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Titel: On the Regularity of the Solution of the Biharmonic Variational Inequality.

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Jahr: 1973

PURL: https://resolver.sub.uni-goettingen.de/purl?365956996_0009|log9

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manuscripta math. 9, 91 - 103 (1973)
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ON THE REGULARITY OF THE SOLUTION OF THE BIHARMONIC VARIATIONAL INEQUALITY

Jens Frehse

It is shown that the solution of the biharmonic variational inequality has bounded second derivatives provided that the obstacle and the data are smooth.

An open problem in the theory of elliptic variational inequalities is to establish regularity theorems for higher order problems. At the present moment (see [3] and [6]), it is only known that a solution u in the Sobolev space H^m of an elliptic variational inequality of order 2m also lies in H^{m+1}_{loc}, provided that the data are sufficiently smooth. One easily constructs counterexamples (first proposed by Lewy and Stampacchia), even in one-dimensional cases in order to illustrate that the solution may in general not be in H^{m+2}_{loc}.

In this paper another contribution to this theory is given. It is shown that the solution of the biharmonic variational inequality has locally bounded second deriva-

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During the preparation of a portion of the paper, the author was a guest of the Scuola Normale Superiore in Pisa, supported by the German Research Association (DFG).

tives provided that the data are smooth. Our result can be extended to fourth order variational inequalities where the corresponding differential operator can be written as a product of two second order differential operators. This was partly done in [4], part 2.

Notations. Let Ω be a bounded open subset of the euclidean space R^n, $n\geq 2$. For any $p\in[1,\infty)$, let $L^p=L^p(\Omega)$ be the usual Lebesgue space over Ω with norm $\|\ \|_p$ defined by $\|u\|_p = (\int\|u\|^p dx)^{1/p}$, $1\leq p<\infty$, $\|u\|_\infty = \text{ess sup}\{|u(x)|, x\in\Omega\}$. The integration $\int$ always runs over Ω , in the sense of Lebesgue.- For any natural number m, let $H^{m,p}=H^{m,p}(\Omega)$ be the Sobolev space over Ω with norm $\|.\|_{m,p}$ defined by

$$\|u\|_{m,p} = \Sigma_\alpha\|u\|_p \ , \ (|\alpha|\leq m).$$

Here we have set $\partial^\alpha = \Pi_i \partial_i^\alpha$, $\partial_i = \partial/\partial x_i$, $(i=1,\ldots,n)$, $|\alpha| = \Sigma_i \alpha_i$, $(i=1,\ldots,n)$, for any multi-index $\alpha=(\alpha_1,\ldots,\alpha_n)$. For any subset $\Omega_o\subset\Omega$ whose closure lies in Ω we write $\Omega_o\subset\subset\Omega$. Furthermore, let L^p_{loc} resp. $H^{m,p}_{loc}$ denote the set of all functions f such that the restrictions $f|\Omega_o \in L^p(\Omega_o)$ resp. $\in H^{m,p}(\Omega_o)$ for any open set $\Omega_o\subset\subset\Omega$. Finally, let $H^{m,p}_o$ be the closure of the space $C^\infty_o(\Omega)$ of test functions on Ω , and let $C^k(\Omega)$ resp. $C^k(\overline{\Omega})$ be the space of all k-times continuously (up to the boundary) differentiable functions on Ω.

The variational inequality: Let V be a closed subspace of $H^{2,2}(\Omega)$ containing $H^{2,2}_o(\Omega)$, and let $\psi\in H^{3,\infty}(\Omega)$ and $f_i\in L^\infty$, $i=1,\ldots,n$, be given functions.- Define $K = \{u\in V | u\geq\psi \text{ in } H^{2,2}\}$

(For the definition of "$u \geq \psi$ in $H^{2,2}(\Omega)$", see [5]).

We consider the problem:

Minimize

(1) $\int [(\Delta u)^2 - \Sigma_i f_i \partial_i u] dx$,$(i=1,\ldots,n)$, $u \in K$.

