f]$ and $\mathcal{W}^f$ coincides with $\mathcal{E}^f$ in E'. Since $E'[\mathcal{E}^f]$ is not regular follows that $G'[\mathcal{W}^f]$ is not regular. q.e.d.

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