This problem is equivalent to the biharmonic variational inequality:

Find $u \in K$ such that

(2) $(\Delta u, \Delta(u-v)) \leq \Sigma_i (f_i, \partial_i (u-v))$,$(i=1,\ldots,n)$,

for all $v \in K$.

Here, Δ denotes the Laplacean, and $(w,z) = \int w \cdot z \, dx$.

If $K = \emptyset$, then (1) or (2) has a unique solution, provided that the form $(\Delta u, \Delta v)$ is K-coercive, which is the case if e.g. $V = H_0^{2,2}$.- If $\psi \in H^{3,2}$, then the solution $u \in H^{3,2}_{loc}$ (see [3] or [6]). In the case of two obstacles, i.e. $K = \{v \in V | \psi_1 \leq v \leq \psi_2\}$, or if ψ has convex corners, then one has to use the method described in [3] in order to obtain $u \in H^{3,2}_{loc}$,too. Up to now further regularity properties of the solution u of (1) are not known. (One expects $\nabla^2 u \in C(\Omega)$).

Thus, the following result is of interest:

THEOREM. Let u be a solution of (1) or (2) and let $\psi \in H^{3,\infty}$ and $f_i \in L^\infty$,$i=1,\ldots,n$. Then $u \in H^{2,\infty}_{loc}$, i.e. the second derivatives of u are uniformly bounded on interior domains of Ω.

REMARKS: (i) Explicit bounds for the second derivatives are given in lemma 2 and 3.

(ii) The theorem holds with the weaker conditions $\psi \in H^{3,n+\varepsilon}$,

$f_i \in L^{n+\varepsilon}$, $\varepsilon>0$, the proof, however, becomes more involved.

For the proof of the theorem, we need some lemmata which we shall prove first.- In the following, let
$[\xi]_\ell^r = |\xi|^{r-1}\xi$ if $|\xi|\leq\ell$ and $[\xi]_\ell^r = |\ell|^r \operatorname{sign}\xi$ if $|\xi|>\ell$, $\xi,\ell\in R^1$.
Furthermore, let

$$\Delta_h z(x) = \Sigma_i \delta_{ih} z(x) \ , \ (i=1,\dots,n),$$
$$\delta_{ih} z(x) = h^{-2}\left[z(x+he_i) - 2z(x) + z(x-he_i)\right] \ , \ h>0,$$

where the e_i denote the unit vectors of R^n.

LEMMA 1. Let $u\in K$ and $\phi\in H^{1,\infty}(\Omega)$, ϕ having compact support in Ω, $0\leq\phi\leq 1$. Then, for every $r\geq 1$, $h,\ell>0$, $h<\frac{1}{2}\operatorname{dist}(\partial\Omega,\operatorname{supp}\phi)$, there exists an $\varepsilon_o>0$ such that the function

$$\varepsilon_o\phi\cdot\left[\Delta_h(u-\psi)\right]_\ell^r$$

is $H^{1,2}$-limit of functions $t_k\in H^{2,2}$ with compact support in Ω and the property $u+t_k\in K$.

PROOF. The function $[\xi]_\ell^r$ is monotone; thus, for any real function z with $z\geq 0$ in Ω, we have

$$[\Delta_h z]_\ell^r \geq \left[-2h^{-2}z\right]_\ell^r \ . \tag{3}$$

Now, if $\varepsilon_o = \varepsilon_o(r,\ell,\phi,h)>0$ is chosen small enough, the first derivates of the function

$$f(\xi) = \xi + \varepsilon_o\phi\cdot\left[-2h^{-2}\xi\right]_\ell^r$$

are nonnegative and so, for any function $z\geq 0$, in view of (3),

$$0 = f(0) \leq f(z) = z + \varepsilon_o\phi\cdot\left[-2h^{-2}z\right]_\ell^r \leq z + \varepsilon_o\phi\cdot\left[\Delta_h z\right]_\ell^r \ . \tag{4}$$

Now, since $u\geq\psi$ in $H^{2,2}$, by definition there exists a sequence of functions $z_j\in H^{2,2}\cap C^\infty(\Omega)$ such that

$z_j \to u - \psi$ in $H^{2,2}$ $(j\to\infty)$ and $z_j \geq 0$ in Ω .

By virtue of (4) we have

(5) $$v_j := z_j + \varepsilon_o \phi \cdot [\Delta_h z_j]^r_\ell \geq 0 \quad \text{in } \Omega .$$

Let $\phi_1 \in C^\infty(R^n)$, $\phi_2 \in C^\infty_o(\Omega)$ such that

$0 \leq \phi_1 \leq 1$, $0 \leq \phi_2 \leq 1$, $\phi_1 + \phi_2 = 1$ in Ω , $\phi_2 = 1$ on supp ϕ .

According to our assumptions, $\phi_1 \cdot z_j \geq 0$ and $\phi_2 \cdot v_j \geq 0$ in Ω .

Let $\omega_\varepsilon *$ denote the usual mollification by convolution with nonnegative mean functions $\in C^\infty_o(R^n)$, see e.g. [1], chap. 1 .

Then $\omega_\varepsilon * (\phi_2 \cdot v_j) \geq 0$ in Ω and

$$s_{\varepsilon j} := \phi_1 \cdot z_j + \omega_\varepsilon * \phi_2 \cdot v_j \geq 0 \quad \text{in } \Omega .$$

Letting $j\to\infty$ we obtain

(6) $s_\varepsilon := \phi_1 \cdot (u-\psi) + \omega_\varepsilon * \phi_2 \cdot (u - \psi + \varepsilon_o \phi \cdot [\Delta_h (u-\psi)]^r_\ell) \geq 0$

in $H^{2,2}$, keeping into account that $s_\varepsilon, s_{\varepsilon j} \in H^{2,2}$. Setting $u_\varepsilon = s_\varepsilon + \psi$, we may write

(7) $u_\varepsilon = u + (-\phi_2 u + \omega_\varepsilon * \phi_2 u) + (\phi_2 \psi - \omega_\varepsilon * \phi_2 \psi) + \varepsilon_o \omega_\varepsilon * \phi \cdot [\Delta_h (u-\psi)]^r_\ell$

where we used $\phi_1 + \phi_2 = 1$.- All the terms on the right-hand side of (7) except the term u have compact support in Ω . Since $u \in V$ and $V \supseteq H^{2,2}_o$, $u_\varepsilon \in V$. Furthermore, $u_\varepsilon - \psi = s_\varepsilon \geq 0$ in $H^{2,2}$ on account of (6) and so, $u_\varepsilon \in K$.

Finally, it is easy to see that

$$u_\varepsilon \to u + \varepsilon_o \phi \cdot [\Delta_h (u-\psi)]^r_\ell \quad \text{in } H^{1,2} \ (\varepsilon \to 0) ,$$

this follows from the fact that

$$\phi \cdot [\Delta_h (u-\psi)]^r_\ell \in H^{1,2}$$

and from the well-known convergence properties of the mollification. Thus, the functions $t_k = u_{\varepsilon(k)} - u$, $(\varepsilon(k) \to 0)$, are the ones desired, the lemma is proved.

The following lemma gives a local bound for Δu :

LEMMA 2. Under the hypotheses of the theorem, $\Delta u \in L^{\infty}_{loc}(\Omega)$, and there exists a constant C only depending on the dimension n such that for every ball $B_R(x) \subset \Omega$ with the property $B_{2R}(x) \subset\subset \Omega$ the following inequality holds

$$\|\Delta(u-\psi)\|^2_{\infty,R} \leq C \cdot \max\{R^{-n} \int_{2R} |\Delta(u-\psi)|^2 dx,\ R^2 \Sigma_{i=1}^n \|f_i\|^2_{\infty;2R}\} .$$

Here $\|\cdot\|_{\infty;R}$ resp. $\int_R$ denotes the L^{∞}-norm resp. the integration taken over B_R .

PROOF OF LEMMA 2. The proof uses the Moser technique, [7]. Let u be the solution of the biharmonic variational inequality. On account of the assumptions put upon the data, $u \in H^{3,2}_{loc}(\Omega)$ and thus

$$(8) \quad -\Sigma_i(\partial_i \Delta u, \partial_i(u-v)) \leq \Sigma_i(f_i, \partial_i(u-v)) , \ v \in K .$$

Here and in the following, the summation Σ_i always runs from i=1 to i=n .- Using the functions t_k constructed in lemma 1, setting $v=t_k$ in (8), and going to the limit $k \to \infty$ we obtain

$$\varepsilon_o \Sigma_i(\partial_i \Delta u, \partial_i(\phi \cdot [\Delta_h(u-\psi)]^r_\ell) \leq -\varepsilon_o \Sigma_i(f_i, \partial_i(\phi \cdot [\Delta_h(u-\psi)]^r_\ell)) .$$

Cancelling $\varepsilon_o > 0$ we may pass to the limit $h \to 0$. This is possible on account of the following consideration. For $g \in H^{1,2}$,

$$(g, \partial_i(\phi \cdot [\Delta_h(u-\psi)]^r_\ell)) = -(\partial_i g, \phi \cdot [\Delta_h(u-\psi)]^r_\ell) \to$$

$$\to -(\partial_i g, \phi \cdot [\Delta(u-\psi)]^r_\ell) = (g, \partial_i(\phi \cdot [\Delta(u-\psi)]^r_\ell) , \quad (h \to 0) .$$

Furthermore, $\partial_i(\phi \cdot [\Delta_h(u-\psi)]^r_\ell)$ is bounded uniformly for $h \to 0$. In fact, $|\partial_i [\Delta_h(u-\psi)]^r_\ell| \leq |\partial_i \Delta_h(u-\psi)| \cdot |[\Delta_h(u-\psi)]^r_\ell|$ and the right-hand side of the last inequality is uniformly bounded in L^2 because of the truncation and the fact $u-\psi \in H^{3,2}_{loc}$.

The convergence desired follows from the Banach-Steinhaus theorem.- From this, we conclude

(9) $\quad \Sigma_i(\partial_i z, \partial_i(\phi\cdot[z]_\ell^r)) \leq \Sigma_i(f_i', \partial_i(\phi\cdot[z]_\ell^r))$

where $z=\Delta u-\Delta\psi$ and $f_i'=-\partial_i\Delta\psi-f_i$ Setting $\phi=\tau^2, \tau\in H^{1,\infty}$, and computing the term $\partial_i(\phi\cdot[z]_\ell^r)$ in (9), we get terms which are estimated in the following way

$$|2\Sigma_i(\partial_i z, \tau\partial_i\tau[z]_\ell^r)| \leq \tfrac{1}{4}r\int_\ell \tau^2|\nabla z|^2|z|_\ell^{r-1}dx + \frac{4}{r}\|\nabla\tau\|_\infty^2\int_*|z|_\ell^{r+1}dx$$

$$r\Sigma_i(f_i', \tau^2\partial_i z[z]_\ell^{r-1}) \leq \tfrac{1}{4}r\int_\ell \tau^2|\nabla z|^2|z|_\ell^{r-1}dx + r\Sigma_i\|f_i'\|_\infty^2\int_*|z|_\ell^{r-1}dx$$

$$2\Sigma_i(f_i', \tau\partial\tau[z]_\ell^r) \leq r^{-1}\|\nabla\tau\|_\infty^2\int_*|z|_\ell^{r+1}dx + r\Sigma_i\|f_i'\|_\infty^2\int_*|z|_\ell^{r-1}dx\ .$$

Here $|z|_\ell^r = |[z]_\ell^r|$, and $|z|_\ell^0 = 1$. The integration $\int_\ell$ is taken over the set where $|z|<\ell$.- Applying the last estimates to (9), we obtain with $\int_* = \int_{\text{supp }\tau}$

(10) $\tfrac{1}{2}r\int_\ell \tau^2|\nabla z|^2|z|_\ell^{r-1}dx \leq 5r^{-1}\|\nabla\tau\|_\infty^2\int_*|z|_\ell^{r+1}dx + 2rF\int_*|z|_\ell^{r-1}dx$

where $F = \Sigma_i\|f_i'\|_\infty^2$. By Sobolev's inequality

$$\left(\int(\tau^2|z|_\ell^r)^\sigma dx\right)^{1/\sigma} \leq S\int|\nabla(\tau|z|_\ell^{r/2})|^2dx \leq$$

$$\leq S(r/2)^2\int_\ell\tau^2|\nabla z|^2|z|^{r-2}dx + S\int|\nabla\tau|^2|z|_\ell^r dx$$

where $\sigma = n/(n-2)$ and S depends only on the dimension if $n\geq 3$; for $n=2$, $\sigma=2$ and $S = S'|\text{supp }\tau|^{1/2}$, S' depending only on the dimension. This, together with (10), yields

(11) $\quad \left(\int(\tau^2|z|_\ell^r)^\sigma dx\right)^{1/\sigma} \leq K\|\nabla\tau\|_\infty^2\int_*|z|_\ell^r dx + KFr^2\int_*|z|_\ell^{r-2}dx\ , r\geq 2.$

Note that we used (10) with r-1 instead of r . Here $K=\kappa S$, κ depending only on the dimension n .

Now, let B_R be a ball of radius R such that the concentric ball B_{2R} of radius 2R is compactly contained in Ω. Let B_j be a sequence of concentric balls such that $B_o=B_{2R}$ and B_j has the radius $(2-(\sigma-1)\Sigma_{k=1}^j\sigma^{-k})R$. Obviously, $\partial B_j\to\partial B_R$ $(j\to\infty)$.- Let $\tau_j\in H^{1,\infty}$ be the sequence of functions with the

property $\tau_j=0$ on $\Omega-B_j$, $\tau_j=1$ on B_{j+1} , $\|\nabla\tau_j\|_\infty\leq\sigma^{j+1}(\sigma-1)^{-1}R^{-1}$. Such functions can easily be constructed since $\operatorname{dist}(\partial B_j,\partial B_{j-1}) = \sigma^j(\sigma-1)R$.- Setting $r=r(j)=2\sigma^j$ and $\tau = \tau_j$ in (11), we obtain

$$(12)\quad (\int_{j+1}|z|_\ell^{r(j+1)}dx)^{1/\sigma} \leq K\sigma^{2j}[\sigma^2(\sigma-1)^{-2}R^{-2}\int_j|z|_\ell^{r(j)}dx + 4F\int_j|z|_\ell^{r(j)-2}dx]$$

Here, $\int_j$ denotes the integration over B_j. For $n=2$, K would be $K=\kappa S=S'|\operatorname{supp}\tau_j|^{1/2}$. Since $|\operatorname{supp}\tau_j|=|B_{2R}|$, we have $K=\kappa'S'R$ in this case, κ' depending only on the dimension. If

$$(13)\quad \sigma^2(\sigma-1)^{-2}R^{-2}\int_j|z|_\ell^{r(j)}dx \leq 4F\int_j|z|_\ell^{r(j)-2}dx$$

for infinitely many j, we have by virtue of Hölder's inequality

$$\int_j|z|_\ell^{r(j)-2}dx \leq \Gamma R^{2n/r(j)}(\int_j|z|_\ell^{r(j)}dx)^{1-2/r(j)} ,$$

with Γ depending only on the dimension, and by (13)

$$(14)\qquad \||z|_\ell\|^2_{r(j),j} \leq 4F\Gamma\sigma^2(\sigma-1)^{-2}R^2R^{2n/r(j)}$$

where $\|\cdot\|_{p,j}$ denotes the L^p-norm taken over B_j . Going to the limit $j\to\infty$ yields

$$(15)\qquad \||z|_\ell\|_{\infty,R} \leq 2R\sqrt{F\Gamma}\sigma(\sigma-1)^{-1}$$

If (13) is not true for infinitely many j , there exists an $m\geq 0$ such that the inequality (13) holds in the opposite direction for $j\geq m+1$. Applying this to (12) we obtain by taking the σ^j-th root

$$\||z|_\ell\|^2_{r(j+1),j+1} \leq \exp[\sigma^{-j}\ell n(2K\sigma^2(\sigma-1)^{-2}\sigma^{2j}R^{-2})]\,\||z|_\ell\|^2_{r(j),j}$$

$j\geq m$.- By recursion, we obtain

$$(16)\qquad \||z|_\ell\|^2_{r(k),k} \leq A_kB_k\||z|_\ell\|^2_{j(m),m}$$

where

$A_k = \exp[\Sigma_{j=m}^{k} \sigma^{-j}(\ell n K')]$, $K' = 2K\sigma^2(\sigma-1)^{-2}R^{-2}$

$B_k = \exp[\Sigma_{j=m}^{k} \sigma^{-j} \ell n \sigma^{2j}]$

For $k\to\infty$, $A_k \to (K')^{\gamma}$, $\gamma = \sigma^{-m+1}/(\sigma-1)$, $B_k \to B_\infty$

where B_∞ is some constant. Thus, in (16) we may pass to the limit $k\to\infty$ and obtain

$$\| |z|_\ell \|^2_{\infty;R} \leq (K')^{\gamma} B_\infty \| |z|_\ell \|^2_{j(m),m} \tag{17}$$

Let m be the minimal index such that the opposite of (13) holds for $j \geq m$. If $m=0$, then, for $n\geq 3$, $\gamma = n/2$ since $\sigma = n/(n-2)$, and $(K')^{\gamma} = CR^{-n}$, $j(m)=j(0)=2$. Therefore,

$$\| |z|_\ell \|^2_{\infty;R} \leq CR^{-n} \| |z|_\ell \|^2_{2;2R} \tag{18}$$

with C depending only on the dimension.

If $n=2$, then $\gamma=2$, and $K=\kappa' S' R$ and $K' = 2K\sigma^2(\sigma-1)^{-2}R^{-2} = 2\sigma^2(\sigma-1)^2\kappa' S' R^{-1}$ and, therefore, (18) holds in the case $n=2$ $(m=0)$, too .-

If $m>0$, then (13) holds with $j=m$, and together with (14) and (17), (for $j=m$), we receive

$$\| |z|_\ell \|^2_{\infty;R} \leq 4(K')^{\gamma} B_\infty F \Gamma \sigma^2(\sigma-1)^{-2} R^{2n/r(m)} R^2 \tag{19}$$

If $n\geq 3$, then $\gamma = \sigma^{-m+1}/(\sigma-1) = (2/r(m))\sigma/(\sigma-1) = n/r(m)$, and if $n=2$, then $\gamma = (2/r(m)) \cdot 2 = 4/r(m)$. Thus, $(R^{-2})^{\gamma} R^{2n/r(m)} = 1$ if $n\geq 3$, and $(R^{-1})^{\gamma} R^{2n/r(m)} = 1$ if $n=2$. Since $K' \alpha R^{-2}$ resp. αR^{-1} if $n\geq 3$ resp. $=2$, it follows that $(K')^{\gamma} R^{2n/r(m)} \leq C'$, with C' depending only on the dimension. Thus we obtain

$$\| |z|_\ell \|^2_{\infty,R} \leq CR^2 F \tag{20}$$

On account of (17) this holds also in the case that there are infinitely many j such that (13) holds.

By (18) and (20), we arrive at the inequality

$$\| |z|_\ell \|^2_{\infty,R} \leq C \max\{R^{-n} \| |z|_\ell \|^2_{2;2R} \; ; \; R^2 F\}$$

and passing to the limit $\ell\to\infty$, we obtain the statement of the lemma.

Since $\Delta u\in L^{\infty}_{loc}$, u is continuously differentiable in Ω , and the touching set $I=\{x|u(x)=\psi(x)\}$ is closed. Of course, Δu is continuous in $\Omega-I$. The estimate of lemma 2 also yields the continuity of Δu in certain points of the touching set, if n=2 :

COROLLARY to lemma 2. Let n=2 and $x\in I$ have a positive density with respect to the touching set, i. e.

$$|I\cap B_R(x)|/|B_R(x)|\geq c_o>0 \quad \text{for } R\to 0 .$$

Then Δu is continuous in x .

PROOF. If $x\in I$ has a positive density with respect to I, then also $|\{x\in B_R|\ \Delta u-\Delta\psi=0\}|/|B_R|\geq c_o>0$ for $R\to 0$. But then there holds Poincaré's inequality

$$R^{-2}\int_{2R}|\Delta u-\Delta\psi|^2dx < K_o\int_{2R}|\nabla\Delta(u-\psi)|^2dx , \quad R\to 0 ,$$

K_o not depending on R. Since $K_o\int_{2R}|\nabla\Delta(u-\psi)|^2dx$ is small for small R, the continuity follows from the estimate in lemma 2.

LEMMA 3. Let $\phi\in C_o^{\infty}(\Omega)$, $\phi\geq 0$, with supp $\phi=G\subset\subset\Omega$. Then, under the hypotheses of the theorem,

$$(21)\quad \partial_j^2(u\phi) > -k_n\left[\|\Delta u\|_{\infty,G}\|\phi\|_{3,\infty}+\|u\|_{\infty,G}\|\phi\|_{5,\infty}+\|u\|_{1,\infty,G}\|\phi\|_{4,\infty}+\right.$$
$$\left.+\Sigma_i\|f_i\|_{\infty;G}(\|\nabla\phi\|_{1,\infty}+1)\right] , \quad j=1,\dots,n,$$

a.e. in Ω with a constant k_n depending only on the dimension.

PROOF. Let $x_o\in\Omega$ and, for $x\neq x_o$, $g(x)=|x-x_o|^{-n+4}$, n=3,5,6,...,

$g(x)=-\ell n|x-x_0|$, $n=4$, $g(x)=-|x-x_0|$, $n=3$, and
$g(x)=|x-x_0|^2(\ell n|x-x_0|-1)$, $n=2$.

By elementary calculations (see also [2], chap.5)

$$\Delta\Delta g = c_n\delta(\cdot-x_0)$$

in the sense of distributions, where $\delta(\cdot-x_0)$ is the Dirac functional at the point x_0 and $c_n>0$ is a constant depending on the dimension. Furthermore, it is elementary to check that

$$(\partial_j^2 - \tfrac{1}{2}\Delta)g(x) \geq 0 \ , \ x\neq x_0 \ , \ n=3,4,\ldots,$$
$$(\partial_j^2 - \tfrac{1}{2}\Delta)g(x) \geq -1, \ x\neq x_0 \ , \ n=2.$$

In fact, this follows from the identity

$\partial_j^2|x|^{-n+4} = (-n+4)\left[|x|^{-n+2} + (-n+2)x_j^2|x|^{-n}\right]$, $n\neq 2$, $n\neq 4$,

$\Delta|x|^{-n-4} = (-n+4)\left[n|x|^{-n+2} + (-n+2)|x|^{-n}\right] = 2(-n+4)|x|^{-n+2}$

$(\partial_j^2 - \tfrac{1}{2}\Delta)|x|^{-n-4} = (-n+4)|x|^{-n}(-n+2)x_j^2 \geq 0$ for $n\geq 5$ resp. ≤ 0 for $n=3$,

$\partial_j^2\ell n|x| = |x|^{-2} - 2x_j^2|x|^{-3}$, $\Delta\ell n|x| = (n-2)|x|^{-2}$,

$(\partial_j^2 - \tfrac{1}{2}\Delta)\ell n|x| = -2x_j^2|x|^{-2} \leq 0$ for $n=4$,

$\partial_j^2(|x|^2\ell n|x|) = 2\ell n|x| + 2x_j^2|x|^{-2} + 1$, $\Delta(|x|^2\ell n|x|) =$

$= 2n\ell n|x| + 2 + n$,

$\Delta(|x|^2(\ell n|x|-1)) = 4\ell n|x|$ for $n=2$,

$(\partial_j^2 - \tfrac{1}{2}\Delta)(|x|^2(\ell n|x|-1)) = 2x_j^2|x|^{-2} - 1 \geq -1$ for $n=2$,

always $x\neq 0$.

Now, let $\phi\in C_0^\infty(\Omega)$, $\phi\geq 0$, and let $\omega_\varepsilon *$ denote the mollification with a mollifier $\omega_\varepsilon\in C_0^\infty(R^n)$, $\omega_\varepsilon\geq 0$, $\int\omega_\varepsilon dx=1$. Then, $t_\varepsilon := \phi\omega_\varepsilon *[(\partial_j^2-\tfrac{1}{2}\Delta)g-1] \geq 0$ in Ω for small ε and $u+t_\varepsilon\in K$. Thus we may insert $u+t_\varepsilon$ into the variational inequality and obtain

$$(\Delta u,\Delta t_\varepsilon) \geq \Sigma_i(f_i,\partial_i t_\varepsilon) \tag{22}$$

Setting $L=\partial_j^2 - \frac{1}{2}\Delta$, we obtain by way of elementary calculations

$(\Delta u,\Delta t_\varepsilon) = (u,\Delta^2 t_\varepsilon) = (\Delta u,\Delta\phi)+(u,\phi\Delta^2 L\omega_\varepsilon * g)+4(u,\nabla\Delta\phi\cdot\nabla\Delta L\omega_\varepsilon * g)+$
$+(u,\Sigma_{\alpha\beta} c_{\alpha\beta}\partial^\alpha\phi\partial^\beta\omega_\varepsilon * g)$, $(|\alpha|+|\beta|=6,\ 2\leq|\beta|\leq 4)$.

The third term in the last sum can be written in the following manner:

$4(u,\nabla\Delta\phi\cdot\nabla\Delta L\omega_\varepsilon * g) = 4(\Delta(u\nabla\Delta\phi),\nabla L\omega_\varepsilon * g)$

and, since $\Delta u\in L^\infty_{loc}$ and $\nabla^3 g\in L^1$, it can be estimated by the right-hand side of (21). The last term can be written in the form $\Sigma_{\alpha\beta\gamma} c_{\alpha\beta\gamma}(\partial^\beta(u\partial^\alpha\phi),\partial^\gamma\omega_\varepsilon * g)$, $|\alpha|+|\beta|+|\gamma|=6, |\gamma|=3, |\beta|\leq 1$ and can thus be estimated by the right-hand side of (21). Finally, it is obvious how to handle the terms. $(f_i,\partial_i(\phi L\omega_\varepsilon * g))$.- Therefore, by (22), we receive the inequality $(u,\phi\Delta^2 L\omega_\varepsilon * g)=(L(u\phi),\Delta^2\omega_\varepsilon * g)=c_n L(u\phi)*\omega_\varepsilon(x_0) \geq R$ where R is the right-hand side of (21). The lemma follows by going to the limit $\varepsilon\to 0$: $L(u\phi) \geq c_n^{-1} R$ a.e. in Ω .

PROOF OF THE THEOREM. Choosing $\phi\in C_0^\infty(\Omega)$ such that $\phi=1$ on $\Omega' c\subset\Omega$ in lemma 3 we obtain that the derivatives $\partial_j^2 u$ are locally bounded from below. On the other hand, $\Sigma_i\partial_i^2 u$ is locally bounded, hence $\partial_i^2 u$ must also be bounded from above The boundedness of the mixed derivatives $\partial_i\partial_k u$ follows from an orthogonal transformation of the problem which yields the boundedness of $(\partial_i\pm\partial_k)^2 u$. This is possible, since Δ^2 is invariant in the case of orthogonal transformations and the lower order term $\Sigma_i(f_i,\partial_i(u-v))$ is transformed into a similar one.

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(Received November 13, 1972)